

## Reverse approximation of gradient flows as Minimizing Movements: a conjecture by De Giorgi

FLORENTINE FLEISSNER AND GIUSEPPE SAVARÉ

**Abstract.** We consider the Cauchy problem for the gradient flow

$$u'(t) = -\nabla\phi(u(t)), \quad t \geq 0; \quad u(0) = u_0, \quad (\star)$$

generated by a continuously differentiable function  $\phi : \mathbb{H} \rightarrow \mathbb{R}$  in a Hilbert space  $\mathbb{H}$  and study the reverse approximation of solutions to  $(\star)$  by the De Giorgi Minimizing Movement approach.

We prove that if  $\mathbb{H}$  has finite dimension and  $\phi$  is quadratically bounded from below (in particular if  $\phi$  is Lipschitz) then for *every* solution  $u$  to  $(\star)$  (which may have an infinite number of solutions) there exist perturbations  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$  ( $\tau > 0$ ) converging to  $\phi$  in the Lipschitz norm such that  $u$  can be approximated by the Minimizing Movement scheme generated by the recursive minimization of  $\Phi(\tau, U, V) := \frac{1}{2\tau}|V - U|^2 + \phi_\tau(V)$ :

$$U_\tau^n \in \operatorname{argmin}_{V \in \mathbb{H}} \Phi(\tau, U_\tau^{n-1}, V) \quad n \in \mathbb{N}, \quad U_\tau^0 := u_0. \quad (\star\star)$$

We show that the piecewise constant interpolations with time step  $\tau > 0$  of *all* possible selections of solutions  $(U_\tau^n)_{n \in \mathbb{N}}$  to  $(\star\star)$  will converge to  $u$  as  $\tau \downarrow 0$ . This result solves a question raised by Ennio De Giorgi in [9].

We also show that even if  $\mathbb{H}$  has infinite dimension the above approximation holds for the distinguished class of minimal solutions to  $(\star)$ , that generate all the other solutions to  $(\star)$  by time reparametrization.

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### 1. Introduction

In his highly inspiring paper [9] Ennio De Giorgi introduced the variational notion of *Minimizing Movement* in order to present a general and unifying approach to a large class of evolution problems in a vector, metric or even topological framework.

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In the case of time-invariant evolutions in a topological space  $\mathbb{H}$ , Minimizing Movements can be characterized by the recursive minimization of a functional  $\Phi : (0, \infty) \times \mathbb{H} \times \mathbb{H} \rightarrow [-\infty, +\infty]$ . For a given initial datum  $u_0 \in \mathbb{H}$  and a parameter  $\tau > 0$  (which plays the role of discrete time step size) one looks for sequences  $(U_\tau^n)_{n \in \mathbb{N}}$  such that  $U_\tau^0 := u_0$  and for every  $n \geq 1$

$$\Phi(\tau, U_\tau^{n-1}, U_\tau^n) = \min_{V \in \mathbb{H}} \Phi(\tau, U_\tau^{n-1}, V), \quad \text{i.e.} \quad U_\tau^n \in \operatorname{argmin} \Phi(\tau, U_\tau^{n-1}, \cdot). \quad (1.1)$$

Any sequence satisfying (1.1) gives rise to a *discrete solution*  $U_\tau : [0, \infty) \rightarrow \mathbb{H}$  at time step  $\tau$ , obtained by piecewise constant interpolation of the values  $(U_\tau^n)_{n \in \mathbb{N}}$ :

$$U_\tau(0) := U_\tau^0 = u_0, \quad U_\tau(t) := U_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau], \quad n \in \mathbb{N}. \quad (1.2)$$

A curve  $u : [0, +\infty) \rightarrow \mathbb{H}$  is called Minimizing Movement associated to  $\Phi$  with initial datum  $u_0$  (short  $u \in \operatorname{MM}(\Phi, u_0)$ ) if there exist discrete solutions  $U_\tau$  (for  $\tau$  in a right neighborhood of 0) to the scheme (1.1) converging pointwise to  $u$  as  $\tau \downarrow 0$ :

$$u(t) = \lim_{\tau \downarrow 0} U_\tau(t) \quad \text{for every } t \geq 0. \quad (1.3)$$

A curve  $u : [0, +\infty) \rightarrow \mathbb{H}$  is more generally called Generalized Minimizing Movement (short  $u \in \operatorname{GMM}(\Phi, u_0)$ ) if there exist a suitable vanishing subsequence  $k \mapsto \tau(k)$  of time steps and corresponding discrete solutions  $U_{\tau(k)}$  at time step  $\tau(k)$  to (1.1) such that

$$u(t) = \lim_{k \rightarrow \infty} U_{\tau(k)}(t) \quad \text{for every } t \geq 0. \quad (1.4)$$

The general notion of Minimizing Movement scheme has proved to be extremely useful in a variety of analytic, geometric and physical contexts; we refer to [2, 4, 10], [6] and [17, 20] for a more detailed account of some applications and developments and to the pioneering paper [1] by Almgren, Taylor, and Wang.

Perhaps the simplest (though still interesting) situation arises if  $\mathbb{H}$  is a Hilbert space and one tries to implement the scheme (1.1) to solve the Cauchy problem for the gradient flow

$$u'(t) = -\nabla \phi(u(t)), \quad t \geq 0, \quad (1.5)$$

with initial datum  $u_0$  and continuously differentiable driving functional  $\phi : \mathbb{H} \rightarrow \mathbb{R}$ . In this case a natural choice for the functional  $\Phi$  is

$$\Phi(\tau, U, V) := \frac{1}{2\tau} |V - U|^2 + \phi(V), \quad (1.6)$$

for which the scheme (1.1) represents a sort of iterated minimization of  $\phi$  perturbed by  $\frac{1}{2\tau} |\cdot - U|^2$ . The last term penalizes the squared distance (induced by the norm  $|\cdot|$  of  $\mathbb{H}$ ) from the previous minimizer  $U$ . The Euler equation associated with the minimum problem (1.1) is then given by

$$\frac{U_\tau^n - U_\tau^{n-1}}{\tau} + \nabla \phi(U_\tau^n) = 0, \quad (1.7)$$

so that the Minimizing Movement scheme can be considered as a variational formulation of the implicit Euler method applied to (1.5). It is then natural to compare the class of solutions to (1.5) and the classes of Minimizing Movements  $\text{MM}(\Phi, u_0)$  and Generalized Minimizing Movements  $\text{GMM}(\Phi, u_0)$  for  $\Phi$  as in (1.6).

If  $\phi$  is a convex (or a quadratic perturbation of a convex) function, it is possible to prove (see, e.g., [2, 4, 7]) that the Minimizing Movement scheme (1.1) is convergent to the unique solution  $u$  of (1.5) with initial datum  $u_0$ , i.e.  $\text{MM}(\Phi, u_0) = \{u\}$ . This fundamental result can be extended to general convex and lower semicontinuous functions  $\phi$ , possibly taking the value  $+\infty$  at some point of  $\mathbb{H}$ , provided (1.5) is suitably formulated as a subdifferential inclusion. Convexity assumptions can also be considerably relaxed [16, 21] as well as the Hilbertian character of the distance (see, e.g., [4, 16, 19]).

### Minimizing Movements and gradient flows governed by $C^1$ functions

If  $\mathbb{H}$  is a finite dimensional Euclidean space and  $\phi$  is a continuously differentiable Lipschitz function, or more generally, a continuously differentiable function satisfying the lower quadratic bound

$$\exists \tau_* > 0, \phi_* \in \mathbb{R} : \quad \frac{1}{2\tau_*}|x|^2 + \phi(x) \geq -\phi_* \quad \text{for every } x \in \mathbb{H}, \quad (1.8)$$

it is not difficult to see that the set  $\text{GMM}(\phi, u_0)$  is not empty and that all its elements are solutions to (1.5).

In general, there are more than one solution to (1.5) with initial datum  $u_0$ . A notable aspect is that the set  $\text{MM}(\phi, u_0)$  may be empty and/or  $\text{GMM}(\phi, u_0)$  merely a proper subset of the class of solutions to (1.5) with initial datum  $u_0$ . Such peculiarities pointed out by De Giorgi can be observed even in one-dimensional examples of gradient flows driven by  $C^1$  Lipschitz functions [9].

It is then natural to look for possible perturbations of the scheme associated with (1.6), generating *all* the solutions to (1.5): this property would deepen our understanding of a gradient flow as a minimizing motion. This kind of question has also been treated in the different context of rate-independent evolution processes [18] from which we borrow the expression *reverse approximation*.

A first contribution [12, 13] in the framework of the Minimizing Movement approach to (1.5) deals with a uniform approximation of  $\Phi$ , based on allowing approximate minimizers in each step of the scheme generated by (1.6).

A much more restrictive class of approximation was proposed by De Giorgi, who made the following conjecture [9, Conjecture 1.1]:

**Conjecture (De Giorgi '93).** Let us suppose that  $\mathbb{H}$  is a finite dimensional Euclidean space and  $\phi : \mathbb{H} \rightarrow \mathbb{R}$  is a continuously differentiable Lipschitz function. A map  $u \in C^1([0, \infty); \mathbb{H})$  is a solution of (1.5) if and only if there exists a family  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$ ,  $\tau > 0$ , of Lipschitz perturbations of  $\phi$  such that

$$\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0, \quad (1.9)$$

and for the corresponding generating functional

$$\Phi(\tau, U, V) := \frac{1}{2\tau} |V - U|^2 + \phi_\tau(V) \quad (1.10)$$

one has  $u \in \text{GMM}(\Phi, u(0))$ .

$\text{Lip}[\cdot]$  in (1.9) denotes the Lipschitz seminorm

$$\text{Lip}[\psi] := \sup_{x, y \in \mathbb{H}, x \neq y} \frac{\psi(y) - \psi(x)}{|y - x|} \quad \text{whenever } \psi : \mathbb{H} \rightarrow \mathbb{R}. \quad (1.11)$$

One of the main difficulties of proving this property concerns the behaviour of  $u$  at critical points  $w \in \mathbb{H}$  where  $\nabla\phi(w) = 0$  vanishes. Since  $\nabla\phi$  is just a  $C^0$  map, it might happen that  $u$  reaches a critical point after finite time, stays there for some amount of time and then leaves the point again. An even worse scenario might happen if the 0 level set of  $\nabla\phi$  is not discrete. Even in the one dimensional case it is possible to construct functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  with a Cantor-like 0 level set  $K \subset \mathbb{R}$  of  $\phi'$  and corresponding solutions  $u$  parametrized by a finite measure  $\mu$  concentrated on  $K$  and singular with respect to the Lebesgue measure (see Appendix A for an explicit example).

A second difficulty arises from the lack of stability of the evolution, due to non-uniqueness: since even small perturbations may generate quite different solutions, one has to find suitable perturbations of  $\phi$  that keep these instability effects under control.

**Aim and plan of the paper** In this paper we address the question raised by De Giorgi and we give a positive answer to the above conjecture, in a stronger form (Theorem 6.4): we will show that it is possible to find Lipschitz perturbations  $\phi_\tau$  of  $\phi$  in such a way that (1.9) holds and

$$\text{MM}(\Phi, u(0)) = \{u\} = \text{GMM}(\Phi, u(0)) \quad (1.12)$$

for the corresponding generating functional  $\Phi$  defined by (1.10). An equivalent characterization of (1.12) can be given in terms of the *discrete solutions* to the scheme: *all* the discrete solutions  $U_\tau$  of (1.1) will converge to  $u$  as  $\tau \downarrow 0$ . Our result also covers the case of a  $C^1$  function  $\phi$  satisfying the lower quadratic bound (1.8).

Moreover, this reverse approximation can also be performed if  $\mathbb{H}$  has infinite dimension, for a particular class of solutions (Theorem 4.8), which is still sufficiently general to generate all the possible solutions by time reparametrization (Theorem 3.6).

In order to obtain an appropriate reverse approximation, we will introduce and apply new techniques that seem of independent interest and give further information on the approximation of the gradient flows (1.5) in a finite and infinite dimensional framework.

In Section 2 we will collect some preliminary material and we will give a detailed account of notions of *approximability* of gradient flows (Section 2.4), in

particular the notion of *strong approximability* (which is equivalent to (1.12) in the finite dimensional case) and the notion of *strong approximability in every compact interval*  $[0, T]$  (which appears to be more fitting in the infinite dimensional setting lacking in compactness).

A first crucial concept in our analysis is a *notion of partial order between solutions* to (1.5). Such notion plays an important role in any situation where non-uniqueness phenomena are present. The basic idea is to study the family of all the solutions  $u$  that share the same range  $R[u] = u([0, \infty))$  in  $\mathbb{H}$ . On this class it is possible to introduce a natural partial order by saying that  $u \succ v$  if there exists an increasing 1-Lipschitz map  $\mathbf{z} : [0, \infty) \rightarrow [0, \infty)$  such that  $u(t) = v(\mathbf{z}(t))$  for every  $t > 0$ .

We will show in Theorem 3.6 that for a given range  $R = R[u]$  there is always a distinguished solution  $v$  (called *minimal*), which induces all the other ones by such time reparametrization. This solution has the remarkable property to cross the critical set of the energy  $\{w \in \mathbb{H} : \nabla\phi(w) = 0\}$  in a Lebesgue negligible set of times (unless it becomes eventually constant after some time  $T_*$ , in that case it has the property in  $[0, T_*]$ ).

This analysis will be carried out for  $C^1$  solutions to the Cauchy problem (1.5) for a gradient flow in an infinite dimensional Hilbert space, but it can be considerably generalized and adapted for abstract evolution problems [11] including general gradient flows in metric spaces (under standard assumptions on the energy functional and on its metric slope as in [4]) and generalized semiflows (which have been introduced in [5]).

In Section 4.1 we will study the general problem to find Lipschitz perturbations of  $\phi$  which confine the discrete solutions of the Minimizing Movement scheme to a given compact set  $\mathcal{U}$ . We will find that a “penalization” with the distance from  $\mathcal{U}$  is sufficient to obtain this property. The important thing here will be a precise quantitative estimate of appropriate “penalty” coefficients depending on the respective time step and on a sort of approximate invariance of  $\mathcal{U}$ .

In Section 4.2 we will obtain a first result on the reverse approximation of gradient flows. We will prove that every minimal solution to (1.5) is approximable in the strong form (1.12) by applying the estimates from Section 4.1 to suitably chosen compact subsets of its range.

This result can be extended to infinite dimensional Hilbert spaces, even if in the infinite dimensional case, the existence of solutions to gradient flows of  $C^1$  functionals is not guaranteed a priori (existence of a solution can be proved if  $\nabla\phi$  is weakly continuous, see [8, Theorem 7]). However, if a solution exists, it always admits a minimal reparametrization and our result can be applied.

The reparametrization technique and the technique of confining discrete solutions to a given compact set provide a foundation for the reverse approximation of the gradient flows. The last crucial step is a reduction to the one dimensional case and its careful analysis. The detailed study of the one dimensional situation will be performed in Section 5. We will find a smoothing argument that allows to approximate any solution to (1.5) by a sequence of minimal solutions for perturbed energies. We can then base our proof of the reverse approximation

(1.12) for arbitrary solutions on the approximation by minimal solutions (which are approximable in the form (1.12)) instead of working directly on the discrete scheme.

In Section 6 the one-dimensional result is ‘lifted’ to arbitrary finite dimension by a careful use of the Whitney extension Theorem (this is the only point where we need a finite dimension): in this way, we will obtain the reverse approximation result (1.12) for arbitrary solutions to (1.5) in finite dimension.

### List of main notation

$\mathbb{H}, \langle \cdot, \cdot \rangle,  \cdot $	Hilbert space, scalar product and norm;
$d_T(u, v), d_\infty(u, v)$	distances between functions in $\mathbb{H}^{[0, T]}$ and $\mathbb{H}^{[0, \infty)}$ , (2.1)-(2.2);
$D_T(v, \mathcal{U}), D_\infty(v, \mathcal{U})$	distances between a map $v$ and a collection of maps $\mathcal{U}$ , (2.4)-(2.5);
$\text{GF}[\phi]$	solutions of the gradient flow equation (GF);
$\text{TGF}[\phi]$	truncated solutions of equation (GF), see before Theorem 3.6;
$\text{GF}_{\min}[\phi]$	class of <i>minimal solutions</i> to (GF), Definition 3.1;
$S[\psi]$	subset of $x \in \mathbb{H}$ where $\nabla \psi(x) = 0$ ;
$T_\star(u)$	minimal time after which $u$ is definitely constant, (2.11);
$U_\tau$	piecewise constant interpolant of a minimizing sequence $(U_\tau^n)_{n \in \mathbb{N}}$ ;
$\text{Lip}[\psi]$	Lipschitz constant of a real map $\psi : \mathbb{H} \rightarrow \mathbb{R}$ , (1.11);
$\succ$	partial order in $\text{GF}[\phi]$ , Definition 3.1;
$\Phi(\tau, U, V)$	functional characterizing the Minimizing Movement scheme, (2.17)-(2.18);
$\text{MM}(\Phi, u_0)$	Minimizing Movements, (2.19);
$\text{GMM}(\Phi, u_0)$	Generalized Minimizing Movements, (2.20);
$\text{MS}_\tau(\psi; u_0), \text{MS}_\tau(\psi; u_0, N)$	Minimizing sequences, (2.13) and Remark 2.3;
$\text{M}_\tau(\psi; u_0), \text{M}_\tau(\psi; u_0, T)$	Discrete solutions, (2.14) and Remark 2.3;
$N(\tau, T)$	$\min\{n \in \mathbb{N} : n\tau \geq T\}$ , Remark 2.3;
$\mathcal{U}(\tau, T)$	sampled values of a map $u$ , (4.23).

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## 2. Notation and preliminary results

### 2.1. Vector valued curves and compact convergence

Throughout the paper, let  $(\mathbb{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space with norm  $|\cdot| := \sqrt{\langle \cdot, \cdot \rangle}$ .

A function  $\psi : \mathbb{H} \rightarrow \mathbb{R}$  is Lipschitz if  $\text{Lip}[\psi] < \infty$ , where  $\text{Lip}[\cdot]$  has been defined in (1.11).  $\text{Lip}(\mathbb{H})$  will denote the vector space of Lipschitz real functions on  $\mathbb{H}$ .

$C^1(\mathbb{H})$  will denote the space of continuously differentiable real functions: by Riesz duality, the differential  $D\psi(x) \in \mathbb{H}'$  of  $\psi \in C^1(\mathbb{H})$  at a point  $x \in \mathbb{H}$  can be represented by a vector  $\nabla\psi(x) \in \mathbb{H}$ . The set of stationary points will be denoted by  $S[\psi] := \{v \in \mathbb{H} : \nabla\psi(v) = 0\}$ . Notice that a function  $\psi \in C^1(\mathbb{H})$  belongs to  $\text{Lip}(\mathbb{H})$  if and only if  $x \mapsto |\nabla\psi(x)|$  is bounded in  $\mathbb{H}$ .

Let  $T \in (0, \infty)$ ; we introduce a distance on the vector space  $\mathbb{H}^{[0, T]}$  (respectively  $\mathbb{H}^{[0, +\infty)}$ ) of curves defined on  $[0, T]$  (respectively  $[0, +\infty)$ ) with values in  $\mathbb{H}$  by setting

$$d_T(u, v) := \sup_{t \in [0, T]} (|u(t) - v(t)| \wedge 1) \quad \text{for every } u, v : [0, T] \rightarrow \mathbb{H}, \quad (2.1)$$

$$d_\infty(u, v) := \sup_{t \geq 0} \frac{1}{1+t} (|u(t) - v(t)| \wedge 1) \quad \text{for every } u, v : [0, \infty) \rightarrow \mathbb{H}. \quad (2.2)$$

$d_T$  clearly induces the topology of uniform convergence on the interval  $[0, T]$ . It is not difficult to show that the distance  $d_\infty$  induces the topology of compact convergence, *i.e.* the topology of uniform convergence on compact sets of  $[0, +\infty)$ : for every  $T > 0$  we have

$$\begin{aligned} (1+T)^{-1} d_T(u|_{[0, T]}, v|_{[0, T]}) &\leq d_\infty(u, v) \\ &\leq d_T(u|_{[0, T]}, v|_{[0, T]}) \vee (1+T)^{-1} \end{aligned} \quad (2.3)$$

so that a net  $(u_\lambda)_{\lambda \in \Lambda}$  in  $\mathbb{H}^{[0, \infty)}$  is  $d_\infty$  convergent if and only if it is convergent in the topology of compact convergence.

We will denote by  $R[u]$  the range of a function.

$C^k([0, +\infty); \mathbb{H})$  will be the vector space of  $C^k$  curves with values in  $\mathbb{H}$ . We will consider  $C^0([0, \infty); \mathbb{H})$  as a (closed) subspace of  $\mathbb{H}^{[0, \infty)}$  with the induced topology. We introduced the distance (2.2) on the bigger space  $\mathbb{H}^{[0, \infty)}$  since we will also consider (discontinuous) piecewise constant paths with values in  $\mathbb{H}$ .

For  $T > 0$  and  $\mathcal{U} \subset \mathbb{H}^{[0, T]}$ ,  $v \in \mathbb{H}^{[0, T]}$  we set

$$D_T(v, \mathcal{U}) := \sup_{u \in \mathcal{U}} d_T(v, u); \quad D_T(v, \mathcal{U}) := +\infty \text{ if } \mathcal{U} \text{ is empty}, \quad (2.4)$$

and, similarly, for  $\mathcal{U} \subset \mathbb{H}^{[0, +\infty)}$  and  $v \in \mathbb{H}^{[0, +\infty)}$  we define

$$D_\infty(v, \mathcal{U}) := \sup_{u \in \mathcal{U}} d_\infty(v, u); \quad D_\infty(v, \mathcal{U}) := +\infty \text{ if } \mathcal{U} \text{ is empty}. \quad (2.5)$$

Notice that  $D_T(v, \mathcal{U})$  (respectively  $D_\infty(v, \mathcal{U})$ ) is the Hausdorff distance between the sets  $\{v\}$  and  $\mathcal{U}$  induced by  $d_T$  (respectively  $d_\infty$ ).

## 2.2. Gradient flows

Let  $\phi \in C^1(\mathbb{H})$  be given.  $\text{GF}[\phi]$  is defined as the collection of all curves  $u \in C^1([0, +\infty); \mathbb{H})$  solving the gradient flow equation

$$u'(t) = -\nabla\phi(u(t)) \quad (\text{GF})$$

in  $[0, \infty)$ . Let us collect some useful properties for  $u \in \text{GF}[\phi]$  which directly follow from the gradient flow equation (GF).

We first observe that  $u$  satisfies

$$|u'(t)|^2 = |\nabla\phi(u(t))|^2 = -\frac{d}{dt}\phi \circ u(t) \quad \text{for every } t \geq 0, \quad (2.6)$$

and thus

$$\begin{aligned} \phi(u(t_1)) - \phi(u(t_2)) &= \int_{t_1}^{t_2} |u'(t)|^2 dt = \int_{t_1}^{t_2} |\nabla\phi(u(t))| |u'(t)| dt \\ &= \int_{t_1}^{t_2} |\nabla\phi(u(t))|^2 dt \end{aligned} \quad (2.7)$$

for every  $0 \leq t_1 \leq t_2$ . In particular  $\phi \circ u$  may take the same value at two points  $t_1 < t_2$  iff  $u$  takes a constant stationary value in  $[t_1, t_2]$ .

If  $\phi$  is Lipschitz it is immediate to check that  $u$  is also Lipschitz and satisfies

$$|u(t_2) - u(t_1)| \leq \text{Lip}[\phi] |t_2 - t_1| \quad \text{for every } t_1, t_2 \in [0, \infty). \quad (2.8)$$

More generally, when  $\phi$  satisfies (1.8), we easily get

$$\frac{d}{dt} \left( \phi(u(t)) + \frac{1}{\tau_*} |u(t)|^2 + \phi_* \right) \leq \frac{1}{\tau_*^2} |u(t)|^2 \leq \frac{2}{\tau_*} \left( \phi(u(t)) + \frac{1}{\tau_*} |u(t)|^2 + \phi_* \right)$$

so that Gronwall lemma and (1.8) yield

$$\begin{aligned} |u(t)|^2 &\leq 2\tau_* C_0 e^{2t/\tau_*}, \quad \phi(u(0)) - \phi(u(t)) \leq C_0 (1 + e^{2t/\tau_*}), \\ C_0 &:= \phi(u(0)) + \frac{1}{\tau_*} |u(0)|^2 + \phi_*. \end{aligned} \quad (2.9)$$

By applying Hölder inequality to (2.7) we thus obtain

$$|u(t_2) - u(t_1)| \leq \sqrt{C_0(1 + e^{2T/\tau_*})} |t_2 - t_1|^{1/2} \quad \text{for every } t_1, t_2 \in [0, T]. \quad (2.10)$$



**Lemma 2.1.** *Let  $u \in \text{GF}[\phi]$ .*

- (i)  $R[u]$  is a connected set and the map  $u : [0, +\infty) \rightarrow R[u]$  is locally invertible around any point  $x \in R[u] \setminus S[\phi]$ ;
- (ii)  $R[u]$  is locally compact and it is compact if and only if  $\phi$  attains its minimum in  $R[u]$  at some point  $\bar{u} = u(\bar{t})$  and  $u$  is constant for  $t \geq \bar{t}$ ;
- (iii) The restriction of  $\phi$  to  $R[u]$  is a homeomorphism with its image  $\phi(R[u]) \subset \mathbb{R}$ ;
- (iv) If  $u$  is not constant, then  $R[u] \setminus S[\phi]$  is dense in  $R[u]$ , and  $\phi(R[u] \setminus S[\phi])$  is dense in  $\phi(R[u])$ .

*Proof.* (i) is obvious. In order to show (ii), let us fix  $\bar{u} = u(\bar{t}) \in R[u]$ ; if  $\phi$  attains its minimum in  $R[u]$  at  $\bar{u}$  then  $\phi(u(t)) = \phi(u(\bar{t}))$  for every  $t \geq \bar{t}$  and (2.7) yields that  $\bar{u} \in S[\phi]$  and  $u(t) \equiv u(\bar{t})$  for every  $t \geq \bar{t}$ , so that  $R[u]$  is compact. The converse implication is obvious.

If  $\phi|_{R[u]}$  does not take its minimum at  $\bar{u} = u(\bar{t}) \in R[u]$ , there exists some  $t_1 > \bar{t}$  such that  $\delta := \phi(\bar{u}) - \phi(u(t_1)) > 0$ . Since  $\phi$  is continuous, the set  $U := \{u \in \mathbb{H} \mid \phi(u) \geq \phi(\bar{u}) - \delta\}$  is a closed neighborhood of  $\bar{u}$  and  $R[u] \cap U = u([0, t_1])$  which is compact.

(iii) By (2.7) the restriction of  $\phi$  to  $R[u]$  is continuous and injective. In order to prove that it is an homeomorphism it is sufficient to prove that  $\phi|_{R[u]}$  is proper, i.e. the counter image of every compact set in  $J := \phi(R[u]) \subset \mathbb{R}$  is compact. This property is obvious if  $R[u]$  is compact; if  $R[u]$  is not compact, then  $\phi$  does not take its minimum on  $R[u]$  and  $J$  is an interval of the form  $(\varphi_-, \varphi_+]$  where  $\varphi_+ = \phi(u(0))$  and  $\varphi_- = \inf_{R[u]} \phi$ . Therefore any compact subset of  $J$  is included in an interval of the form  $[\phi(u(\bar{t})), \varphi_+]$  and its counter image is a closed subset of the compact set  $u([0, \bar{t}])$  (recall that  $u$  is constant in each interval where  $\phi \circ u$  is constant).

(iv) Let  $\varphi_i = \phi(u(t_i))$ ,  $i = 1, 2$ , be two distinct points in  $\phi(R[u] \cap S[\phi])$ . Assuming that  $t_1 < t_2$ , (2.7) shows that there exists a point  $\bar{t} \in (t_1, t_2)$  such that  $\nabla \phi(u(\bar{t})) \neq 0$ , so that  $\bar{\varphi} = \phi(u(\bar{t}))$  belongs to  $(\varphi_2, \varphi_1) \setminus \phi(S[\phi])$ . We deduce that  $\phi(R[u] \setminus S[\phi])$  is dense in  $\phi(R[u])$  and, by the previous claim, that  $R[u] \setminus S[\phi]$  is dense in  $R[u]$ .  $\square$

**Notation 2.2.** If  $u \in C^0([0, \infty); \mathbb{H})$  we will set

$$T_\star(u) := \inf \{t \in [0, +\infty) : u(s) = u(t) \text{ for every } s \geq t\} \quad (2.11)$$

with the usual convention  $T_\star(u) := +\infty$  if the argument of the infimum in (2.11) is empty.

It is not difficult to check that the map  $\mathcal{T}_\star : C^0([0, \infty); \mathbb{H}) \rightarrow [0, +\infty]$ ,  $u \mapsto T_\star(u)$ , is lower semicontinuous with respect to the topology of compact convergence in  $C^0([0, \infty); \mathbb{H})$ .

By Lemma 2.1(ii), if  $u \in \text{GF}[\phi]$  then

$$T_\star(u) < \infty \quad \text{if and only if} \quad R[u] \text{ is compact;} \quad (2.12)$$

if  $R[u]$  is compact then  $u(t) = u_\star := u(T_\star(u))$  for every  $t \geq T_\star(u)$  and  $u_\star$  is a stationary point.

### 2.3. Minimizing movements

Let a function  $\psi : \mathbb{H} \rightarrow \mathbb{R}$ , a time step  $\tau > 0$ , and an initial value  $u_0 \in \mathbb{H}$  be given.

We consider the (possibly empty) set  $\text{MS}_\tau(\psi; u_0)$  of *Minimizing Sequences*  $(U_\tau^n)_{n \in \mathbb{N}}$  such that  $U_\tau^0 = u_0$  and

$$\begin{aligned} & \frac{1}{2\tau} |U_\tau^n - U_\tau^{n-1}|^2 + \psi(U_\tau^n) \\ & \leq \frac{1}{2\tau} |V - U_\tau^{n-1}|^2 + \psi(V) \quad \text{for every } V \in \mathbb{H}, \quad n \geq 1. \end{aligned} \quad (2.13)$$

We can associate a discrete sequence satisfying (2.13) with its piecewise constant interpolation  $U_\tau : [0, \infty) \rightarrow \mathbb{H}$  given by

$$U_\tau(0) := u_0, \quad U_\tau(t) := U_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau]. \quad (2.14)$$

(2.14) can be equivalently expressed as

$$U_\tau(t) := \sum_{n \in \mathbb{N}} U_\tau^n \chi(t/\tau - (n-1)) \quad \text{for } t > 0,$$

in which  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  denotes the characteristic function of the interval  $(0, 1]$ . We call  $\mathbf{M}_\tau(\psi; u_0)$  the class of discrete solutions  $U_\tau$  at time step  $\tau > 0$ , which admit the previous representation (2.14) in terms of solutions to (2.13).

**Remark 2.3 (Bounded intervals).** Sometimes it will also be useful to deal with approximations defined in a bounded interval  $[0, T]$ , involving finite minimizing sequences. For  $N \in \mathbb{N}$  we call

$$\text{MS}_\tau(\psi; u_0, N) \text{ the set of sequences } (U_\tau^n)_{0 \leq n \leq N} \text{ satisfying (2.13).} \quad (2.15)$$

Similarly, for a given final time  $T \in (0, +\infty)$  we set

$$N(\tau, T) := \min\{n \in \mathbb{N} : n\tau \geq T\}, \quad (2.16)$$

and we define  $\mathbf{M}_\tau(\psi; u_0, T)$  as the collection of all the piecewise constant functions  $U_\tau : [0, T] \rightarrow \mathbb{H}$  satisfying (2.14) in their domain of definition, for some  $(U_\tau^n)_n \in \text{MS}_\tau(\psi; u_0, N(\tau, T))$ .

Let us now assign a family of functions  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$  depending on the parameter  $\tau \in (0, \tau_0)$  and define the functional  $\Phi : (0, \tau_0) \times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$  as

$$\Phi(\tau, U, V) := \frac{1}{2\tau} |V - U|^2 + \phi_\tau(V). \quad (2.17)$$

(2.13) for the choice  $\psi := \phi_\tau$  is equivalent to

$$U_\tau^0 := u_0; \quad U_\tau^n \in \operatorname{argmin} \Phi(\tau, U_\tau^{n-1}, \cdot) \quad \text{for every } n \geq 1. \quad (2.18)$$

According to [9], a curve  $u : [0, +\infty) \rightarrow \mathbb{H}$  is called a Minimizing Movement associated to  $\Phi$  if there exist discrete solutions  $U_\tau \in \mathbf{M}_\tau(\phi_\tau; u_0)$  such that

$$\lim_{\tau \downarrow 0} U_\tau(t) = u(t) \quad \text{for every } t \geq 0. \quad (2.19)$$

$\text{MM}(\Phi, u_0)$  denotes the collection of all the Minimizing Movements.

A curve  $u : [0, +\infty) \rightarrow \mathbb{H}$  is called a Generalized Minimizing Movement [9] associated to  $\Phi$  if there exist a decreasing sequence  $k \mapsto \tau(k) \downarrow 0$  and corresponding  $U_{\tau(k)} \in \mathbf{M}_{\tau(k)}(\phi_{\tau(k)}; u_0)$  such that

$$u(t) = \lim_{k \rightarrow \infty} U_{\tau(k)}(t) \quad \text{for every } t \geq 0. \quad (2.20)$$

$\text{GMM}(\Phi, u_0)$  denotes the collection of all the Generalized Minimizing Movements. It clearly holds that  $\text{MM}(\Phi, u_0) \subset \text{GMM}(\Phi, u_0)$ .

**Remark 2.4 (Quadratic lower bounds).** If for some  $\tau_* > 0$

$$\inf_V \Phi(\tau_*, u_0, V) = -A > -\infty \quad (2.21)$$

(this happens, in particular, if  $\text{MS}_{\tau_*}(\phi_{\tau_*}; u_0, 1)$  is nonempty) and  $\text{Lip}[\phi_{\tau_*} - \phi] \leq \ell$ , then  $\phi$  satisfies the quadratic lower bound

$$\phi(x) \geq (\phi(u_0) - \phi_{\tau_*}(u_0)) - A - \ell/2 - \frac{\ell\tau_* + 1}{2\tau_*} |x - u_0|^2 \quad \text{for every } x \in \mathbb{H}. \quad (2.22)$$

This shows that the lower quadratic bound of (1.8) is a natural assumption in the framework of minimizing movements. (2.22) follows by the fact that  $\phi(x) - \phi(u_0) - (\phi_{\tau_*}(x) - \phi_{\tau_*}(u_0)) \geq -\text{Lip}[\phi - \phi_{\tau_*}] |x - u_0| \geq -\ell |x - u_0|$ .

Similarly, if  $\phi$  satisfies (1.8) and  $\text{Lip}[\phi_\tau - \phi] \leq \ell$ , we get

$$\phi_\tau(x) \geq (\phi_\tau(0) - \phi(0)) - \phi_* - \frac{\ell}{2} - \frac{\ell\tau_* + 1}{2\tau_*} |x|^2 \quad \text{for every } x \in \mathbb{H}. \quad (2.23)$$

It is a well known fact that if  $\phi \in C^1(\mathbb{H}) \cap \text{Lip}(\mathbb{H})$  and  $\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0$ , then every  $u \in \text{GMM}(\Phi, u_0)$  solves (GF) with initial datum  $u_0$ . We present here the proof of this statement (including the case of  $\phi \in C^1(\mathbb{H})$  satisfying (1.8)) and a few related results that will turn to be useful in the following.

**Lemma 2.5 (A priori estimates for minimizing sequences).** *Let  $\phi \in C^1(\mathbb{H})$  satisfy (1.8), let  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$  be such that  $\ell_\tau := \text{Lip}[\phi_\tau - \phi] < \infty$ , let  $T > 0$  and  $(U_\tau^n)_{0 \leq n \leq N} \in \text{MS}_\tau(\phi_\tau; u_0, N)$ ,  $1 \leq N \leq N(\tau, T)$ .*

(i) *For every  $1 \leq n \leq N$  we have*

$$\left| \frac{U_\tau^n - U_\tau^{n-1}}{\tau} + \nabla \phi(U_\tau^n) \right| \leq \ell_\tau; \quad (2.24)$$

- (ii) Suppose that  $\phi(u_0) \vee |u_0|^2 \leq F$ ,  $\ell_\tau \leq 1$ ,  $\tau \leq \tau_*/16$ . There exists a positive constant  $C = C(\phi_*, \tau_*, F, T)$  only depending on  $\phi_*, \tau_*, F, T$ , such that

$$\sup_{0 \leq n \leq N} |U_\tau^n|^2 \leq C, \quad \frac{1}{2\tau} \sum_{n=1}^N |U_\tau^n - U_\tau^{n-1}|^2 \leq \phi_\tau(u_0) - \phi_\tau(U_\tau^N) \leq C. \quad (2.25)$$

*Proof.* (i) Let us set  $\psi_\tau := \phi_\tau - \phi$ . The minimality condition (2.18) yields for every  $W \in \mathbb{H}$

$$\begin{aligned} & \phi(W) + \frac{1}{2\tau} |U_\tau^{n-1} - W|^2 - \phi(U_\tau^n) - \frac{1}{2\tau} |U_\tau^{n-1} - U_\tau^n|^2 \\ & \geq \psi_\tau(U_\tau^n) - \psi_\tau(W) \geq -\ell_\tau |U_\tau^n - W|. \end{aligned}$$

We can choose  $W := U_\tau^n + \theta v$ , divide the above inequality by  $\theta > 0$  and pass to the limit as  $\theta \downarrow 0$  obtaining

$$\left\langle \tau^{-1}(U_\tau^n - U_\tau^{n-1}) + \nabla \phi(U_\tau^n), v \right\rangle \geq -\ell_\tau |v| \quad \text{for every } v \in \mathbb{H},$$

which yields (2.24).

- (ii) It follows by [4, Lemma 3.2.2], by using (2.23) and  $u_* := 0$ . Up to the addition of a constant to  $\phi_\tau$ , it is not restrictive to assume that  $\phi_\tau(0) = \phi(0)$ .  $\square$

**Lemma 2.6.** Let  $\phi \in C^1(\mathbb{H})$  and let  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$ ,  $\tau \in (0, \tau_o)$ , such that

$$\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0.$$

- (i) If there exist a vanishing decreasing sequence  $k \mapsto \tau(k)$  and discrete solutions in a bounded interval  $U_{\tau(k)} \in \mathbf{M}_{\tau(k)}(\phi_{\tau(k)}; u_0, T)$  such that  $u(t) = \lim_{k \rightarrow \infty} U_{\tau(k)}(t)$  for every  $t \in [0, T]$ , then  $u \in C^1([0, T]; \mathbb{H})$  is a solution to (GF) with initial datum  $u(0) = u_0$ ;
- (ii) If  $u \in \text{GMM}(\Phi, u_0)$  then  $u \in \text{GF}[\phi]$ ;
- (iii) Let  $T > 0$  and  $U_\tau \in \mathbf{M}_\tau(\phi_\tau; u_0, T)$ ,  $\tau \in (0, \tau_o)$ , be a family of discrete solutions taking values in a compact subset  $\mathcal{K} \subset \mathbb{H}$ . Then for every decreasing and vanishing sequence  $k \mapsto \tau(k)$  there exist a further subsequence (still denoted by  $\tau(k)$ ) and a limit function  $u \in C^1([0, T]; \mathbb{H})$  such that

$$\lim_{k \rightarrow \infty} d_T(U_{\tau(k)}, u) = 0, \quad u \text{ is a solution to (GF) in } [0, T]; \quad (2.26)$$

- (iv) Let  $U_\tau \in \mathbf{M}_\tau(\phi_\tau; u_0)$ ,  $\tau \in (0, \tau_o)$ , be a family of discrete solutions satisfying the following property: for every  $T > 0$  there exist  $\bar{\tau} \in (0, \tau_o)$  and a compact set  $\mathcal{K} \subset \mathbb{H}$  such that  $U_\tau([0, T]) \subset \mathcal{K}$  for every  $\tau \in (0, \bar{\tau})$ . Then for every decreasing and vanishing sequence  $k \mapsto \tau(k)$  there exist a further subsequence (still denoted by  $\tau(k)$ ) and a limit function  $u \in C^1([0, \infty); \mathbb{H})$  such that

$$\lim_{k \rightarrow \infty} d_\infty(U_{\tau(k)}, u) = 0, \quad u \in \text{GF}[\phi]. \quad (2.27)$$

*Proof.* (i) Let us call  $\hat{U}_\tau(t)$  the piecewise linear interpolant of the values  $U_\tau^n$ ,  $0 \leq n \leq N(\tau, T)$  of the minimizing sequence associated to  $U_\tau$ :  $\hat{U}_\tau(t) = \sum_{n=0}^{N(\tau, T)} U_\tau^n \hat{\chi}(t/\tau - n)$  where  $\hat{\chi}(t) = (1 - |t|) \vee 0$ .

Since  $\text{MS}_{\tau(k)}(\phi_{\tau(k)}; u_0, 1)$  are not empty and  $\text{Lip}[\phi_\tau - \phi] \rightarrow 0$ , by Remark 2.4 we deduce that  $\phi$  satisfies the lower quadratic bound (1.8). By Lemma 2.5(ii) we deduce that there exists  $\tau_\star \in (0, \tau_0)$  sufficiently small such that any curve  $\hat{U}_\tau$  is equi Hölder continuous for  $\tau \leq \tau_\star$ , i.e. there exists a constant  $C$  independent of  $\tau$  such that

$$|\hat{U}_\tau(t) - \hat{U}_\tau(s)| \leq C|t - s|^{1/2} \quad \text{for every } s, t \in [0, T], \tau \in (0, \tau_\star), \quad (2.28)$$

and

$$\mathbf{d}_T(\hat{U}_\tau, U_\tau) \leq C\sqrt{\tau} \quad \tau \in (0, \tau_\star). \quad (2.29)$$

(2.29) shows that  $\hat{U}_\tau$  has the same limit points of  $U_\tau$ ; since  $\hat{U}_\tau$  is equi-Hölder, the pointwise convergence  $U_{\tau(k)} \rightarrow u(t)$  as  $k \rightarrow \infty$  for every  $t \in [0, T]$  implies the uniform convergence of  $U_{\tau(k)}$  and of  $\hat{U}_{\tau(k)}$  to the same limit  $u$ , which belongs to  $C^0([0, T]; \mathbb{H})$ .

Since  $\hat{U}'_\tau(t) = \tau^{-1}(U_\tau^n - U_\tau^{n-1})$  in each interval  $((n-1)\tau, n\tau)$ , we obtain from (2.24)

$$\begin{aligned} & |\hat{U}'_\tau(t) + \nabla\phi(U_\tau(t))| \\ & \leq \text{Lip}[\phi_\tau - \phi] \quad \text{for every } t \in [0, T] \setminus \{h\tau : 0 \leq h \leq N(\tau, T)\}. \end{aligned} \quad (2.30)$$

We can then pass to the limit in (the integrated version of) (2.30) for  $\tau = \tau(k)$  to obtain that

$$u(t) = u_0 - \int_0^t \nabla\phi(u(r)) \, dr \quad \text{for every } t \in [0, T], \quad (2.31)$$

which shows that  $u \in C^1([0, T]; \mathbb{H})$  is a solution to (GF).

Let us remark that when  $\phi$  is Lipschitz a reinforced version of (2.28) and (2.29) follows directly from (2.24), which yields

$$\begin{aligned} & |U_\tau^n - U_\tau^{n-1}| \leq L_\tau \tau \quad \text{for } 1 \leq n \leq N(\tau, T), \\ & \mathbf{d}_T(U_\tau, \hat{U}_\tau) \leq L_\tau \tau, \quad \text{Lip}(\hat{U}_\tau) \leq L_\tau, \end{aligned} \quad (2.32)$$

where  $L_\tau := \text{Lip}[\phi] + \ell_\tau$ .

(ii) The proof is completely analogous to (i).

(iii) We observe that  $\hat{U}_{\tau(k)}$  takes values in the closed convex hull  $\overline{\text{co}(\mathcal{K})}$ , which is still a compact subset of  $\mathbb{H}$  (see, e.g., [22, Theorem 3.20]). Since  $k \mapsto \hat{U}_{\tau(k)}$  is eventually equi-Hölder by (2.28), Ascoli-Arzelà Theorem yields the relative compactness of the sequence in the uniform topology. We conclude by applying (2.29) and the previous Claim (i).

(iv) We can apply the previous point (iii) and a standard diagonal argument.  $\square$

## 2.4. De Giorgi conjecture and notions of approximability

If the generating function  $\Phi$  of (2.17) is induced by perturbations  $\phi_\tau$  converging to  $\phi \in C^1(\mathbb{H})$  in the Lipschitz seminorm, then Lemma 2.6(ii) shows that  $\text{GMM}(\Phi, u_0) \subset \text{GF}[\phi]$ .

The most challenging part of De Giorgi's conjecture deals with the opposite direction. It can be equivalently formulated in the following way:

**Conjecture.** Suppose that  $\mathbb{H}$  has finite dimension,  $\phi \in C^1(\mathbb{H}) \cap \text{Lip}(\mathbb{H})$ , and let a solution  $u \in \text{GF}[\phi]$  be given. There exist a family of functions  $\phi_\tau \in \text{Lip}(\mathbb{H})$ , a decreasing sequence  $k \mapsto \tau(k) \downarrow 0$  and corresponding  $U_{\tau(k)} \in M_{\tau(k)}(\phi_{\tau(k)}; u(0))$  such that

$$\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0, \quad \lim_{k \rightarrow \infty} d_\infty(u, U_{\tau(k)}) = 0. \quad (2.33)$$

We will introduce a stronger property, based on the set distance introduced in (2.5).

**Definition 2.7 (Strongly approximable solutions).** Let  $\phi \in C^1(\mathbb{H})$ . We say that a solution  $u \in \text{GF}[\phi]$  is a strongly approximable solution if there exists a family of perturbations  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$ ,  $\tau \in (0, \tau_0)$ , such that

$$\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0, \quad \lim_{\tau \downarrow 0} D_\infty(u, M_\tau(\phi_\tau; u(0))) = 0. \quad (2.34)$$

We denote by  $\text{AGF}[\phi]$  the class of strongly approximable solutions.

The second part of (2.34) is equivalent to the following property: for every  $\tau > 0$  sufficiently small the set  $M_\tau(\phi_\tau; u(0))$  is nonempty and *all* the possible selections  $U_\tau \in M_\tau(\phi_\tau; u(0))$  will converge to  $u$  in the topology of compact convergence as  $\tau \downarrow 0$ . We note that (2.34) implies

$$\text{MM}(\Phi, u(0)) = \{u\} = \text{GMM}(\Phi, u(0)) \quad (2.35)$$

for the generating functional  $\Phi$  of (2.17). In the finite dimensional case, (2.35) is indeed equivalent to the second part of (2.34), due to the  $d_\infty$ -compactness of every sequence  $(U_\tau)$  of discrete solutions.

It is clear that any  $u \in \text{AGF}[\phi]$  satisfies the property expressed by De Giorgi's conjecture, and we will prove that in the finite dimensional Euclidean setting, indeed *every* solution  $u \in \text{GF}[\phi]$  is strongly approximable.

In a few situations ( $\mathbb{H}$  has infinite dimension and  $\phi$  is not bounded from below) we will also consider approximations on bounded intervals, recalling the notation introduced in Remark 2.3.

**Definition 2.8.** Let  $\phi \in C^1(\mathbb{H})$ . We say that a solution  $u \in \text{GF}[\phi]$  is strongly approximable in every compact interval if there exists a family of Lipschitz perturbations  $\phi_\tau \in \text{Lip}(\mathbb{H})$ ,  $\tau \in (0, \tau_0)$ , such that

$$\begin{aligned} \lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] &= 0, \\ \lim_{\tau \downarrow 0} D_T(u|_{[0,T]}, M_\tau(\phi_\tau; u(0), T)) &= 0 \quad \text{for every } T > 0. \end{aligned} \quad (2.36)$$

The notion of *strong approximability in every compact interval* slightly differs from the notion of *strong approximability* since we do not require the existence of elements in  $M_\tau(\phi_\tau; u(0))$  for  $\tau > 0$  small enough and we work with  $M_\tau(\phi_\tau; u(0), T)$  instead. The next remark better clarifies the relation between the two notions.

**Remark 2.9 (Strong approximability).** If a solution  $u \in \text{GF}[\phi]$  is strongly approximable in every compact interval and *for every sufficiently small  $\tau > 0$  the set of minimizing sequences  $\text{MS}_\tau(\phi_\tau; u(0))$  is nonempty*, then  $u$  is strongly approximable according to Definition 2.7: it is a simple consequence of (2.3) and of the fact that for every  $U \in M_\tau(\phi_\tau; u(0))$  the restriction  $U|_{[0, T]}$  belongs to  $M_\tau(\phi_\tau; u(0), T)$ .

Conversely, if  $u$  is strongly approximable and for every  $\tau > 0$  sufficiently small and *for every  $N > 0$  any minimizing sequence in  $\text{MS}_\tau(\phi_\tau; u(0), N)$  can be extended to a minimizing sequence in  $\text{MS}_\tau(\phi_\tau; u(0))$* , then  $u$  is strongly approximable in every compact interval  $[0, T]$ , according to Definition 2.8.

*In the finite dimensional Euclidean case the two notions of approximability are equivalent*, since the minimization problems (2.18) are always solvable for  $\tau$  sufficiently small and  $\phi$  quadratically bounded from below.

At the end of this preliminary section, we want to show that the class of strongly approximable solutions is closed with respect to Lipschitz convergence of the functionals and compact convergence of the solutions. We make use of an equivalent characterization of  $\text{AGF}[\phi]$  provided by the next lemma.

**Lemma 2.10.** *Let  $\phi \in C^1(\mathbb{H})$ .  $u \in \text{AGF}[\phi]$  if and only if for every  $\varepsilon > 0$  there exist  $\bar{\tau} > 0$  and a family  $\phi_{\varepsilon, \tau} : \mathbb{H} \rightarrow \mathbb{R}$ ,  $0 < \tau \leq \bar{\tau}$ , such that*

$$\text{Lip}[\phi_{\varepsilon, \tau} - \phi] \leq \varepsilon, \quad D_\infty(u, M_\tau(\phi_{\varepsilon, \tau}; u(0))) \leq \varepsilon \quad \text{for every } \tau \in (0, \bar{\tau}]. \quad (2.37)$$

*Proof.* Since it is obvious that any  $u \in \text{AGF}[\phi]$  satisfies the condition stated in the lemma, we only consider the inverse implication.

Let us fix a decreasing sequence  $\varepsilon_n \downarrow 0$ ; we can find a corresponding sequence  $\bar{\tau}_n$  and functions  $\phi_{\varepsilon_n, \tau}$  satisfying (2.37) for  $0 < \tau \leq \bar{\tau}_n$ . By possibly replacing  $\bar{\tau}_n$  with  $\tilde{\tau}_n := 2^{-n} \wedge \min_{1 \leq m \leq n} \bar{\tau}_m$ , it is not restrictive to assume that  $\bar{\tau}_n$  is also decreasing and converging to 0. We can thus define

$$\phi_\tau := \phi_{\varepsilon_n, \tau} \quad \text{whenever } \tau \in (\bar{\tau}_{n+1}, \bar{\tau}_n], \quad (2.38)$$

and it is easy to check that this choice satisfies (2.34). The fact that  $u \in \text{GF}[\phi]$  follows by (2.35) and Lemma 2.6(ii); hence,  $u$  is a strongly approximable solution according to Definition 2.7.  $\square$

**Lemma 2.11.** *The class of strongly approximable solutions satisfies the following closure property: if  $\phi, \phi_k \in C^1(\mathbb{H})$  and  $u_k \in \text{AGF}[\phi_k]$ ,  $k \in \mathbb{N}$ , with the same initial datum  $\bar{u} = u_k(0)$ , satisfy*

$$\lim_{k \rightarrow \infty} \text{Lip}[\phi_k - \phi] = 0, \quad \lim_{k \rightarrow \infty} d_\infty(u_k, u) = 0 \quad (2.39)$$

*then  $u \in \text{AGF}[\phi]$ .*

*Proof.* Let us fix  $\varepsilon > 0$ ; according to (2.39), we can find  $k \in \mathbb{N}$  such that

$$\text{Lip}[\phi_k - \phi] \leq \varepsilon/2, \quad d_\infty(u, u_k) \leq \varepsilon/2. \quad (2.40)$$

Since  $u_k \in \text{AGF}[\phi_k]$  we can also find  $\bar{\tau} > 0$  and a family of functions  $\phi_{k,\varepsilon,\tau} : \mathbb{H} \rightarrow \mathbb{R}$ ,  $\tau \in (0, \bar{\tau})$ , such that

$$\begin{aligned} \text{Lip}[\phi_{k,\varepsilon,\tau} - \phi_k] &\leq \varepsilon/2, \quad D_\infty(u_k, M_\tau(\phi_{k,\varepsilon,\tau}; u(0))) \leq \varepsilon/2 \\ \text{for every } \tau &\in (0, \bar{\tau}). \end{aligned} \quad (2.41)$$

The family  $\phi_{k,\varepsilon,\tau}$  obeys (2.37), since the triangle inequality yields

$$\begin{aligned} \text{Lip}[\phi_{k,\varepsilon,\tau} - \phi] &\leq \text{Lip}[\phi_{k,\varepsilon,\tau} - \phi_k] + \text{Lip}[\phi_k - \phi] \leq \varepsilon, \\ D_\infty(u, M_\tau(\phi_{k,\varepsilon,\tau}; u(0))) &\leq d_\infty(u, u_k) + D_\infty(u_k, M_\tau(\phi_{k,\varepsilon,\tau}; u(0))) \leq \varepsilon. \quad \square \end{aligned}$$

### 3. The minimal gradient flow

In this section, we define and study a particular class of solutions to (GF) for a function  $\phi \in C^1(\mathbb{H})$ , which we call *minimal gradient flows*. Let us first introduce a partial order in  $\text{GF}[\phi]$ .

**Definition 3.1.** If  $u, v \in \text{GF}[\phi]$  we say that  $u \succ v$  if  $R[v] \subset \overline{R[u]}$  and there exists an increasing 1-Lipschitz map  $\mathbf{z} : [0, +\infty) \rightarrow [0, +\infty)$  with  $\mathbf{z}(0) = 0$  such that

$$\begin{aligned} 0 \leq \mathbf{z}(t) - \mathbf{z}(s) &\leq t - s \quad \text{for every } 0 \leq s \leq t, \\ u(t) &= v(\mathbf{z}(t)) \quad \text{for every } 0 \leq t. \end{aligned} \quad (3.1)$$

An element  $u \in \text{GF}[\phi]$  is minimal if for every  $v \in \text{GF}[\phi]$ ,  $u \succ v$  yields  $u = v$ . We will denote by  $\text{GF}_{\min}[\phi]$  the collection of all the minimal solutions.

As it appears from (3.1), by “increasing” we mean that  $\mathbf{z}(s) \leq \mathbf{z}(t)$  for all  $s \leq t$ ; if we want to require a strict inequality, we will use the term “strictly increasing”. The same goes for “decreasing” and “strictly decreasing”.

**Remark 3.2 (Range inclusion).** Notice that if  $u \succ v$  then  $R[u] \subset R[v] \subset \overline{R[u]}$ ; the inclusion  $R[u] \subset R[v]$  is guaranteed by (3.1).

The condition  $R[v] \subset \overline{R[u]}$  prevents some arbitrariness in the extension of a candidate minimal solution. In order to understand its role, consider the classical 1-dimensional example given by  $\phi'(x) = 2\sqrt{|x|}$ . For a given  $T_* > 0$  the curve  $u(t) := ((T_* - t) \vee 0)^2$  belongs to  $\text{GF}[\phi]$  and it is minimal according to the previous definition (it is an easy consequence of the next Theorem 3.6(5)). However, the curve  $v(t) := u(t) - ((t - 2T_*) \vee 0)^2$  still belongs to  $\text{GF}[\phi]$  and satisfies (3.1) by choosing  $\mathbf{z}(t) = t \wedge T_*$ .  $u \not\succ v$  since  $R[v] \not\subset \overline{R[u]} = [0, T_*^2]$ .

We note that constant solutions are minimal by definition.



**Remark 3.3 ( $\succ$  is a partial order in  $\text{GF}[\phi]$ ).** It is easy to check that the relation  $\succ$  is reflexive and transitive; let us show that it is also antisymmetric. If  $u, v \in \text{GF}[\phi]$  satisfy  $u \succ v$  and  $v \succ u$ , we can find increasing and 1-Lipschitz maps  $Z_1, Z_2 : [0, \infty) \rightarrow [0, \infty)$  such that  $u(t) = v(Z_1(t))$  and  $v(t) = u(Z_2(t))$  for every  $t \in [0, \infty)$ ; in particular  $u = u \circ Z$  where  $Z = Z_2 \circ Z_1$  is also an increasing and 1-Lipschitz map satisfying  $Z(t) \leq t$ . Notice that the inequalities  $Z_i(t) \leq t$  and the monotonicity of  $Z_i$  yield

$$Z(t) \leq Z_i(t) \leq t \quad \text{for every } t \geq 0, \quad i = 1, 2.$$

Let us fix  $t \in [0, \infty)$ : if  $Z(t) = t$  then  $Z_i(t) = t$  so that  $u(t) = v(Z_1(t)) = v(t)$ . If  $Z(t) < t$  then  $u$  is constant in the interval  $[Z(t), t]$  so that  $u(t) = u(Z_2(t)) = v(t)$  as well.

**Remark 3.4 (Derivative of the time reparametrization map).** If  $u, v \in \text{GF}[\phi]$  satisfy  $u \succ v$ , then the corresponding time reparametrization map  $Z : [0, +\infty) \rightarrow [0, +\infty)$  is a.e. differentiable (since it is 1-Lipschitz) and for a.e.  $t \in [0, T_\star(u))$  it holds that

$$\begin{aligned} -|\nabla\phi(u(t))|^2 &= (\phi \circ u)'(t) = (\phi \circ v \circ Z)'(t) \\ &= -|\nabla\phi(v(Z(t)))|^2 Z'(t) = -|\nabla\phi(u(t))|^2 Z'(t), \end{aligned}$$

implying  $Z' = 1$  a.e. in  $\{t \in [0, T_\star(u)) : \nabla\phi(u(t)) \neq 0\}$ .

The next result collects a list of useful properties concerning minimal solutions. Recall that  $T_\star(u)$  has been defined by (2.11). We introduce the class of truncated solutions  $\text{TGF}[\phi] \supset \text{GF}[\phi]$  whose elements are solutions in  $\text{GF}[\phi]$  or curves  $v : [0, \infty) \rightarrow \mathbb{H}$  of the form  $v(t) := \tilde{v}(t \wedge T)$  for some  $\tilde{v} \in \text{GF}[\phi]$  and  $T \in [0, \infty)$ . The set  $\text{GF}[\phi]$  is closed in  $C^0([0, \infty); \mathbb{H})$ .

**Remark 3.5.** If  $v : [0, S] \rightarrow \mathbb{H}$  solves (GF) in  $[0, S]$  for some  $S > 0$  and there exists  $u \in \text{GF}[\phi]$  and  $T > 0$  so that  $u(T) = v(S)$ , then  $v$  can be identified with an element in  $\text{TGF}[\phi]$  since  $\tilde{v} \in \text{GF}[\phi]$  where  $\tilde{v}(t) := v(t)$  if  $t \in [0, S]$  and  $\tilde{v}(t) := u(t - S + T)$  if  $t \in (S, +\infty)$ .

**Theorem 3.6.** Let  $\phi \in C^1(\mathbb{H})$  satisfy (1.8).

- (1) For every  $R = R[y] \subset \mathbb{H}$  which is the range of  $y \in \text{GF}[\phi]$  there exists a unique  $u \in \text{GF}_{\min}[\phi]$  such that  $R \subset R[u] \subset \overline{R}$ . If  $v \in \text{GF}[\phi]$  and  $R \subset R[v] \subset \overline{R}$ , then  $v \succ u$ . In particular, for every  $v \in \text{GF}[\phi]$  there exists a unique  $u \in \text{GF}_{\min}[\phi]$  such that  $v \succ u$ ;
- (2)  $u \in \text{GF}[\phi]$  is minimal if and only if for every  $v \in \text{TGF}[\phi]$  with  $v(0) = u(0)$  and  $R[v] \subset \overline{R[u]}$  the following holds: if  $u(t_0) = v(t_1)$  for some  $t_0, t_1 \geq 0$  then  $t_0 \wedge T_\star(u) \leq t_1$ ;
- (3)  $u \in \text{GF}[\phi]$  is minimal if and only if for every  $v \in \text{TGF}[\phi]$  with  $v(0) = u(0)$  and  $R[v] \subset \overline{R[u]}$  we have  $\phi(v(t)) \geq \phi(u(t))$  for every  $t \geq 0$ ;
- (4) If  $u \in \text{GF}_{\min}[\phi]$ ,  $v \in \text{TGF}[\phi]$  with  $v(0) = u(0)$ ,  $R[v] \subset R[u]$  and  $\phi(v(t)) \leq \phi(u(t))$  for every  $t \in [0, T_\star(v))$ , then  $v(t) = u(t)$  for every  $t \in [0, T_\star(v))$ ;

- (5)  $u \in \text{GF}[\phi]$  is minimal if and only if the restriction of  $u$  to  $[0, T_\star(u))$  crosses the set  $S[\phi]$  of critical points of  $\phi$  in an  $\mathcal{L}^1$ -negligible set of times, i.e.

$$\mathcal{L}^1\left(\{t \in [0, T_\star(u)) : \nabla\phi(u(t)) = 0\}\right) = 0. \quad (3.2)$$

If  $u \in \text{GF}_{\min}[\phi]$  and  $T_\star(u) > 0$ , the map  $t \mapsto (\phi \circ u)(t)$  is strictly decreasing in  $[0, T_\star(u))$  and the map  $t \mapsto u(t)$  is injective in  $[0, T_\star(u))$ ;

- (6) A non-constant solution  $u \in \text{GF}[\phi]$  is minimal if and only if there exists a locally absolutely continuous map

$$\begin{aligned} \psi : \left( \inf_{R[u]} \phi, \phi(u(0)) \right] &\rightarrow [0, T_\star(u)) \quad \text{such that} \\ t = \psi(\phi(u(t))) &\text{ for every } t \in [0, T_\star(u)). \end{aligned}$$

*Proof.* (1) Let us fix  $v, y \in \text{GF}[\phi]$ ,  $R = R[y]$  with  $R \subset R[v] \subset \bar{R}$ ; it is not restrictive to assume that  $T_\star := T_\star(v) > 0$  (otherwise  $v$  is constant and  $R$  is reduced to one stationary point). We set  $\varphi_\star := \inf_R \phi$  and we select a sequence  $r_n \in R \setminus S[\phi]$  so that  $\varphi_n = \phi(r_n)$  is decreasing and converging to  $\varphi_\star$ . We can find a corresponding increasing sequence of points  $T_n \rightarrow T_\star$  such that  $v(T_n) = r_n$  and we set  $R_n := \{r \in R : \phi(r) \geq \varphi_n\} = v([0, T_n])$ . We consider the class

$$\mathcal{G}[R_n] := \left\{ w \in \text{TGF}[\phi] : R[w] = R_n \right\}.$$

$\mathcal{G}[R_n]$  is not empty since it contains the function  $t \mapsto v(t \wedge T_n)$ , and the sublevel sets  $\{w \in \mathcal{G}[R_n] : T_\star(w) \leq c\}$ ,  $c > 0$ , are compact in  $C^0([0, \infty); \mathbb{H})$  by the Arzelà-Ascoli Theorem. It follows that  $T_\star$  admits a minimizer in  $\mathcal{G}[R_n]$  that we will denote by  $u_n$ , with  $S_n := T_\star(u_n) \leq T_n$ . We now define

$$Z_n(t) := \min \left\{ s \in [0, S_n] : u_n(s) = v(t) \right\}, \quad t \in [0, T_n], \quad (3.3)$$

and we claim that  $Z_n$  is increasing, surjective and 1-Lipschitz from  $[0, T_n]$  to  $[0, S_n]$ . Since  $\phi$  is an homeomorphism between  $v([0, T_n])$  and the interval  $[\phi(v(T_n)), \phi(v(0))]$ ,  $Z_n$  can be equivalently defined as  $\min \{s \in [0, S_n] : \phi(u_n(s)) = \phi(v(t))\}$ , which shows that  $Z_n$  is increasing. In order to prove that  $Z_n$  is 1-Lipschitz, we argue by contradiction and suppose that there exist times  $0 \leq t_1 < t_2 \leq T_n$  with  $\delta_Z = Z_n(t_2) - Z_n(t_1) > t_2 - t_1 = \delta_t$ . Since by construction  $v(t) = u_n(Z_n(t))$ , we can consider a new curve

$$w(r) := \begin{cases} u_n(r) & \text{if } 0 \leq r \leq Z_n(t_1) \\ v(r + t_1 - Z_n(t_1)) & \text{if } Z_n(t_1) \leq r \leq \delta_t + Z_n(t_1) \\ u_n(r + \delta_Z - \delta_t) & \text{if } r \geq \delta_t + Z_n(t_1). \end{cases}$$

Defining  $W_n := S_n - \delta_Z + \delta_t < S_n$  it is easy to check that  $w(r) \equiv u_n(S_n) = v(T_n)$  for every  $r \geq W_n$  and  $w$  is a solution to (GF) in the interval  $[0, W_n]$ , so that  $w \in \mathcal{G}[R_n]$  and  $T_\star(w) = W_n < S_n$  which contradicts the minimality of  $u_n$ .

The same argument shows that  $u_n$  is in fact the unique minimizer of  $\mathcal{T}_\star$  in  $\mathcal{G}[R_n]$ : another minimizer  $\tilde{u}_n$  will also belong to  $\mathcal{G}[R_n]$  with  $T_\star(\tilde{u}_n) = S_n$ , so that there exists an increasing 1-Lipschitz map  $r : [0, S_n] \rightarrow [0, S_n]$  such that  $u_n(s) = \tilde{u}_n(r(s))$  for every  $s \in [0, S_n]$ . Since  $r(S_n) = S_n$   $r$  should be the identity so that  $\tilde{u}_n$  coincides with  $u_n$ .

Let us now show that

$$\begin{aligned} S_n < S_{n+1}, \quad u_n(s) = u_{n+1}(s), \quad z_n(t) = z_{n+1}(t) \\ \text{for every } s \in [0, S_n], \quad t \in [0, T_n]. \end{aligned} \quad (3.4)$$

In fact, for every  $\bar{t} \in [0, T_n]$  with  $z_n(\bar{t}) = \bar{s} \in [0, S_n]$  there exists  $z_{n+1}(\bar{t}) = s' \in (0, S_{n+1})$  such that  $u_{n+1}(s') = v(\bar{t}) = u_n(\bar{s})$ ; if  $s' < \bar{s}$  we would conclude that the map

$$\hat{w}(s) := \begin{cases} u_{n+1}(s) & \text{if } s \in [0, s'] \\ u_n(s - s' + \bar{s}) & \text{if } s \geq s' \end{cases}$$

belongs to  $\mathcal{G}[R_n]$  with  $T_\star(\hat{w}) = S_n + s' - \bar{s} < S_n$  contradicting the minimality of  $u_n$ . Choosing  $\bar{s} = S_n$  this in particular shows that  $S_{n+1} > S_n$ . If  $s' > \bar{s}$  we could define

$$\tilde{w}(s) := \begin{cases} u_n(s) & \text{if } 0 \leq s \leq \bar{s} \\ u_{n+1}(s - \bar{s} + s') & \text{if } s \geq \bar{s} \end{cases}$$

obtaining a function  $\tilde{w} \in \mathcal{G}[R_{n+1}]$  with  $T_\star(\tilde{w}) = S_{n+1} - (s' - \bar{s}) < S_{n+1}$ , contradicting the minimality of  $u_{n+1}$ . We thus get  $s' = \bar{s}$ , and therefore  $u_{n+1}(s) = u_n(s)$  in  $[0, S_n]$  and  $z_n(t) = z_{n+1}(t)$  in  $[0, T_n]$ .

Let us now set  $S_\star := \sup S_n$ . Due to (3.4) we can define the maps

$$\begin{aligned} u(s) &:= \begin{cases} u_n(s) & \text{if } s \in [0, S_n] \text{ for some } n \in \mathbb{N} \\ u_\star & \text{if } s \in [S_\star, \infty) \end{cases} \\ z(t) &:= \begin{cases} z_n(t) & \text{if } t \in [0, T_n] \text{ for some } n \in \mathbb{N} \\ S_\star & \text{if } t \in [T_\star, \infty) \end{cases} \end{aligned} \quad (3.5)$$

with  $u_\star := \lim_{s \uparrow S_\star} u(s)$  if  $S_\star < \infty$ . The curve  $u$  solves (GF) in  $[0, S_\star]$ ; due to (2.10), the limit  $u_\star$  is well-defined for  $S_\star < \infty$ . If  $T_\star < \infty$  then  $u_\star = v(T_\star)$ ; if  $T_\star = +\infty$  then  $u_\star = \lim_{t \uparrow \infty} v(t)$  is a critical point of  $\phi$  as (2.7) yields

$$\int_0^\infty |\nabla(\phi(v(t)))|^2 dt < +\infty$$

in this case. So we constructed an element  $u \in \text{GF}[\phi]$  with  $v \succ u$  and  $R \subset R[u] \subset \bar{R}$ .

Notice that by construction  $u$  just depends on  $R$ . Suppose now that there exists  $\bar{u} \in \text{GF}[\phi]$  with  $u \succ \bar{u}$ : in particular  $R \subset R[\bar{u}] \subset \bar{R}$  by Remark 3.2, so that the above argument shows that  $\bar{u} \succ u$  and therefore  $\bar{u} \equiv u$ . This property shows that  $u \in \text{GF}_{\min}[\phi]$ .

(2) Let us first observe that if  $u \in \text{GF}_{\min}[\phi]$  is non-constant, then the map  $t \mapsto u(t)$  is injective in  $[0, T_*(u))$ . In fact, if  $u(t_0) = u(t_0 + \delta)$  for some  $0 \leq t_0 < t_0 + \delta < T_*(u)$ , then  $\phi \circ u$  and thus  $u$  is constant in  $[t_0, t_0 + \delta]$  so that the curve

$$w(t) := \begin{cases} u(t) & \text{if } t \in [0, t_0] \\ u(t + \delta) & \text{if } t \geq t_0 \end{cases}$$

belongs to  $\text{GF}[\phi]$ , satisfies  $R[w] = R[u]$  and  $u(t) = w(Z(t))$  where  $Z(t) = t \wedge t_0 + (t - (t_0 + \delta))_+$ . This yields  $u \succ w$  so that  $u \equiv w$  by the minimality of  $u$ ; we deduce  $u(t) = u(t + \delta)$  for every  $t \geq t_0$ , which implies that  $T_*(u) \leq t_0$ , a contradiction.

Let us now suppose that  $u \in \text{GF}_{\min}[\phi]$ ,  $v \in \text{TGF}[\phi]$  with  $v(0) = u(0)$  and  $R[v] \subset \overline{R[u]}$ . It is not restrictive to assume  $T_*(u) > 0$ . We fix  $t_0 \geq 0$  and  $t_1 \in [0, T_*(v)]$  such that  $u(t_0) = v(t_1)$ , and we define the curve

$$w(t) := \begin{cases} v(t) & \text{if } 0 \leq t \leq t_1 \\ u(t - t_1 + t_0) & \text{if } t \geq t_1. \end{cases}$$

We clearly have  $w \in \text{GF}[\phi]$ ; moreover Lemma 2.1(iii) yields  $u([0, t_0]) = v([0, t_1])$  so that  $R[w] = R[u]$ . By the previous point (i) we deduce that  $w \succ u$  and there exists an increasing 1-Lipschitz map  $Z : [0, \infty) \rightarrow [0, \infty)$  such that  $w(t) = u(Z(t))$ . In particular,  $w(t_1) = v(t_1) = u(Z(t_1)) = u(t_0)$  and  $Z(t_1) \leq t_1$ . On the other hand, since  $t \mapsto u(t)$  is injective in  $[0, T_*(u))$  we deduce that  $t_0 = Z(t_1)$  or  $Z(t_1) \geq T_*(u)$ . If  $t_1 > T_*(v)$  we simply replace  $t_1$  by  $T_*(v)$ , since  $v(t_1) = v(T_*(v))$ .

The converse implication is a simple consequence of the previous claim: if  $u \in \text{GF}[\phi]$  we can construct the unique minimal flow  $v \in \text{GF}_{\min}[\phi]$  with  $R[u] \subset R[v] \subset \overline{R[u]}$ , so that  $u(t) = v(Z(t))$  for a suitable 1-Lipschitz map satisfying  $Z(0) = 0$ . By assumption,  $t \wedge T_*(u) \leq Z(t)$  but the 1-Lipschitz property yields  $t \geq Z(t)$  so that  $Z$  is the identity on  $[0, T_*(u)]$ . If  $T_*(u) = +\infty$  we deduce immediately that  $u \equiv v$ ; if  $T_*(u) < \infty$  we deduce that  $v(t) = u(t)$  for every  $t \in [0, T_*(u)]$  and then  $v \equiv u$  since  $R[v] \subset R[u] = u([0, T_*(u)])$ . In particular  $u \in \text{GF}_{\min}[\phi]$ .

(3) Let  $u \in \text{GF}_{\min}[\phi]$ ,  $v \in \text{TGF}[\phi]$  with  $v(0) = u(0)$  and  $R[v] \subset \overline{R[u]}$ ; it is not restrictive to assume  $T_*(u) > 0$ . For every  $t \in [0, T_*(v))$  there exists  $s \in [0, T_*(u))$  such that  $v(t) = u(s)$ . Claim (2) yields  $s \leq t$  so that  $\phi(v(t)) = \phi(u(s)) \geq \phi(u(t))$ . If  $T_*(v) < \infty$  we get by continuity  $\phi(v(T_*(v))) \geq \phi(u(T_*(v)))$  and therefore  $\phi(v(t)) = \phi(v(T_*(v))) \geq \phi(u(T_*(v))) \geq \phi(u(t))$  for every  $t \geq T_*(v)$ .

In order to prove the converse implication, we argue as in the previous claim and we construct the minimal solution  $v \in \text{GF}_{\min}[\phi]$  with  $R[u] \subset R[v] \subset \overline{R[u]}$ , so that  $u(t) = v(Z(t))$  for a suitable 1-Lipschitz map satisfying  $Z(0) = 0$ . Since  $Z(t) \leq t$  we get  $\phi(u(t)) = \phi(v(Z(t))) \geq \phi(v(t))$ , so that we deduce  $\phi(u(t)) = \phi(v(t))$  for every  $t \geq 0$ ; since  $\phi$  is injective on  $R[v] \supset R[u]$  we obtain  $u(t) = v(t)$ .

(4) Is an immediate consequence of the previous point (3) and Lemma 2.1(iii).

(5) We first prove that a solution  $u \in \text{GF}[\phi]$  satisfying (3.2) is minimal. In fact, if  $u \succ v$  we can find a 1-Lipschitz increasing map  $Z$  such that  $u(t) = v(Z(t))$ . Since the map  $Z$  is differentiable a.e. in  $[0, \infty)$  and  $u, v$  are solutions to (GF) we

obtain for a.e.  $t \in [0, T_*(u))$

$$\begin{aligned} -|\nabla\phi(u(t))|^2 &= (\phi \circ u)'(t) = (\phi \circ v \circ \mathbf{z})'(t) \\ &= -|\nabla\phi(v(\mathbf{z}(t)))|^2 \mathbf{z}'(t) = -|\nabla\phi(u(t))|^2 \mathbf{z}'(t). \end{aligned}$$

By (3.2) we deduce  $\mathbf{z}'(t) = 1$  for a.e.  $t \in [0, T_*(u))$ , so that  $\mathbf{z}(t) = t$  in  $[0, T_*(u))$  and  $v \equiv u$ .

Let us now prove that every  $u \in \text{GF}_{\min}[\phi]$  satisfies (3.2). Let  $T_* := T_*(u) > 0$ . Starting from  $u$  we construct a solution  $w \in \text{GF}[\phi]$  with the same range as  $u$  and which crosses  $S[\phi]$  in an  $\mathcal{L}^1$ -negligible set of times. For this purpose, we introduce the map

$$\mathbf{x} \in C^1([0, +\infty)), \quad \mathbf{x}(t) := \int_0^t |u'(s)| \, ds \quad \text{with} \quad X := \int_0^\infty |u'(s)| \, ds = \lim_{t \uparrow +\infty} \mathbf{x}(t),$$

and we consider the dense open set  $\Omega := \{t \in (0, T_*) : \mathbf{x}'(t) = |u'(t)| > 0\}$ . Notice that  $\mathbf{x}$  is strictly increasing in  $[0, T_*)$ , since  $\mathbf{x}(t_0) = \mathbf{x}(t_1)$  for some  $0 \leq t_0 < t_1 < T_*$  yields  $u$  constant in  $(t_0, t_1)$  which is not allowed by the minimality of  $u$ . We can thus define the continuous and strictly increasing inverse map  $\mathbf{y} : [0, X) \rightarrow [0, T_*)$  such that  $\mathbf{y}(\mathbf{x}(t)) = t$  for every  $t \in [0, T_*)$ . We notice that the set

$$\Xi := \{x \in [0, X) : u(\mathbf{y}(x)) \in S[\phi]\} = \mathbf{x}(\{t \in [0, T_*) : \mathbf{x}'(t) = 0\}) \quad (3.6)$$

has Lebesgue measure 0 by the Morse-Sard Theorem and that the map  $\mathbf{y}$  is differentiable on its complement  $[0, X) \setminus \Xi$  with

$$\mathbf{y}'(x) = \frac{1}{|u'(\mathbf{y}(x))|} = \frac{1}{|\nabla\phi(u(\mathbf{y}(x)))|}.$$

Since  $\mathbf{y}$  is continuous and increasing, its derivative belongs to  $L^1(0, X')$  for every  $X' < X$ . We can thus consider the strictly increasing and locally absolutely continuous function

$$\begin{aligned} \vartheta : [0, X) \rightarrow [0, \Theta), \quad \vartheta(x) &:= \int_{[0, x] \setminus \Xi} \frac{1}{|\nabla\phi(u(\mathbf{y}(r)))|} \, dr, \\ \Theta &:= \int_{[0, X) \setminus \Xi} \frac{1}{|\nabla\phi(u(\mathbf{y}(r)))|} \, dr. \end{aligned}$$

It holds that  $\vartheta'(x) = \mathbf{y}'(x) > 0$  for every  $x \in (0, X) \setminus \Xi$  and  $0 < \vartheta(x_1) - \vartheta(x_0) \leq \mathbf{y}(x_1) - \mathbf{y}(x_0)$  for every  $0 \leq x_0 < x_1 < X$ , so that the composition  $\mathbf{z} := \vartheta \circ \mathbf{x}$  satisfies

$$0 < \mathbf{z}(t_1) - \mathbf{z}(t_0) \leq t_1 - t_0 \quad \text{for every } 0 \leq t_0 < t_1 < T_*. \quad (3.7)$$

$\mathbf{z}$  is 1-Lipschitz and differentiable a.e.; moreover,  $\mathbf{z}$  is differentiable in  $\Omega$  with

$$\mathbf{z}'(t) = \vartheta'(\mathbf{x}(t))\mathbf{x}'(t) = 1 \quad \text{for every } t \in \Omega, \quad \mathbf{z}'(t) = 0 \quad \text{a.e. in } [0, T_*) \setminus \Omega \quad (3.8)$$

(see, e.g., [15, Theorem 3.44] for the chain rule for absolutely continuous functions).

We will denote by  $\mathbf{t} : [0, \Theta) \rightarrow [0, T_\star)$  the continuous inverse map of  $\mathbf{z}$  which is differentiable in the dense open set  $\mathbf{z}(\Omega)$  with derivative 1. Since  $\mathbf{t}$  is increasing, it is of bounded variation in every compact interval  $[0, \Theta']$  with  $\Theta' < \Theta$ . For every  $h \in \mathbb{H}$  we set  $u_h(t) := \langle u(t), h \rangle$ ,  $w := u \circ \mathbf{t} : [0, \Theta) \rightarrow \mathbb{H}$ , and  $w_h := u_h \circ \mathbf{t} : [0, \Theta) \rightarrow \mathbb{R}$ . Since  $u_h$  is locally Lipschitz,  $w_h$  is a function of bounded variation in every compact interval  $[0, \Theta']$  with  $\Theta' < \Theta$ : we want to show that  $w_h$  is absolutely continuous in  $[0, \Theta']$ . To this aim, we use the chain rule for BV functions (see, e.g., [3, Theorem 3.96]) and the facts that  $u_h, \mathbf{t}$  are continuous,  $u_h$  is Lipschitz in  $[0, \mathbf{t}(\Theta')]$ , and that  $\mathbf{t}$  is continuously differentiable on the open set  $\mathbf{z}(\Omega)$ ; the Cantor part  $D^c \mathbf{t}$  of the distributional derivative of  $\mathbf{t}$  is therefore concentrated on the set  $(0, \Theta') \setminus \mathbf{z}(\Omega)$  and the BV chain rule yields

$$\begin{aligned} D^c w_h &= (u'_h \circ \mathbf{t}) D^c \mathbf{t} \\ \text{where } u'_h(t) &:= \langle u'(t), h \rangle = \langle -\nabla \phi(u(t)), h \rangle = -\nabla_h \phi(u(t)). \end{aligned} \quad (3.9)$$

On the other hand, for every  $s \in (0, \Theta') \setminus \mathbf{z}(\Omega)$  we have  $\mathbf{t}(s) \in (0, T_\star) \setminus \Omega$  and thus  $\nabla \phi(u(\mathbf{t}(s))) = 0$ . We conclude that  $D^c w_h = 0$  and  $w_h$  is locally absolutely continuous. The same argument shows that the pointwise derivative of  $w_h$  vanishes a.e. in  $(0, \Theta) \setminus \mathbf{z}(\Omega)$ , whereas the computation of the derivative of  $w$  in  $\mathbf{z}(\Omega)$  yields

$$w'(s) = u'(\mathbf{t}(s))\mathbf{t}'(s) = u'(\mathbf{t}(s)) = -\nabla \phi(u(\mathbf{t}(s))) = -\nabla \phi(w(s)).$$

Summarizing, we obtain

$$w'_h(s) = -\nabla_h \phi(w(s)) \quad \text{a.e. in } (0, \Theta); \quad (3.10)$$

since the righthand side of (3.10) is continuous we deduce that  $w_h$  is a  $C^1$  function and (3.10) holds in fact everywhere in  $[0, \Theta)$ . Being  $w$  continuous and scalarly  $C^1$ , we deduce that  $w$  is of class  $C^1$  in  $[0, \Theta)$  and  $w$  is a solution of (GF) satisfying  $w(s) = u(\mathbf{t}(s))$ . If  $\Theta$  is finite, the uniform Hölder estimate (2.10) shows that  $w$  admits the limit  $\bar{w} := \lim_{s \uparrow \Theta} w(s) = \lim_{t \uparrow +\infty} u(t)$ . It follows that  $\bar{w}$  is a stationary point of  $\phi$ , so that extending  $w$  by the constant value  $\bar{w}$  for  $t \geq \Theta$  still yields a solution to (GF). If we have  $T_\star < \infty$ , we can extend  $\mathbf{z}$  by the constant value  $\Theta = \lim_{t \uparrow T_\star} \mathbf{z}(t) < \infty$  for  $t \geq T_\star$ . Since we have  $R[w] \subset \overline{R[u]}$  and  $u(t) = w(\mathbf{z}(t))$  for every  $t \geq 0$ , we deduce that  $u \succ w$ . Since  $u$  is minimal, we should have  $w \equiv u$  so that  $\mathbf{z}(t) \equiv t$  for  $t \in [0, T_\star)$ . (3.8) then yields that  $[0, T_\star) \setminus \Omega$  has 0 Lebesgue measure and (3.2) holds.

(6) If  $u \in \text{GF}_{\min}[\phi]$  and  $T_\star(u) > 0$ , we know that the map  $\varphi : t \mapsto \phi(u(t))$  is of class  $C^1$ , strictly decreasing with  $\varphi'(t) < 0$  a.e. in  $(0, T_\star)$ . It follows that it has a locally absolutely continuous inverse  $\psi$ . Conversely, if  $\varphi$  has a locally absolutely continuous left inverse  $\psi$  (which is then also the inverse) then  $\varphi'(t) = -|\nabla \phi(u(t))|^2 \neq 0$  a.e. in  $(0, T_\star)$ , so that (3.2) holds and  $u \in \text{GF}_{\min}[\phi]$  by the previous claim (5).  $\square$

We conclude this section with a definition and a simple remark.

**Definition 3.7 (Eventually minimal solutions).** We say that a solution  $u \in \text{GF}[\phi]$  is *eventually minimal* if there exists a time  $T > 0$  such that  $u'(T) \neq 0$  and the curve  $t \mapsto u(t + T)$  is a minimal non-constant solution.

**Remark 3.8 (Approximation by eventually minimal solutions).** Any non-constant  $u \in \text{GF}[\phi]$  may be locally uniformly approximated by a sequence of eventually minimal solutions keeping the same initial data. For every  $n \in \mathbb{N}$  it is sufficient to choose an increasing sequence  $t_n \uparrow T_\star(u)$  with  $u'(t_n) \neq 0$  and replace the curve  $v_n := u(\cdot + t_n)$  with the unique minimal solution  $w_n$  such that  $v_n \succ w_n$ , given by Theorem 3.6. The curves

$$u_n(t) := \begin{cases} u(t) & \text{if } 0 \leq t \leq t_n, \\ w_n(t - t_n) & \text{if } t > t_n, \end{cases} \quad (3.11)$$

are eventually minimal and converge to  $u$  uniformly on compact intervals. The uniform convergence on compact intervals follows directly from (3.11) for  $T_\star(u) = +\infty$ ; if  $T_\star(u) < +\infty$ , we can estimate

$$|u_n(t) - u(t)| \leq \max_{s \in [0, T_\star(u)]} |\nabla \phi(u(s))| |T_\star(u) - t_n| \quad \text{for all } t \geq 0.$$

Any constant  $u \in \text{GF}[\phi]$  is minimal.

Minimal gradient flows will play a crucial role in the proof of De Giorgi's conjecture. Roughly speaking, the conjecture can be proved directly for this class of gradient flows, and in addition, any other gradient flow can be approximated by a sequence of minimal gradient flows.

## 4. Approximation of the minimal gradient flow

In this section we study a particular family of perturbations that will be extremely useful to approximate minimal gradient flows. As a first step, we present a general strategy to force a discrete solution of the minimizing movement scheme to stay in a prescribed compact set. We will always assume that  $\phi \in C^1(\mathbb{H})$  satisfies the uniform quadratic bound (1.8), so that

$$\inf_{y \in \mathbb{H}} \frac{1}{2\tau} |x - y|^2 + \phi(y) > -\infty \quad \text{for every } x \in \mathbb{H}, \tau \in (0, \tau_\star). \quad (4.1)$$

### 4.1. Distance penalizations from compact sets

Let a time step  $\tau > 0$  and a nonempty compact set  $\mathcal{U} \subset \mathbb{H}$  be fixed. We denote by  $\psi_{\mathcal{U}} : \mathbb{H} \rightarrow \mathbb{R}$  the distance function

$$\psi_{\mathcal{U}}(x) := \text{dist}(x, \mathcal{U}) = \min_{y \in \mathcal{U}} |x - y|, \quad (4.2)$$

by  $\Gamma_{\mathcal{U}}$  the closed convex set

$$\Gamma_{\mathcal{U}} := \left\{ (a, b) \in [0, \infty) \times [0, \infty) : |\nabla\phi(x) - \nabla\phi(y)| \wedge 1 \leq a + b|x - y| \text{ for every } x \in \mathcal{U}, y \in \mathbb{H} \right\} \quad (4.3)$$

and by  $\omega_{\mathcal{U}} : [0, \infty) \rightarrow [0, \infty)$  the concave modulus of continuity

$$\omega_{\mathcal{U}}(r) := \inf \left\{ a + br : (a, b) \in \Gamma_{\mathcal{U}} \right\}. \quad (4.4)$$

Notice that

$$\omega_{\mathcal{U}} \text{ is increasing, bounded by 1, concave, and satisfies } \lim_{r \downarrow 0} \omega_{\mathcal{U}}(r) = 0, \quad (4.5)$$

with

$$|\nabla\phi(x) - \nabla\phi(y)| \wedge 1 \leq \omega_{\mathcal{U}}(|x - y|) \quad \text{whenever } x \in \mathcal{U}, y \in \mathbb{H}. \quad (4.6)$$

In order to prove the limit property of (4.5), we can argue by contradiction; let us assume that we have instead  $\inf_{r>0} \omega_{\mathcal{U}}(r) = \bar{a} \in (0, 1]$ . Choosing  $r = \bar{a}/(4n)$ ,  $n \in \mathbb{N}$ , we see that the couple  $(\bar{a}/2, n)$  does not belong to  $\Gamma_{\mathcal{U}}$ , so that for every  $n \in \mathbb{N}$  there exist  $x_n \in \mathcal{U}$  and  $y_n \in \mathbb{H}$  such that

$$1 \wedge |\nabla\phi(x_n) - \nabla\phi(y_n)| - n|x_n - y_n| > \bar{a}/2. \quad (4.7)$$

In particular  $|x_n - y_n| \leq 1/n$  so that  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ . Since  $x_n \in \mathcal{U}$  and  $\mathcal{U}$  is compact, we can extract a subsequence  $k \mapsto n(k)$  such that  $\lim_{k \rightarrow \infty} x_{n(k)} = x \in \mathcal{U}$ , and thus  $\lim_{k \rightarrow \infty} y_{n(k)} = x$  as well and therefore  $\lim_{k \rightarrow \infty} |\nabla\phi(x_{n(k)}) - \nabla\phi(y_{n(k)})| = 0$  by the continuity of  $\nabla\phi$ , a contradiction with (4.7).

We consider a family of perturbations of the function  $\phi$  depending on a parameter  $\lambda \geq 0$  and on a compact set  $\mathcal{U} \subset \mathbb{H}$ . It is given by

$$\varphi_{\lambda, \mathcal{U}}(x) := \phi(x) + \lambda \psi_{\mathcal{U}}(x), \quad \Phi_{\lambda, \mathcal{U}}(\tau, x, y) := \frac{1}{2\tau} |x - y|^2 + \varphi_{\lambda, \mathcal{U}}(y), \quad (4.8)$$

$$\mathcal{J}_{\tau, \lambda, \mathcal{U}}(x) := \operatorname{argmin} \Phi_{\lambda, \mathcal{U}}(\tau, x, \cdot). \quad (4.9)$$

Our aim is to give a sufficient condition on the choice of  $\lambda$  in dependence of  $\tau$  and  $\mathcal{U}$  in order to be sure that whenever  $x \in \mathcal{U}$  the minimizing set  $\mathcal{J}_{\tau, \lambda, \mathcal{U}}(x)$  is nonempty and it is contained in  $\mathcal{U}$  as well.

In Lemma 4.1, a rough estimate of  $|\nabla\phi(y)|$  of an approximate minimizer  $y$  of  $\Phi_{\lambda, \mathcal{U}}(\tau, x, \cdot)$  is given.

**Lemma 4.1.** *There exists  $\tau_{\mathcal{U}} \in (0, \tau_*)$  so that for every  $y \in \mathbb{H}$ ,  $x \in \mathcal{U}$ ,  $\tau \in (0, \tau_{\mathcal{U}})$  satisfying*

$$\phi(y) + \frac{1}{2\tau} |x - y|^2 \leq \phi(x) + |x - y|, \quad (4.10)$$

*it holds that*

$$|\nabla\phi(y) - \nabla\phi(x)| \leq \frac{1}{2}. \quad (4.11)$$



*Proof.* Since  $\lim_{r \downarrow 0} \omega_{\mathcal{U}}(r) = 0$ , there exists  $\bar{r} > 0$  such that  $\omega_{\mathcal{U}}(r) \leq \frac{1}{2}$  for every  $0 \leq r < \bar{r}$ . In view of (4.6), it is sufficient to prove that there exists  $\tau_{\mathcal{U}} \in (0, \tau_*)$  such that  $|x - y| < \bar{r}$  whenever  $y \in \mathbb{H}$ ,  $x \in \mathcal{U}$  satisfy (4.10) for some  $\tau \in (0, \tau_{\mathcal{U}})$ .

Let us suppose that (4.10) holds for  $y \in \mathbb{H}$ ,  $x \in \mathcal{U}$ ,  $\tau \in (0, \tau_*)$ . We apply [4, Lemma 2.2.1], and (1.8) in order to obtain

$$\begin{aligned} |x - y|^2 &\leq \frac{4\tau\tau_*}{\tau_* - \tau} \left( \phi(y) + \frac{1}{2\tau}|x - y|^2 + \phi_* + \frac{1}{\tau_* - \tau}|x|^2 \right) \\ &\leq \frac{4\tau\tau_*}{\tau_* - \tau} \left( \max_{z \in \mathcal{U}} \phi(z) + \frac{1}{2} + \frac{1}{2}|x - y|^2 + \phi_* + \frac{1}{\tau_* - \tau} \max_{z \in \mathcal{U}} |z|^2 \right). \end{aligned}$$

The claim now easily follows.  $\square$

The following Lemma 4.2 is a typical result for nonsmooth analysis of the distance function.

**Lemma 4.2.** *Let  $L := \max_{\mathcal{U}} |\nabla \phi| \vee 1$ ,  $0 \leq \eta \leq \lambda < 1/4$ ,  $\tau \in (0, \tau_{\mathcal{U}})$ ,  $x \in \mathcal{U}$  and  $y \in \mathbb{H}$  be an approximate  $\eta$ -minimizer of  $\Phi_{\lambda, \mathcal{U}}(\tau, x, \cdot)$ , i.e.*

$$\Phi_{\lambda, \mathcal{U}}(\tau, x, y) \leq \Phi_{\lambda, \mathcal{U}}(\tau, x, w) + \eta|w - y| \quad \text{for every } w \in \mathbb{H}. \quad (4.12)$$

*Then the vector  $\xi := \frac{y-x}{\tau} + \nabla \phi(y)$  satisfies*

$$|\xi| \leq \lambda + \eta, \quad |y - x| \leq (L + 1/2 + \lambda + \eta)\tau \leq 2L\tau. \quad (4.13)$$

*Moreover, if  $y \notin \mathcal{U}$ , then  $|\xi| \geq \lambda - \eta$ .*

*Proof.* Since  $\psi_{\mathcal{U}}$  is 1-Lipschitz, the minimality condition (4.12) yields for every  $w \in \mathbb{H}$

$$\begin{aligned} \phi(w) + \frac{1}{2\tau}|x - w|^2 - \phi(y) - \frac{1}{2\tau}|x - y|^2 \\ \geq \lambda\psi_{\mathcal{U}}(y) - \lambda\psi_{\mathcal{U}}(w) - \eta|w - y| \geq -(\lambda + \eta)|y - w|. \end{aligned}$$

We can choose  $w := y + \theta v$ , divide the above inequality by  $\theta > 0$  and pass to the limit as  $\theta \downarrow 0$  obtaining

$$\langle \xi, v \rangle \geq -(\lambda + \eta)|v| \quad \text{for every } v \in \mathbb{H},$$

which yields the first part of (4.13). The second part of (4.13) then follows from the estimate  $|y - x| \leq \tau(|\xi| + |\nabla \phi(y) - \nabla \phi(x)| + |\nabla \phi(x)|)$  and (4.11).

If we choose  $w := (1 - \theta)y + \theta \hat{y}$  with  $\hat{y} \in \mathcal{U}$  satisfying  $|y - \hat{y}| = \psi_{\mathcal{U}}(y) > 0$ , we also obtain  $\psi_{\mathcal{U}}(w) = |(1 - \theta)y + \theta \hat{y} - \hat{y}| = (1 - \theta)|y - \hat{y}|$  and  $|y - w| = \theta|y - \hat{y}|$  so that

$$\begin{aligned} \phi(w) + \frac{1}{2\tau}|x - w|^2 - \phi(y) - \frac{1}{2\tau}|x - y|^2 \\ \geq \lambda(\psi_{\mathcal{U}}(y) - \psi_{\mathcal{U}}(w)) - \eta|y - w| = \theta(\lambda - \eta)|y - \hat{y}| \end{aligned}$$

and therefore

$$\langle \xi, \hat{y} - y \rangle \geq (\lambda - \eta)|y - \hat{y}|$$

which yields  $|\xi| \geq \lambda - \eta$ .  $\square$

The next lemma provides a suitable condition on the choice of  $\lambda$ .

**Lemma 4.3.** *Let  $\mathcal{U}$  be a compact subset of  $\mathbb{H}$ ,  $L := \max_{\mathcal{U}} |\nabla \phi| \vee 1$ ,  $x, z \in \mathcal{U}$ ,  $\tau \in (0, \tau_{\mathcal{U}})$ , and  $\lambda, \delta \in [0, 1/4)$ , satisfy*

$$\left| \frac{z - x}{\tau} + \nabla \phi(z) \right| \leq \delta, \quad (4.14)$$

$$\lambda^2 > 14 L \omega_{\mathcal{U}}(3L\tau) + 2\delta^2. \quad (4.15)$$

Then  $\mathcal{J}_{\tau, \lambda, \mathcal{U}}(x)$  is nonempty and contained in  $\mathcal{U}$ .

*Proof.* We argue by contradiction and we suppose that

$$\text{there exists } y \in \mathbb{H} \setminus \mathcal{U} \text{ such that } \Phi_{\lambda, \mathcal{U}}(\tau, x, y) \leq \min_{u \in \mathcal{U}} \Phi_{\lambda, \mathcal{U}}(\tau, x, u). \quad (4.16)$$

We can apply Ekeland variational principle in  $\mathbb{H}$  to the continuous function

$$w \mapsto \Phi_{\lambda, \mathcal{U}}(\tau, x, w)$$

which is bounded below by (4.1). For every  $\eta > 0$  we can find  $y_{\eta} \in \mathbb{H}$  satisfying the properties

$$\Phi_{\lambda, \mathcal{U}}(\tau, x, y_{\eta}) + \eta|y_{\eta} - y| \leq \Phi_{\lambda, \mathcal{U}}(\tau, x, y), \quad (4.17)$$

$$\Phi_{\lambda, \mathcal{U}}(\tau, x, y_{\eta}) \leq \Phi_{\lambda, \mathcal{U}}(\tau, x, w) + \eta|y_{\eta} - w| \quad \text{for every } w \in \mathbb{H}. \quad (4.18)$$

(4.17) and (4.16) yield that  $y_{\eta} \notin \mathcal{U}$  and

$$\phi(y_{\eta}) + \frac{1}{2\tau}|y_{\eta} - x|^2 + \lambda\psi_{\mathcal{U}}(y_{\eta}) \leq \phi(z) + \frac{1}{2\tau}|z - x|^2. \quad (4.19)$$

Choosing  $\eta$  sufficiently small so that  $\lambda + \delta + \eta \leq 1/2$ , (4.13) and (4.14) yield

$$|y_{\eta} - x| \leq (L + 1/2 + \lambda + \eta)\tau \leq 2L\tau, \quad |z - x| \leq (L + \delta)\tau, \quad (4.20)$$

and therefore

$$|y_{\eta} - z| \leq (2L + 1/2 + \lambda + \delta + \eta)\tau \leq 3L\tau. \quad (4.21)$$

Since  $\omega_{\mathcal{U}}(3L\tau) < \lambda^2 \leq 1$  by (4.15), we get the estimate

$$\begin{aligned} |\nabla \phi((1-t)y_{\eta} + tz) - \nabla \phi(y_{\eta})| &\leq |\nabla \phi((1-t)y_{\eta} + tz) - \nabla \phi(z)| \\ &\quad + |\nabla \phi(z) - \nabla \phi(y_{\eta})| \\ &\leq 2\omega_{\mathcal{U}}(|y_{\eta} - z|) \quad \text{for every } t \in [0, 1]. \end{aligned}$$

The integral mean value theorem

$$\phi(z) - \phi(y_\eta) - \langle \nabla \phi(y_\eta), z - y_\eta \rangle = \int_0^1 \langle \nabla \phi((1-t)y_\eta + tz) - \nabla \phi(y_\eta), z - y_\eta \rangle dt$$

yields

$$|\phi(z) - \phi(y_\eta) - \langle \nabla \phi(y_\eta), z - y_\eta \rangle| \leq 2|z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|). \quad (4.22)$$

So, combining (4.19) and (4.22) we obtain

$$\begin{aligned} & \frac{1}{2\tau}|y_\eta - x|^2 - \frac{1}{2\tau}|z - x|^2 - \langle \nabla \phi(y_\eta), z - y_\eta \rangle + \lambda \psi_{\mathcal{U}}(y_\eta) \\ & \leq \phi(z) - \phi(y_\eta) - \langle \nabla \phi(y_\eta), z - y_\eta \rangle \\ & \leq 2|z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|). \end{aligned}$$

Using the identity  $|a|^2 - |b|^2 = \langle a + b, a - b \rangle$  and neglecting the positive term  $\lambda \psi_{\mathcal{U}}(y_\eta)$  we get

$$\frac{1}{2\tau} \langle y_\eta - x + 2\tau \nabla \phi(y_\eta) + z - x, y_\eta - z \rangle \leq 2|z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|).$$

Setting  $\xi_\eta := \frac{y_\eta - x}{\tau} + \nabla \phi(y_\eta)$  as in Lemma 4.2 we get

$$y_\eta - z = y_\eta - x + x - z = \tau \xi_\eta - \tau \nabla \phi(y_\eta) + x - z.$$

Thus, we obtain

$$\frac{1}{2\tau} \langle \tau \xi_\eta + \tau \nabla \phi(y_\eta) + z - x, \tau \xi_\eta - \tau \nabla \phi(y_\eta) - (z - x) \rangle \leq 2|z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|),$$

yielding

$$\frac{\tau}{2} |\xi_\eta|^2 \leq \frac{1}{2\tau} |\tau \nabla \phi(y_\eta) - (x - z)|^2 + 2|z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|).$$

Using (4.14) and the fact that  $|\xi_\eta| \geq \lambda - \eta$  if  $\eta \leq \lambda$  by Lemma 4.2, we obtain

$$\begin{aligned} |\lambda - \eta|^2 & \leq 2 \left( |\nabla \phi(y_\eta) - \nabla \phi(z)|^2 + \delta^2 \right) + \frac{4}{\tau} |z - y_\eta| \omega_{\mathcal{U}}(|z - y_\eta|) \\ & \leq 2 \left( \omega_{\mathcal{U}}^2(3L\tau) + \delta^2 \right) + 12L \omega_{\mathcal{U}}(3L\tau) \leq 14L \omega_{\mathcal{U}}(3L\tau) + 2\delta^2, \end{aligned}$$

where we used (4.21) and the fact that  $\omega_{\mathcal{U}} \leq 1$ . Since  $\eta$  can be chosen arbitrarily small, we get a contradiction with (4.15).  $\square$

Notice that the use of Ekeland variational principle in the previous proof is only needed when  $\mathbb{H}$  has infinite dimension. If  $\mathbb{H}$  has finite dimension, one can directly select  $y_\eta$  as the minimizer of  $\Phi_{\lambda, \mathcal{U}}(\tau, x, \cdot)$  in  $\mathbb{H}$  setting  $\eta = 0$ .

**Corollary 4.4.** *Let  $\mathcal{U} \subset \mathbb{H}$  be a compact set,  $L := 1 \vee \max_{\mathcal{U}} |\nabla \phi|$ ,  $\lambda, \delta \in [0, 1/4)$ ,  $\tau \in (0, \tau_{\mathcal{U}})$ . If (4.15) holds and for every  $x \in \mathcal{U}$  there exists  $z \in \mathcal{U}$  satisfying (4.14), then for every initial choice of  $u_0 \in \mathcal{U}$  the set  $\text{MS}_\tau(\varphi_{\lambda, \mathcal{U}}; u_0)$  is nonempty and every discrete solution  $U \in \mathbf{M}_\tau(\varphi_{\lambda, \mathcal{U}}; u_0)$  takes values in  $\mathcal{U}$ .*

## 4.2. Strong approximation of minimal solutions

We can now apply Lemma 4.3 and Corollary 4.4 in order to construct good discrete solutions by choosing suitable compact subsets of the range of  $u \in \text{GF}[\phi]$ . We distinguish two cases: the next lemma contains the fundamental estimates in the case when  $\phi$  is bounded on the range of a solution  $u$ ; Lemma 4.7 will deal with solutions  $u$  for which  $\phi(u(t)) \rightarrow -\infty$  as  $t \rightarrow +\infty$ .

We introduce the following notation (recall Remark 2.3): if  $u \in \text{GF}[\phi]$ ,  $T > 0$ ,  $\tau > 0$  we set

$$\mathcal{U}(\tau, T) := \{u(n\tau) : 0 \leq n \leq N(\tau, T)\}. \quad (4.23)$$

**Lemma 4.5.** *Let  $u \in \text{GF}[\phi]$  such that*

$$\inf_{t \geq 0} \phi(u(t)) = \lim_{t \uparrow \infty} \phi(u(t)) > -\infty. \quad (4.24)$$

*For every  $\varepsilon \in (0, 1/4)$  there exist  $T = T(\varepsilon) \geq \varepsilon^{-1}$  and  $\bar{\tau} = \bar{\tau}(\varepsilon) \in (0, 1)$  such that for every  $0 < \tau \leq \bar{\tau}$  the set  $M_\tau(\varphi_\varepsilon, \mathcal{U}(\tau, T); u(0))$  is nonempty, every element  $U \in M_\tau(\varphi_\varepsilon, \mathcal{U}(\tau, T); u(0))$  takes values in  $\mathcal{U}(\tau, T) \subset u([0, T + 1])$  and satisfies*

$$\phi(U(t)) \leq \phi(u(t \wedge T)) \quad \text{for every } t \geq 0. \quad (4.25)$$

*Moreover, for every  $S > 0$ , it holds that*

$$M_\tau(\varphi_\varepsilon, \mathcal{U}(\tau, T); u(0), S) = \{U|_{[0, S]} \mid U \in M_\tau(\varphi_\varepsilon, \mathcal{U}(\tau, T); u(0))\}. \quad (4.26)$$

*Proof.* Since  $u$  satisfies (4.24), the identity (2.7) yields  $\int_0^\infty |\nabla \phi(u(t))|^2 dt < \infty$  and therefore

$$\liminf_{t \uparrow \infty} |\nabla \phi(u(t))| = 0. \quad (4.27)$$

We select  $T \geq \varepsilon^{-1}$  such that  $|\nabla \phi(u(T))| \leq \varepsilon/4$  and consider the compact set  $\mathcal{K} := u([0, T + 1])$ ; notice that  $\mathcal{U}(\tau, T) \subset \mathcal{K}$  for every  $\tau \leq 1$ .

We set  $L := 1 \vee \max_{\mathcal{K}} |\nabla \phi|$  and we choose  $\delta := \varepsilon/2$  and  $\bar{\tau} < \tau_{\mathcal{K}} \wedge 1$  (with  $\tau_{\mathcal{K}}$  as in Lemma 4.1) so that  $(14L + 1)\omega_{\mathcal{K}}(3L\bar{\tau}) < \varepsilon^2/2$ ; in particular

$$14L\omega_{\mathcal{K}}(3L\bar{\tau}) + 2\delta^2 < \varepsilon^2, \quad \omega_{\mathcal{K}}(L\bar{\tau}) \leq \delta/2. \quad (4.28)$$

We observe that for every  $x = u((n-1)\tau) \in \mathcal{U}(\tau, T)$ ,  $1 \leq n \leq N$ ,  $N = N(\tau, T)$ ,  $\tau \in (0, \bar{\tau}]$ , the choice  $z := u(n\tau)$  satisfies (4.14) since

$$\begin{aligned} \frac{z-x}{\tau} + \nabla \phi(z) &= \frac{u(n\tau) - u((n-1)\tau)}{\tau} + \nabla \phi(u(n\tau)) \\ &= \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} \left( \nabla \phi(u(n\tau)) - \nabla \phi(u(r)) \right) dr \end{aligned} \quad (4.29)$$

and therefore

$$\left| \frac{z-x}{\tau} + \nabla \phi(z) \right| \leq \omega_{\mathcal{K}}(L\tau) \leq \omega_{\mathcal{K}}(L\bar{\tau}) \leq \delta/2 \quad (4.30)$$

by (4.28). Notice that  $|u'(t)| = |\nabla\phi(u(t))| \leq L$  for  $t \in [0, T + 1]$  so that  $|u(n\tau) - u(r)| \leq L\tau$  whenever  $r \in ((n-1)\tau, n\tau]$ .

For  $x = u(N\tau)$  we can choose  $z = x = u(N\tau)$ , since in this case

$$|\nabla\phi(z)| \leq |\nabla\phi(z) - \nabla\phi(u(T))| + |\nabla\phi(u(T))| \leq \omega_K(L\tau) + \frac{\delta}{2} \leq \delta.$$

Since  $\omega_{\mathcal{U}(\tau, T)}(r) \leq \omega_K(r)$ , we can apply Lemma 4.3 with the choice  $\lambda := \varepsilon$  thanks to (4.28): we obtain the fact that  $M_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, T)}; u(0))$  is nonempty, every element  $U \in M_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, T)}; u(0))$  takes values in  $\mathcal{U}(\tau, T)$  and (4.26) holds.

In order to prove (4.25) we write  $U(t) = \sum_n U_\tau^n \chi(t/\tau - (n-1))$  for  $t > 0$  and we observe that (4.25) is equivalent to

$$\phi(U_\tau^n) \leq \phi(u(n\tau \wedge T)) \quad \text{for every } n \in \mathbb{N}, \quad (4.31)$$

thanks to the monotonicity of  $t \mapsto \phi(u(t))$ .

We argue by induction, observing that (4.31) is clearly true for  $n = 0$ .

If  $\phi(U_\tau^{n-1}) \leq \phi(u((n-1)\tau))$  for some  $1 \leq n \leq N$ , then we can deduce that  $U_\tau^{n-1} = u(k\tau)$  for some  $k \geq n-1$ .

If  $k > n-1$  then we easily get  $\phi(U_\tau^n) \leq \phi(U_\tau^{n-1}) \leq \phi(u(k\tau)) \leq \phi(u(n\tau))$ .

It remains to consider the case  $k = n-1$ , i.e.  $U_\tau^{n-1} = u((n-1)\tau)$ . If  $\phi(u(n\tau)) = \phi(u((n-1)\tau))$ , the induction step is obvious. If  $\phi(u(n\tau)) < \phi(u((n-1)\tau))$ , then it is sufficient to observe that  $\Phi_{\lambda, \mathcal{U}}(\tau, u((n-1)\tau), u(n\tau)) < \phi(u((n-1)\tau))$ . Indeed, it then holds by (2.7) that

$$\begin{aligned} & \phi(u(n\tau)) + \frac{1}{2\tau} |u(n\tau) - u((n-1)\tau)|^2 \leq \phi(u(n\tau)) + \frac{1}{2} \int_{(n-1)\tau}^{n\tau} |u'(r)|^2 dr \\ & = \phi(u(n\tau)) + \frac{1}{2} (\phi(u((n-1)\tau)) - \phi(u(n\tau))) \\ & = \phi(u((n-1)\tau)) - \frac{1}{2} (\phi(u((n-1)\tau)) - \phi(u(n\tau))) < \phi(u((n-1)\tau)), \end{aligned}$$

so that  $U_\tau^n$  belongs to  $\{u(k\tau) : n \leq k \leq N\}$  and thus satisfies  $\phi(U_\tau^n) \leq \phi(u(n\tau))$ .

Eventually, for  $n > N$ , the induction step is trivial.  $\square$

**Remark 4.6.** The proof shows that the statement of Lemma 4.5 in fact holds for every  $u \in \text{GF}[\phi]$  satisfying (4.27).

We now consider the case when  $\phi$  is unbounded on  $\mathbb{R}[u]$ .

**Lemma 4.7.** *Let  $u \in \text{GF}[\phi]$  such that*

$$\inf_{t \geq 0} \phi(u(t)) = \lim_{t \uparrow \infty} \phi(u(t)) = -\infty. \quad (4.32)$$

*For every  $\varepsilon \in (0, 1/4)$ ,  $T > 0$  there exist  $\bar{\tau} = \bar{\tau}(\varepsilon, T) \in (0, 1)$  and  $\bar{T} = \bar{T}(T) \geq T$  such that for every  $0 < \tau \leq \bar{\tau}$  the set  $M_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), T)$  is nonempty, every*

element  $U \in \mathbf{M}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), T)$  takes values in  $\mathcal{U}(\tau, \bar{T}) \subset u([0, \bar{T} + 1])$  and satisfies

$$\phi(U(t)) \leq \phi(u(t)) \quad \text{for every } t \in [0, T]. \quad (4.33)$$

Moreover, for every  $0 \leq S \leq T$ , it holds that

$$\mathbf{M}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), S) = \{U|_{[0, S]} \mid U \in \mathbf{M}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), T)\}. \quad (4.34)$$

*Proof.* The argument of the proof is quite similar to the one of Lemma 4.5: the only difference is that we cannot find a compact set containing the range of the whole discrete solutions.

Let us set  $F := \phi(u(0)) \vee |u(0)|^2$  and let  $C = C(\phi_*, \tau_*, F, T)$  the constant provided by Lemma 2.5(ii). By (4.32) we can select a time  $\bar{T} \geq T$  such that

$$\phi(u(\bar{T})) < \phi(u(0)) - C \quad (4.35)$$

and we set  $\mathcal{K} := u([0, \bar{T} + 1])$ ,  $L := 1 \vee \max_{\mathcal{K}} |\nabla \phi|$ ,  $\bar{N} = N(\tau, \bar{T})$ ,  $\delta = \varepsilon/2$  and  $\bar{\tau} \in (0, 1 \wedge \tau_*/16 \wedge \tau_{\mathcal{K}})$  sufficiently small so that (4.28) holds.

Since  $\mathcal{U}(\tau, \bar{T}) \subset \mathcal{K}$ , the same calculations of (4.29) and (4.30) show that for every  $x \in \{u(k\tau) : 0 \leq k < \bar{N}\}$  there exists  $z \in \mathcal{U}(\tau, \bar{T})$  satisfying (4.14).

We can then apply Lemma 4.3 and the same induction argument of the previous proof to prove that an integer  $M \geq 1$  and a sequence

$$(U_\tau^n)_{0 \leq n \leq M} \in \mathbf{MS}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), M)$$

exist such that  $U_\tau^n \in \{u(k\tau) : 0 \leq k \leq \bar{N}\}$  and  $U_\tau^M = u(\bar{N}\tau)$ . Since  $\phi(U_\tau^M) = \phi(u(\bar{N}\tau)) \leq \phi(u(\bar{T})) < \phi(u(0)) - C$  and (2.25) yields

$$\phi(U_\tau^n) \geq \phi(u(0)) - C \quad \text{for every } 1 \leq n \leq N(\tau, T), \quad (4.36)$$

we deduce that  $N(\tau, T) < M$  so that  $\mathbf{M}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), T)$  is not empty.

If now  $U$  is any element of  $\mathbf{M}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), T)$  corresponding to a sequence  $(U_\tau^n)_{0 \leq n \leq N} \in \mathbf{MS}_\tau(\varphi_{\varepsilon, \mathcal{U}(\tau, \bar{T})}; u(0), N)$ ,  $N = N(\tau, T)$ , then Lemma 4.3, the same induction argument of the previous proof and (4.36) show that  $U$  take values in  $\mathcal{U}(\tau, \bar{T})$  and (4.33) holds. The same arguments show that (4.34) holds for every  $0 \leq S \leq T$ .  $\square$

We are now able to state the main result of this section.

**Theorem 4.8.** *Every minimal solution  $u \in \mathbf{GF}_{\min}[\phi]$  is strongly approximable in every compact interval, according to Definition 2.8.*

*If in addition  $\mathbb{H}$  has finite dimension or (4.27) is satisfied, then  $u$  is strongly approximable according to Definition 2.7.*

*Proof.* We pick a decreasing sequence  $\varepsilon_n \downarrow 0$  and an increasing sequence  $T_n := \varepsilon_n^{-1} \uparrow +\infty$ .

If (4.24) holds, we can apply Lemma 4.5 and we set  $\bar{\tau}_n := \bar{\tau}(\varepsilon_n)$ ,  $\bar{T}_n := T(\varepsilon_n) \geq T_n$ .

If (4.32) holds, we set  $\bar{\tau}_n := \bar{\tau}(\varepsilon_n, T_n) > 0$ ,  $\bar{T}_n := \bar{T}(T_n) \geq T_n$  provided by Lemma 4.7.

We can find a decreasing sequence  $\sigma_n \downarrow 0$  satisfying  $\sigma_n \leq \min_{1 \leq m \leq n} \bar{\tau}_m$  and a family  $\phi_\tau$  by choosing

$$\phi_\tau := \varphi_{\varepsilon_n, \mathcal{U}(\tau, \bar{T}_n)} \quad \text{whenever } \sigma_{n+1} < \tau \leq \sigma_n.$$

By construction

$$\text{Lip}[\phi_\tau - \phi] \leq \varepsilon_n \quad \text{if } \sigma_{n+1} < \tau \leq \sigma_n,$$

so that  $\lim_{\tau \downarrow 0} \text{Lip}[\phi_\tau - \phi] = 0$ .

We first consider the case  $T_\star(u) < +\infty$ . If  $T_\star(u) < +\infty$ , then the range  $R[u]$  is compact and (4.24) holds. Lemma 4.5 shows that  $M_\tau(\phi_\tau; u(0))$  is not empty for  $\tau \in (0, \sigma_1)$ . Moreover, if  $U_\tau \in M_\tau(\phi_\tau; u(0))$  is any selection depending on  $\tau \in (0, \sigma_1)$ , we have  $U_\tau([0, +\infty)) \subset R[u]$  and  $\phi(U_\tau(t)) \leq \phi(u(t \wedge T_n))$  for every  $t \geq 0$ ,  $\sigma_{n+1} < \tau \leq \sigma_n$ . By Lemma 2.6(iv), every decreasing vanishing sequence  $k \mapsto \tau(k)$  admits a further subsequence (still denoted by  $\tau(k)$ ) such that  $U_{\tau(k)}$  converges in the topology of compact convergence to a limit  $v \in \text{GF}[\phi]$ . It holds that  $\phi(v(t)) \leq \phi(u(t))$  for all  $t \geq 0$ , which implies  $u(t) = v(t)$  for all  $t \geq 0$  by Theorem 3.6(4) since  $u$  is minimal and  $R[v] = R[u]$ . As the limit is unique, we obtain

$$\lim_{\tau \downarrow 0} D_\infty(u, M_\tau(\phi_\tau; u(0))) = 0,$$

showing that  $u$  is strongly approximable according to Definition 2.7. For every  $T > 0$ ,  $\tau \in (0, \sigma_1)$ , it holds that  $M_\tau(\phi_\tau; u(0), T) = \{U|_{[0, T]} \mid U \in M_\tau(\phi_\tau; u(0))\}$ ; hence, by Remark 2.9,  $u$  is also strongly approximable in every compact interval.

Now, we consider the case  $T_\star(u) = +\infty$ . Let us fix  $T > 0$  and take  $\bar{n} = \min\{n \in \mathbb{N} : T_n \geq T + 1\}$ . Lemma 4.5 and 4.7 show that  $M_\tau(\phi_\tau; u(0), T + 1)$  is not empty whenever  $\tau \leq \sigma_{\bar{n}}$ . Moreover, if  $U_\tau \in M_\tau(\phi_\tau; u(0), T + 1)$  is any selection depending on  $\tau \in (0, \sigma_{\bar{n}})$ , we have  $\phi(U_\tau(t)) \leq \phi(u(t))$  for every  $t \in [0, T + 1]$ . According to Lemma 2.5(ii) and to (2.28) and (2.29), there exist  $\tau_\star \in (0, \sigma_{\bar{n}})$  and a constant  $C > 0$  independent of  $\tau$  such that

$$|U_\tau(t) - U_\tau(s)| \leq 2C\sqrt{\tau} + C|t - s|^{1/2} \quad \text{for every } s, t \in [0, T + 1], \tau \in (0, \tau_\star). \quad (4.37)$$

We define  $S_\tau := \inf\{t \in [0, T + 1] \mid \phi(U_\tau(t)) \leq \phi(u(T + 1))\}$  for  $\tau \in (0, \sigma_{\bar{n}})$  and  $\tilde{S} := \liminf_{\tau \downarrow 0} S_\tau$ . The varying times  $S_\tau$  serve as auxiliary final times in order to prove convergence of  $U_\tau$ . We set  $\gamma_\tau := (T + 1 - S_\tau) \wedge \tau$ . As the piecewise constant functions  $U_\tau$  are left-continuous by definition and  $\phi(U_\tau(T + 1)) \leq \phi(u(T + 1))$ , it holds that  $S_\tau < T + 1$  (thus  $\gamma_\tau > 0$ ) and  $\phi(U_\tau(S_\tau + \gamma_\tau)) \leq \phi(u(T + 1))$ . The plan is as follows. We show that  $\tilde{S} > 0$ , we prove that  $U_\tau$  converges to  $u$  uniformly in  $[0, S]$  for every  $0 < S < \tilde{S}$ , and we conclude by proving that  $\tilde{S} = T + 1$ .

There exists a vanishing sequence  $l \mapsto \tau(l)$  such that  $\lim_{l \uparrow \infty} S_{\tau(l)} = \tilde{S}$ . A contradiction argument shows that  $\tilde{S} > 0$ . Suppose that  $\tilde{S} = 0$ ; then  $U_{\tau(l)}(S_{\tau(l)} + \gamma_{\tau(l)})$  converges to  $u(0)$  by (4.37) and  $\phi(u(0)) = \lim_{l \uparrow \infty} \phi(U_{\tau(l)}(S_{\tau(l)} + \gamma_{\tau(l)})) \leq \phi(u(T+1))$  in contradiction to  $\phi(u(T+1)) < \phi(u(0))$  by the minimality of  $u$  and Theorem 3.6(5). Hence,  $\tilde{S} > 0$ . For every  $0 < S < \tilde{S}$  and sufficiently small  $\tau$ , it holds that  $U_{\tau}([0, S]) \subset u([0, T+1])$  so that by Lemma 2.6(iii), every decreasing vanishing sequence  $k \mapsto \tau(k)$  admits a further subsequence (still denoted by  $\tau(k)$ ) such that  $U_{\tau(k)}$  converges uniformly in  $[0, S]$  to a limit  $v \in C^1([0, S], \mathbb{H})$  solving (GF) in  $[0, S]$ . Moreover, since we have  $v([0, S]) \subset R[u]$  and  $\phi(v(t)) \leq \phi(u(t))$  for all  $t \in [0, S]$ , we deduce that  $u(t) = v(t)$  for all  $t \in [0, S]$  by the minimality of  $u$ , Remark 3.5 and Theorem 3.6(4). Since the limit is unique, we can now infer that  $\lim_{\tau \downarrow 0} d_S(U_{\tau}, u) = 0$  for every  $S < \tilde{S}$ . Using (4.37), we obtain

$$\begin{aligned} & \limsup_{l \uparrow \infty} |U_{\tau(l)}(S_{\tau(l)}) - u(\tilde{S})| \\ & \leq \limsup_{l \uparrow \infty} (|U_{\tau(l)}(S_{\tau(l)}) - U_{\tau(l)}(S)| + |U_{\tau(l)}(S) - u(\tilde{S})|) \\ & \leq \limsup_{l \uparrow \infty} (2C\sqrt{\tau(l)} + C|S_{\tau(l)} - S|^{1/2} + |U_{\tau(l)}(S) - u(\tilde{S})|) \\ & \leq C|\tilde{S} - S|^{1/2} + |u(S) - u(\tilde{S})| \end{aligned}$$

for every  $S < \tilde{S}$  and therefore  $u(\tilde{S}) = \lim_{l \uparrow \infty} U_{\tau(l)}(S_{\tau(l)}) = \lim_{l \uparrow \infty} U_{\tau(l)}(S_{\tau(l)} + \gamma_{\tau(l)})$ . It follows that  $\phi(u(\tilde{S})) = \lim_{l \uparrow \infty} \phi(U_{\tau(l)}(S_{\tau(l)} + \gamma_{\tau(l)})) \leq \phi(u(T+1))$  which implies  $u(\tilde{S}) = u(T+1)$  as  $\tilde{S} \leq T+1$ . Since the minimal solution  $u$  is injective for  $T_*(u) = +\infty$  by Theorem 3.6(5), it follows that  $\tilde{S} = T+1 = \lim_{\tau \downarrow 0} S_{\tau}$ . So we obtain

$$\lim_{\tau \downarrow 0} D_T(u|_{[0, T]}, M_{\tau}(\phi_{\tau}; u(0), T)) = 0 \quad (4.38)$$

by the preceding argument and the fact that  $M_{\tau}(\phi_{\tau}; u(0), T) = \{U|_{[0, T]} \mid U \in M_{\tau}(\phi_{\tau}; u(0), T+1)\}$  for  $\tau \in (0, \sigma_{\tilde{n}})$ . This shows that  $u$  is strongly approximable in every compact interval.

If  $\mathbb{H}$  has finite dimension, then Remark 2.9 shows that  $u$  is also strongly approximable.

If (4.24) holds, then Lemma 4.5 shows that  $M_{\tau}(\phi_{\tau}; u(0))$  is not empty for  $\tau \in (0, \sigma_1)$ ; hence, according to Remark 2.9,  $u$  is also strongly approximable. The same can be shown if (4.27) holds, see Remark 4.6.  $\square$

The next step in the proof of De Giorgi's conjecture is to show that we can approximate any gradient flow curve by a sequence of minimal gradient flows for slightly (in the Lipschitz norm) modified energies, and then to combine that convergence result and Theorem 4.8 by Lemma 2.11. This will be first considered in the one-dimensional setting.



## 5. The one-dimensional setting

In this section we want to study the one-dimensional case  $\mathbb{H} = \mathbb{R}$ . Just for this section, we will call  $E := -\phi$  and we consider a continuously differentiable function  $E : \mathbb{R} \rightarrow \mathbb{R}$  with derivative  $f := E'$ .

**Proposition 5.1.** *Let  $u \in \text{GF}[-E]$  be an eventually minimal solution (see Definition 3.7), i.e.*

$$\text{there exists } T > 0 \text{ with } u'(T) \neq 0 \text{ and } t \mapsto u(t + T) \text{ is minimal.} \quad (5.1)$$

*Then there exist a sequence of energies  $E_\varepsilon \in C^1(\mathbb{R})$  and a sequence of curves  $u_\varepsilon \in \text{GF}_{\min}[-E_\varepsilon]$  with  $u_\varepsilon(0) = u(0)$  such that*

$$E'_\varepsilon = E' \quad \text{in } \mathbb{R} \setminus u([0, T]), \quad \lim_{\varepsilon \downarrow 0} \text{Lip}[E_\varepsilon - E] = 0, \quad \lim_{\varepsilon \downarrow 0} d_\infty(u, u_\varepsilon) = 0. \quad (5.2)$$

*Proof.* In order to simplify the notation, we may assume without loss of generality that  $u(0) = 0$ . We notice that  $u$  is a monotone function. This can be shown by contradiction: suppose that  $u$  is not monotone and choose  $a, b \in (0, \infty)$  with  $u'(a) > 0$  and  $u'(b) < 0$ , without loss of generality  $a < b$ . Then there exists  $\gamma > 0$  such that  $u$  is strictly increasing on  $[a, a + \gamma]$  and strictly decreasing on  $[b - \gamma, b]$ . It holds that  $u(a) < u(b)$ : otherwise there would be  $s \in (a + \gamma, b]$  with  $u(s) = u(a)$  which, by (2.7), would imply  $u(t) = u(a)$  for all  $t \in [a, s]$  contradicting the strict monotonicity of  $u$  in  $[a, a + \gamma]$ . A similar argument yields  $u(b) < u(a)$ , a contradiction. Hence,  $u$  is either increasing or decreasing. If Proposition 5.1 holds for increasing solutions, then, by obvious reflection arguments, it also holds for decreasing solutions, and thus for all solutions  $u$ . So we may assume that  $u$  is an increasing function whose range  $R[u]$  is an interval of the form  $[0, R)$ ,  $R \in (0, \infty]$  or  $[0, R]$ ,  $R \in (0, \infty)$ . We define the left-continuous pseudo-inverse map  $\mathbf{t} : [0, R) \rightarrow [0, T_\star(u))$

$$\mathbf{t}(x) := \min\{t \geq 0 : u(t) = x\}, \quad \text{satisfying } u(\mathbf{t}(x)) = x \quad \text{for every } x \in [0, R). \quad (5.3)$$

The map  $\mathbf{t}$  is an increasing function, in particular it is a function of bounded variation in any compact interval of  $[0, R)$ ; (5.3) yields that the set  $D$  of points in  $[0, R)$  where  $\mathbf{t}$  is differentiable coincides with the set  $\{x \in [0, R) : E'(x) > 0\}$  and  $u'(\mathbf{t}(x))\mathbf{t}'(x) = 1$  for every  $x \in D$ . Lebesgue differentiation theorem shows that  $D$  has full measure in  $[0, R)$ . Since  $u' = E'(u)$  we deduce that

$$\mathbf{t}'(x) = \frac{1}{u'(\mathbf{t}(x))} = \frac{1}{E'(\mathbf{t}(x))} = \frac{1}{f(\mathbf{t}(x))} \quad \text{for every } x \in D,$$

and the property

$$\int_{[0, x] \cap D} \frac{1}{f(y)} dy \leq \mathbf{t}(x) < \infty \quad \text{for every } x < R. \quad (5.4)$$

(5.1) and Theorem 3.6(5) yield

$$\mathcal{L}^1(\{t \in (T, T_\star(u)) : u(t) \notin D\}) = 0, \quad (5.5)$$

so that  $\mathbf{t}$  is locally absolutely continuous in the interval  $[u(T), R)$ . Since the distributional derivative of  $\mathbf{t}$  is a Radon measure on  $[0, R)$ , there exists a nonnegative finite Borel measure  $\mu$  supported on  $[0, u(T)]$  such that

$$\mathbf{t}(x) = \int_{[0,x] \cap D} \frac{1}{f(r)} dr + \mu([0, x)) \quad \text{for every } x \in [0, R).$$

Notice that  $\mu([0, R)) \leq T$ . We can approximate  $\mu$  by convolution (we will still denote by  $\mu$  its trivial extension to 0 outside the interval  $[0, R)$ )

$$m_\varepsilon(x) := \mu * \kappa_\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \kappa((x-y)/\varepsilon) d\mu(y)$$

where  $\kappa$  is a shifted standard  $C_c^\infty$  mollifier (see, *e.g.*, [3, page 41]) with support in  $[0, 1]$  and we define

$$\mathbf{t}_\varepsilon(x) := \int_0^x \frac{1}{f(r)} + m_\varepsilon(r) dr = \int_0^x \frac{1 + m_\varepsilon f}{f}(r) dr$$

for  $x \in [0, R)$ .

We denote by  $J_t \subset [0, u(T)]$  the at most countable set of discontinuity points of  $\mathbf{t}$ , which coincides with the set of atomic points of  $\mu$  (*i.e.*  $\{x \in [0, u(T)] : \mu(\{x\}) > 0\}$ ). Since  $m_\varepsilon \mathcal{L}^1$  converge to  $\mu$  as  $\varepsilon \downarrow 0$  in the weak topology of finite positive measures, we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^x m_\varepsilon(r) dr = \mu([0, x)) \quad \text{for every } x \in [0, R) \setminus J_t,$$

see, *e.g.*, [3, Proposition 1.62(b) and Theorem 2.2]. We used the fact that the support of  $m_\varepsilon$  is contained in  $[0, u(T) + \varepsilon]$ . The convergence

$$\mathbf{t}_\varepsilon(x) \rightarrow \mathbf{t}(x) \text{ as } \varepsilon \rightarrow 0$$

for all  $x \in [0, R) \setminus J_t$  directly follows. Moreover, there exists  $\bar{\varepsilon} > 0$  such that  $(u(T) - \bar{\varepsilon}, u(T)) \subset D$ ; hence for  $\varepsilon \in (0, \bar{\varepsilon})$ , the support of  $m_\varepsilon$  is contained in  $[0, u(T)]$  and

$$\mathbf{t}_\varepsilon(x) = \mathbf{t}(x) \text{ for every } x \geq u(T), \quad (5.6)$$

since  $\int_0^{u(T)} m_\varepsilon(r) dr = \mu([0, u(T)])$ . Let us now consider the map  $\mathbf{t}_\varepsilon$  for  $\varepsilon \in (0, \bar{\varepsilon})$  fixed. It is locally absolutely continuous, strictly increasing and differentiable for  $\mathcal{L}^1$ -a.e.  $x \geq 0$  with

$$\mathbf{t}'_\varepsilon(x) = \frac{1 + m_\varepsilon(x)f(x)}{f(x)} > 0, \quad \lim_{x \uparrow R} \mathbf{t}_\varepsilon(x) = T_\star(u) =: T_\star.$$

Thus the inverse map  $u_\varepsilon : [0, T_\star) \rightarrow [0, R)$  is locally absolutely continuous with

$$u'_\varepsilon(t) = \frac{f(u_\varepsilon(t))}{1 + m_\varepsilon(u_\varepsilon(t))f(u_\varepsilon(t))} \text{ for } \mathcal{L}^1\text{-a.e. } t, \quad u_\varepsilon(t) = u(t) \quad \text{for } t \geq T. \quad (5.7)$$

Moreover, if  $T_\star < \infty$ , we see that  $\lim_{t \uparrow T_\star} u_\varepsilon(t) = R = u(T_\star)$  and we can extend  $u_\varepsilon$  to the whole real line by setting  $u_\varepsilon(t) = u(T_\star)$  for  $t \geq T_\star$ .

So we obtain that  $u_\varepsilon$  satisfies  $u'_\varepsilon(t) = E'_\varepsilon(u_\varepsilon(t))$  for all  $t \in [0, \infty)$ , for the energy  $E_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  with

$$E'_\varepsilon = \frac{f}{1 + m_\varepsilon f}$$

(and initial value  $u_\varepsilon(0) = 0$ ). Moreover, since  $t_\varepsilon$  is absolutely continuous, the set

$$\{t \in [0, T_\star) : E'_\varepsilon(u_\varepsilon(t)) = 0\} \subset t_\varepsilon([0, R) \setminus D)$$

has Lebesgue measure 0; by Theorem 3.6(5), it follows that  $u_\varepsilon \in \text{GF}_{\min}[-E_\varepsilon]$ .

Since  $E'_\varepsilon$  is uniformly bounded in every bounded interval, the family  $u_\varepsilon$  is uniformly Lipschitz in every bounded interval by (5.7); in order to prove that it converges to  $u$  as  $\varepsilon \downarrow 0$  it is sufficient to characterize its limit  $\tilde{u}$  along a convergent subsequence  $k \mapsto u_{\varepsilon(k)}$ ,  $\varepsilon(k) \downarrow 0$  (which exists by Ascoli-Arzelà theorem).

Since for all  $x \in [0, R) \setminus J_t$  we have

$$u(t(x)) = x = u_{\varepsilon(k)}(t_{\varepsilon(k)}(x)) \rightarrow \tilde{u}(t(x)) \quad \text{as } k \uparrow \infty;$$

since  $t$  is left-continuous, we get

$$\tilde{u}(t) = u(t) \text{ for all } t \in t([0, R)).$$

Since  $u$  is continuous and locally constant in the interior of  $[0, T_\star) \setminus t([0, R))$  and  $\tilde{u}$  is monotone, we conclude that  $u \equiv \tilde{u}$  on  $[0, \infty)$ . Hence,  $u_\varepsilon$  is converging uniformly to  $u$ .

In the last part of the proof, we show that  $E'_\varepsilon =: f_\varepsilon$  is converging uniformly to  $E' = f$  on  $\mathbb{R}$ . We notice that the support of  $\mu$  is a compact set included in  $\tilde{N} := [0, u(T)] \setminus D$ , where  $f$  vanishes. Hence, the support of  $m_\varepsilon$  is contained in the  $\varepsilon$ -neighborhood  $\tilde{N}_\varepsilon := \{x \in [0, u(T)] : \text{dist}(x, \tilde{N}) \leq \varepsilon\}$  of  $\tilde{N}$  for  $\varepsilon \in (0, \bar{\varepsilon})$  so that  $f = f_\varepsilon$  in the complement of  $\tilde{N}_\varepsilon$ . On the other hand, since  $0 \leq f_\varepsilon \leq f$  on  $[0, R)$ , we get

$$\sup_{x \in \mathbb{R}} |f(x) - f_\varepsilon(x)| \leq 2 \sup_{x \in \tilde{N}_\varepsilon} f(x) \downarrow 0 \quad \text{as } \varepsilon \downarrow 0,$$

since  $f$  is uniformly continuous in every compact subset of  $\mathbb{R}$  and  $f \equiv 0$  on  $\tilde{N}$ .  $\square$

By applying Lemma 2.11, Remark 3.8 and Theorem 4.8, we can now easily prove that in the one dimensional case any solution of (GF) is strongly approximable. In the next section, we will use Proposition 5.1 as an inspiring guide to study the problem in an arbitrary finite dimensional setting.

## 6. Strongly approximable solutions

We consider an arbitrary non-constant solution  $u \in \text{GF}[\phi]$  for  $\phi \in C^1(\mathbb{H})$ . Let  $v \in \text{GF}_{\min}[\phi]$  be the unique minimal solution with  $u \succ v$  (see Theorem 3.6) and  $R[u] \subset R[v] \subset \overline{R[u]}$ , and set  $T_\star := T_\star(v)$ .

We know that there exists an increasing 1-Lipschitz map  $\mathbf{z} : [0, +\infty) \rightarrow [0, +\infty)$  such that  $u(t) = v(\mathbf{z}(t))$ . We also know that the restriction of  $v$  to  $[0, T_\star)$  is an homeomorphism with

$$R := \begin{cases} R[v] & \text{if } T_\star = +\infty, \\ R[v] \setminus \{v(T_\star)\} & \text{if } T_\star < +\infty \end{cases} \quad (6.1)$$

whose inverse will be denoted by  $\tilde{\mathbf{t}} : R \rightarrow [0, T_\star)$ . We will set

$$\mathbf{x}(t) := \int_0^t |v'(s)| \, ds, \quad L_\star := \lim_{t \uparrow T_\star} \mathbf{x}(t) = \int_0^{+\infty} |v'(s)| \, ds; \quad (6.2)$$

notice that  $\mathbf{x} \in C^1([0, T_\star))$  with  $\mathbf{x}'(t) > 0$  a.e. so that it admits a locally absolutely continuous inverse that we will denote by  $\mathbf{t} : [0, L_\star) \rightarrow [0, T_\star)$ .

The arc-length parametrization of  $R$  is then given by

$$\tilde{\mathbf{x}} : R \rightarrow [0, L_\star), \quad \tilde{\mathbf{x}}(y) := \mathbf{x}(\tilde{\mathbf{t}}(y)) = \int_0^{\tilde{\mathbf{t}}(y)} |v'(s)| \, ds, \quad y \in R. \quad (6.3)$$

Notice that  $\tilde{\mathbf{x}}$  is an homeomorphism between  $R$  and  $[0, L_\star)$  which associates to every point  $u(t) \in R$  the length of the curve  $u([0, t])$ ; in particular  $\tilde{\mathbf{t}}(y) = \mathbf{t}(\tilde{\mathbf{x}}(y))$ .

Its inverse  $\mathbf{y} := (\tilde{\mathbf{x}})^{-1} : [0, L_\star) \rightarrow R$  is the arc-length parametrization of the curve  $v$ , defined by

$$\mathbf{y}(x) = v(\mathbf{t}(x)). \quad (6.4)$$

We can now consider the one-dimensional energy obtained by rectifying the graph of  $v$

$$E : [0, L_\star) \rightarrow \mathbb{R}, \quad E := -\phi \circ \mathbf{y}, \quad E(\mathbf{x}(t)) = -\phi(v(t)), \quad (6.5)$$

which is continuously differentiable with derivative

$$E'(x) = -\langle \nabla \phi(\mathbf{y}(x)), v'(\mathbf{t}(x)) \rangle \mathbf{t}'(x) = |\nabla \phi(\mathbf{y}(x))|^2 \frac{1}{|v'(\mathbf{t}(x))|} = |\nabla \phi(\mathbf{y}(x))|. \quad (6.6)$$

If  $T_\star < +\infty$ , then  $L_\star < +\infty$  and  $E$  is Lipschitz, so there is a continuous extension of the energy to  $[0, L_\star]$  which we still denote by  $E$ .

The next lemma shows that  $u$  gives rise to a solution of  $\text{GF}[-E]$  via a suitable rescaling.

**Lemma 6.1.** *The curve*

$$\mathbf{u} : [0, \infty) \rightarrow \mathbb{R}, \quad \mathbf{u}(t) := \begin{cases} \tilde{\mathbf{x}}(u(t)) & \text{for } t < T_\star(u) \\ L_\star & \text{for } t \geq T_\star(u) \text{ if } T_\star(u) < +\infty \end{cases} \quad (6.7)$$

belongs to  $\text{GF}[-E]$ , i.e.,

$$\mathbf{u}'(t) = E'(\mathbf{u}(t)) \quad \text{for all } t \geq 0. \quad (6.8)$$

Moreover, if  $u$  is eventually minimal, then  $\mathbf{u}$  is also eventually minimal.

*Proof.* In order to check (6.8) we first observe that  $\mathbf{u}(t) = \mathbf{x}(\tilde{\mathbf{t}}(u(t))) = \mathbf{x}(\tilde{\mathbf{t}}(v(\mathbf{z}(t)))) = \mathbf{x}(\mathbf{z}(t))$  for  $t \in [0, T_\star(u))$  and

$$\mathbf{x}'(t) = |v'(t)| = |\nabla\phi(v(t))| = |\nabla\phi(\mathbf{y}(\mathbf{x}(t)))| = E'(\mathbf{x}(t)) \quad \text{for } t \in [0, T_\star(u))$$

since  $v(t) = \mathbf{y}(\mathbf{x}(t))$ . Therefore

$$\mathbf{u}'(t) = \mathbf{x}'(\mathbf{z}(t))\mathbf{z}'(t) = E'(\mathbf{x}(\mathbf{z}(t)))\mathbf{z}'(t) = E'(\mathbf{u}(t))\mathbf{z}'(t)$$

for a.e.  $t \in [0, T_\star(u))$ . On the other hand, we know that  $\mathbf{z}'(t) = 1$  whenever  $|\nabla\phi(u(t))| \neq 0$ , i.e. if  $|\nabla\phi(\mathbf{y}(\mathbf{u}(t)))| = E'(\mathbf{u}(t)) \neq 0$ . If  $T_\star(u) < +\infty$ , the limit  $L_\star = \lim_{t \uparrow T_\star(u)} \mathbf{x}(\mathbf{z}(t))$  is finite and we can extend  $E$  to  $[0, L_\star]$  with  $E'(L_\star) = \lim_{x \uparrow L_\star} E'(x) = 0$ . Our calculations show that  $\mathbf{u} \in \text{GF}[-E]$ .

Finally, let us assume that  $u$  is eventually minimal; this is equivalent to say that for some  $T < T_\star(u)$  with  $u'(T) \neq 0$  we have  $\mathbf{z}'(t) \equiv 1$  in  $(T, T_\star(u))$ , so that  $\mathbf{u}$  is also eventually minimal.  $\square$

Let us assume that  $u$  is eventually minimal, according to Definition 3.7; in particular  $\mathbf{u}$  then satisfies (5.1). Arguing as in Proposition 5.1, we associate to  $E$  and  $\mathbf{u}$  energies  $E_\varepsilon : [0, L_\star] \rightarrow \mathbb{R}$  (with continuous extension to  $[0, L_\star]$  if  $T_\star \vee L_\star < +\infty$ ) and curves  $\mathbf{u}_\varepsilon : [0, \infty) \rightarrow \mathbb{R}$  satisfying

$$E'_\varepsilon(x) = \frac{|\nabla\phi(\mathbf{y}(x))|}{1 + m_\varepsilon(x)|\nabla\phi(\mathbf{y}(x))|} \quad (6.9)$$

with  $m_\varepsilon$  chosen as in the proof of Proposition 5.1, and

$$\mathbf{u}'_\varepsilon(t) = E'_\varepsilon(\mathbf{u}_\varepsilon(t)) \quad \text{for all } t \in [0, +\infty). \quad (6.10)$$

The set

$$\{t \in [0, T_\star(u)) : E'_\varepsilon(\mathbf{u}_\varepsilon(t)) = 0\}$$

has Lebesgue measure 0, and  $\mathbf{u}_\varepsilon$  is converging locally uniformly to  $\mathbf{u}$  as  $\varepsilon \rightarrow 0$ . We observe that  $E_\varepsilon$  satisfies up to an additive constant

$$\begin{aligned} E_\varepsilon(x) &= \int_0^x \frac{|\nabla\phi(\mathbf{y}(r))|}{1 + m_\varepsilon(r)|\nabla\phi(\mathbf{y}(r))|} dr \\ &= \int_0^{t(x)} \frac{|\nabla\phi(v(t))|}{1 + m_\varepsilon(\mathbf{x}(t))|\nabla\phi(v(t))|} |v'(t)| dt. \end{aligned} \quad (6.11)$$

Now, we translate this one dimensional setting with the approximation by  $E_\varepsilon$  and  $u_\varepsilon$  back to the initial situation with  $\phi$  and  $u$ .

**Lemma 6.2.** *Let us suppose that  $u$  is eventually minimal and that there exist  $\phi_\varepsilon \in C^1(\mathbb{H})$  satisfying*

$$\phi_\varepsilon(y) = -E_\varepsilon(\tilde{\mathbf{x}}(y)) = - \int_0^{\tilde{\mathbf{t}}(y)} \frac{|\nabla\phi(v(t))|}{1 + m_\varepsilon(\mathbf{x}(t))|\nabla\phi(v(t))|} |v'(t)| dt \quad \text{on } R, \quad (6.12)$$

and

$$\nabla\phi_\varepsilon(y) = \frac{\nabla\phi(y)}{1 + m_\varepsilon(\tilde{\mathbf{x}}(y))|\nabla\phi(y)|} \quad \text{for all } y \in R. \quad (6.13)$$

Then the curve

$$u_\varepsilon : [0, +\infty) \rightarrow \mathbb{H}, \quad u_\varepsilon(t) := \begin{cases} \mathbf{y}(u_\varepsilon(t)) & \text{for } t < T_\star(u), \\ u(T_\star(u)) & \text{for } t \geq T_\star(u) \text{ if } T_\star(u) < +\infty \end{cases} \quad (6.14)$$

is a minimal gradient flow for  $\phi_\varepsilon$ . Moreover,  $u_\varepsilon$  is converging locally uniformly to  $u$  as  $\varepsilon \rightarrow 0$ .

*Proof.* We just observe that for a.e.  $x \in [0, L_\star)$

$$\mathbf{y}'(x) = v'(\mathbf{t}(x))\mathbf{t}'(x) = -\nabla\phi(v(\mathbf{t}(x))) \frac{1}{|v'(\mathbf{t}(x))|} = -\frac{\nabla\phi(\mathbf{y}(x))}{|\nabla\phi(\mathbf{y}(x))|} \quad (6.15)$$

so that

$$u'_\varepsilon(t) = \mathbf{y}'(u_\varepsilon(t))\mathbf{u}'_\varepsilon(t) = -\frac{\nabla\phi(u_\varepsilon(t))}{|\nabla\phi(u_\varepsilon(t))|} \frac{|\nabla\phi(u_\varepsilon(t))|}{1 + m_\varepsilon(u_\varepsilon(t))|\nabla\phi(u_\varepsilon(t))|} = -\nabla\phi_\varepsilon(u_\varepsilon(t)).$$

We can apply Theorem 3.6(5) in order to conclude from  $u_\varepsilon \in \text{GF}_{\min}[-E_\varepsilon]$  that  $u_\varepsilon \in \text{GF}_{\min}[\phi_\varepsilon]$ . The convergence of  $u_\varepsilon$  is a consequence of the convergence of  $u_\varepsilon$ .  $\square$

It remains to show that there indeed exist energies  $\phi_\varepsilon : \mathbb{H} \rightarrow \mathbb{R}$  satisfying the assumptions of Lemma 6.2 and converging to  $\phi$  in the Lipschitz seminorm. Note that (6.12) which is not used in the proof of Lemma 6.2 should give an idea of how to construct  $\phi_\varepsilon$ .

**Lemma 6.3.** *Let us suppose that  $\mathbb{H}$  has finite dimension and  $u$  is an eventually minimal solution to (GF). There exist continuously differentiable functions  $\phi_\varepsilon : \mathbb{H} \rightarrow \mathbb{R}$  such that (6.12) (up to an additive constant) and (6.13) is satisfied and*

$$\limsup_{\varepsilon \downarrow 0} \sup_{\mathbb{H}} |\phi_\varepsilon - \phi| + \text{Lip}[\phi_\varepsilon - \phi] = 0. \quad (6.16)$$

*Proof.* Let us fix a time  $T < T_*(u)$  such that  $m_\varepsilon(\tilde{\mathbf{x}}(y)) = 0$  for all  $y \in R \setminus u([0, T])$  and  $\varepsilon > 0$  and choose  $T_1 \in (T, T_*(u))$  so that  $\phi(u(T)) > \phi(u(T_1)) > \inf_{R[u]} \phi$ . We consider the compact sets  $K := u([0, T])$  and  $K_1 := u([0, T_1])$  and the open set  $A := \{w \in \mathbb{H} : \phi(w) > \phi(u(T_1))\}$  which contains  $K$ . We can find a smooth function  $\psi : \mathbb{H} \rightarrow [0, 1]$  such that

$$\psi(w) \equiv 1 \quad \text{on } K, \quad \psi(w) = 0 \quad \text{on } \mathbb{H} \setminus A, \quad (6.17)$$

and

$$\nabla \psi \equiv 0 \quad \text{on } K, \quad \sup_{\mathbb{H}} |\nabla \psi| < +\infty. \quad (6.18)$$

The construction of  $\psi$  is standard: there exists  $\delta > 0$  such that the distance function  $\text{dist}(x, K)$  satisfies

$$\text{dist}(x, K) \leq 4\delta \quad \Rightarrow \quad x \in A.$$

The composition of  $\text{dist}(x, K)$  with  $\eta(d) := \frac{1}{\delta}(\delta - (d - 2\delta)_+)_+$  then yields a  $\frac{1}{\delta}$ -Lipschitz function taking value 1 in a neighborhood of radius  $2\delta$  around  $K$  and vanishing if  $\text{dist}(x, K) \geq 3\delta$ . Taking the convolution of  $\eta \circ \text{dist}(\cdot, K)$  with a smooth kernel with support in  $\{|x| \leq \delta\}$ , we obtain a suitable function  $\psi$ .

Let us define  $\delta_\varepsilon : K_1 \rightarrow \mathbb{R}$ ,  $\delta_\varepsilon(w) := -E_\varepsilon(\tilde{\mathbf{x}}(w)) - \phi(w)$ . Applying Whitney's Extension Theorem (see, e.g., [14, Theorem 2.3.6]), we aim to extend  $\delta_\varepsilon$  to a  $C^1$  function in  $\mathbb{H}$  with gradient  $Q_\varepsilon : \mathbb{H} \rightarrow \mathbb{H}$  satisfying  $Q_\varepsilon(w) = F_\varepsilon(w) - F(w)$  on  $K_1$ , in which

$$F_\varepsilon(w) := \frac{\nabla \phi(w)}{1 + m_\varepsilon(\tilde{\mathbf{x}}(w))|\nabla \phi(w)|}, \quad F(w) := \nabla \phi(w).$$

For that purpose, since  $\delta_\varepsilon$  and  $Q_\varepsilon$  are continuous and  $\phi \in C^1(\mathbb{H})$ , we only need to check if for  $w_n, \bar{w}_n \in K_1$  with  $w_n \neq \bar{w}_n$ ,  $\lim_{n \rightarrow \infty} |\bar{w}_n - w_n| = 0$ , it holds that

$$\lim_{n \rightarrow \infty} \frac{-E_\varepsilon(\tilde{\mathbf{x}}(\bar{w}_n)) + E_\varepsilon(\tilde{\mathbf{x}}(w_n)) - \langle F_\varepsilon(w_n), \bar{w}_n - w_n \rangle}{|\bar{w}_n - w_n|} = 0. \quad (6.19)$$

Up to extracting a subsequence, it is not restrictive to assume that  $\bar{w}_n$  and  $w_n$  converge to a common limit point  $w$ . By using the minimal flow  $v$  we can also find points  $t_n = \mathbf{t}(w_n)$ ,  $\bar{t}_n = \mathbf{t}(\bar{w}_n)$  converging to some  $t$  such that  $\bar{w}_n = v(\bar{t}_n)$ ,  $w_n = v(t_n)$ ,  $w = v(t)$ . Notice that

$$\begin{aligned} E_\varepsilon(\tilde{\mathbf{x}}(w_n)) - E_\varepsilon(\tilde{\mathbf{x}}(\bar{w}_n)) &= \int_{\bar{t}_n}^{t_n} \frac{|\nabla \phi(v(r))|}{1 + m_\varepsilon(\mathbf{x}(r))|\nabla \phi(v(r))|} |v'(r)| \, dr \\ \langle F_\varepsilon(w_n), \bar{w}_n - w_n \rangle &= \frac{\langle \nabla \phi(v(t_n)), v(\bar{t}_n) - v(t_n) \rangle}{1 + m_\varepsilon(\mathbf{x}(t_n))|\nabla \phi(v(t_n))|}. \end{aligned}$$

If  $\nabla \phi(w) = 0$ , then (6.19) directly follows from the fact that

$$|E_\varepsilon(\tilde{\mathbf{x}}(w_n)) - E_\varepsilon(\tilde{\mathbf{x}}(\bar{w}_n))| \leq \left| \int_{\bar{t}_n}^{t_n} |\nabla \phi(v(r))| |v'(r)| \, dr \right| = |\phi(\bar{w}_n) - \phi(w_n)|,$$

so that

$$\limsup_{n \rightarrow \infty} \frac{|E_\varepsilon(\tilde{\mathbf{x}}(w_n)) - E_\varepsilon(\tilde{\mathbf{x}}(\bar{w}_n))|}{|w_n - \bar{w}_n|} \leq \limsup_{n \rightarrow \infty} \frac{|\phi(\bar{w}_n) - \phi(w_n)|}{|w_n - \bar{w}_n|} = 0,$$

and

$$\limsup_{n \rightarrow \infty} \frac{|\langle F_\varepsilon(w_n), \bar{w}_n - w_n \rangle|}{|\bar{w}_n - w_n|} \leq \limsup_{n \rightarrow \infty} |\nabla \phi(w_n)| = 0.$$

If  $|\nabla \phi(w)| \neq 0$ , then

$$\lim_{n \rightarrow \infty} \frac{v(t_n) - v(\bar{t}_n)}{t_n - \bar{t}_n} = v'(t) = -\nabla \phi(v(t)) = -\nabla \phi(w) \neq 0,$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{-E_\varepsilon(\tilde{\mathbf{x}}(\bar{w}_n)) + E_\varepsilon(\tilde{\mathbf{x}}(w_n)) - \langle F_\varepsilon(w_n), \bar{w}_n - w_n \rangle}{t_n - \bar{t}_n} \\ &= \frac{|\nabla \phi(v(t))|}{1 + m_\varepsilon(\mathbf{x}(t))|\nabla \phi(v(t))|} |v'(t)| + \frac{\langle \nabla \phi(v(t)), v'(t) \rangle}{1 + m_\varepsilon(\mathbf{x}(t))|\nabla \phi(v(t))|} = 0. \end{aligned}$$

So  $\delta_\varepsilon : K_1 \rightarrow \mathbb{R}$  can be extended to a continuously differentiable function  $\delta_\varepsilon : \mathbb{H} \rightarrow \mathbb{R}$  with gradient  $\nabla \delta_\varepsilon = Q_\varepsilon$  on  $K_1$ . Moreover, there exists a constant  $C$  only depending on  $K_1$  such that (see [14, (2.3.8) in Theorem 2.3.6])

$$\begin{aligned} & \sup_{\mathbb{H}} |\delta_\varepsilon| + \sup_{\mathbb{H}} |\nabla \delta_\varepsilon| \\ & \leq C \left( \sup_{x, y \in K_1} W_\varepsilon(x, y) + \sup_{x, y \in K_1} |Q_\varepsilon(x) - Q_\varepsilon(y)| + \sup_{K_1} |\delta_\varepsilon| + \sup_{K_1} |Q_\varepsilon| \right), \end{aligned} \quad (6.20)$$

where

$$W_\varepsilon(x, y) := \frac{|\delta_\varepsilon(x) - \delta_\varepsilon(y) - \langle Q_\varepsilon(y), x - y \rangle|}{|x - y|} \text{ if } x \neq y, \quad W_\varepsilon(x, x) = 0.$$

Since  $E_\varepsilon$  is determined up to an additive constant, we may assume that  $E_\varepsilon(\mathbf{u}(T)) = E(\mathbf{u}(T))$  and thus that  $\delta_\varepsilon$  is converging uniformly to 0 on  $K_1$  and  $\delta_\varepsilon \equiv 0$  on  $K_1 \setminus K$ . Moreover, it is not difficult to check that  $Q_\varepsilon$  is converging uniformly to 0 on  $K_1$ . Now, in order to show that  $W_\varepsilon$  is converging uniformly to 0 on  $K_1 \times K_1$ , it is sufficient to prove that  $W_\varepsilon(x_\varepsilon, y_\varepsilon) \rightarrow 0$  whenever  $|x_\varepsilon - y_\varepsilon| \rightarrow 0$ ,  $x_\varepsilon \neq y_\varepsilon$ ,  $x_\varepsilon, y_\varepsilon \in K_1$ . For this, we repeat the arguments as in the proof of (6.19) combined with the argument at the end of the proof of Proposition 5.1: We may assume without loss of generality that  $x_\varepsilon = v(t_\varepsilon)$  and  $y_\varepsilon = v(\bar{t}_\varepsilon)$  converge to a common limit point  $w = v(t)$  and  $t_\varepsilon, \bar{t}_\varepsilon$  converge to  $t$ . If  $\nabla \phi(w) = 0$ , we may copy the corresponding arguments from the proof of (6.19) as they are independent of  $\varepsilon$  in this case. If  $\nabla \phi(w) \neq 0$ , then a contradiction argument shows that there exists  $\bar{\varepsilon} > 0$  such



that  $m_\varepsilon(\mathbf{x}(r)) = 0$  for all  $\min\{t_\varepsilon, \tilde{t}_\varepsilon\} \leq r \leq \max\{t_\varepsilon, \tilde{t}_\varepsilon\}$ ,  $0 < \varepsilon < \bar{\varepsilon}$  (because  $m_\varepsilon(\mathbf{x}(r_\varepsilon)) \neq 0$  would imply  $\nabla\phi(\mathbf{y}(s_\varepsilon)) = 0$  for some  $s_\varepsilon$  with  $|s_\varepsilon - \mathbf{x}(r_\varepsilon)| \leq \varepsilon$ ), and again, we may argue as in the proof of (6.19). The claim then follows.

Therefore, we infer from (6.20) that  $\delta_\varepsilon$  and  $\nabla\delta_\varepsilon$  are converging uniformly to 0 on  $\mathbb{H}$ . We set

$$\phi_\varepsilon := \phi + \psi\delta_\varepsilon,$$

where  $\psi$  has been introduced in (6.17) and (6.18). The functions  $\phi_\varepsilon : \mathbb{H} \rightarrow \mathbb{R}$  have all the desired properties.  $\square$

**Theorem 6.4.** *Let us suppose that  $\mathbb{H}$  is a finite dimensional Euclidean space,  $\phi \in C^1(\mathbb{H})$  satisfies the quadratic lower bound (1.8) and  $u : [0, +\infty) \rightarrow \mathbb{H}$  is a solution to (GF). Then  $u$  is strongly approximable, according to Definition 2.7, i.e. there exist functions  $\phi_\tau : \mathbb{H} \rightarrow \mathbb{R}$  ( $\tau > 0$ ) such that  $\text{Lip}[\phi_\tau - \phi] \rightarrow 0$  as  $\tau \downarrow 0$  and  $\text{MM}(\Phi, u(0)) = \{u\} = \text{GMM}(\Phi, u(0))$  for*

$$\Phi(\tau, U, V) := \phi_\tau(V) + \frac{1}{2\tau}|V - U|^2.$$

*Proof.* Lemma 2.11 shows that the class of strongly approximable solutions is closed with respect to Lipschitz convergence of the functionals and locally uniform convergence of the solutions. By Theorem 4.8, every minimal solution is strongly approximable; combining these results with the results from Lemma 6.2 and 6.3 we obtain that the class of eventually minimal solutions is also strongly approximable. By Remark 3.8 we conclude.  $\square$

**Remark 6.5.** If  $\phi \in \text{Lip}(\mathbb{H})$ , then it clearly satisfies the quadratic lower bound (1.8).

## Appendix A. Diffuse critical points for one-dimensional gradient flows

In this section we give an example of a solution to a one dimensional gradient flow generated by a function whose derivative vanishes in a Cantor set. In particular, the example shows that the strict monotonicity of the energy along a solution curve is not sufficient to guarantee its minimality.

Let us start from the continuous function  $f = -\phi' : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) := \begin{cases} \pi\sqrt{x(1-x)} & \text{if } x \in (0, 1), \\ 0 & \text{elsewhere.} \end{cases}$$

One can check by direct calculation that the curve  $u : [0, 1] \rightarrow \mathbb{R}$

$$u(t) := \frac{1}{2} + \frac{1}{2} \sin\left(\pi\left(t - \frac{1}{2}\right)\right), \quad \text{satisfies } u'(t) = f(u(t)), \quad u(0) = 0, \quad u(1) = 1.$$

Let  $C \subset [0, 1]$  be the classical Cantor set and decompose  $[0, 1] \setminus C$  into the disjoint union of countable open intervals  $I_n = (a_n, b_n)$  ( $n \in \mathbb{N}$ ) with  $l_n := b_n - a_n$ . We denote by  $L_n : \mathbb{R} \rightarrow [0, 1]$  the continuous and piecewise linear map transforming  $I_n$  into  $(0, 1)$ , which is constant outside  $I_n$  (i.e.  $L_n(x) = 0$  if  $x \leq a_n$  and  $L_n(x) = 1$  if  $x \geq b_n$ ).

Since  $\sum_n l_n = 1 < \infty$ , we can choose  $\beta_n > 0$  so that

$$\alpha_n := \beta_n^{-1} l_n \rightarrow 0, \quad B := \sum_n \beta_n < \infty.$$

We set

$$f_n(x) := \beta_n^{-1} l_n f(L_n(x))$$

and define  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) := \sum_n f_n(x).$$

Note that  $g$  is well-defined and continuous since  $\sup_x \sum_{n=m}^M f_n(x) \leq \pi \sup_{n \geq m} \alpha_n$  and every  $f_n$  is continuous.

Now, let us define the map  $R : [0, 1] \rightarrow \mathbb{R}$ ,

$$R(x) := \sum_n \beta_n L_n(x), \quad R'(x) = \frac{\beta_n}{l_n} \text{ on } I_n,$$

which is absolutely continuous since  $\sum_n \beta_n < \infty$ , with  $R'(x) > 0$  a.e.. Hence,  $R$  possesses an inverse map  $R^{-1} =: S : [0, B] \rightarrow [0, 1]$ , which is also absolutely continuous.

The intervals  $\tilde{I}_n := R(I_n) = (\tilde{a}_n, \tilde{b}_n)$  are disjoint, covering  $[0, B] \setminus R(C)$ . Note that  $R(C)$  has Lebesgue measure 0. Setting  $\tilde{L}_n := L_n \circ S$ , we have

$$\tilde{L}_n(t) = \frac{t - \tilde{a}_n}{\beta_n} \text{ if } t \in \tilde{I}_n, \quad \tilde{L}_n(t) = 0 \text{ if } t \leq \tilde{a}_n, \quad \tilde{L}_n(t) = 1 \text{ if } t \geq \tilde{b}_n.$$

We define  $v : [0, B] \rightarrow \mathbb{R}$ ,

$$v(t) := \sum_n l_n u(\tilde{L}_n(t)).$$

It is not difficult to check that  $v$  is of class  $C^1$ , and that

$$\{t \in [0, B] : v'(t) = 0\} = R(C).$$

Moreover, it holds that  $v' = g \circ v$ : by density and continuity, it is sufficient to select  $t \in \tilde{I}_n$ ; in this case, we have

$$v(t) = a_n + l_n u((t - \tilde{a}_n)/\beta_n)$$

and

$$v'(t) = \beta_n^{-1} l_n u'(\tilde{L}_n(t)) = f_n(l_n u'(\tilde{L}_n(t)) + a_n) = f_n(v(t)) = g(v(t)).$$

So if we denote by  $G$  the primitive of  $g$ , then  $v$  is a minimal gradient flow for  $-G$ .

Let  $\mu$  be a positive finite Cantor measure concentrated on  $R(C)$ , in particular  $\mu(\{x\}) = 0$  for all  $x$  and  $\mu(R(C)) > 0$ . We define

$$\psi(t) := t + \mu([0, t)).$$

The map  $\psi : [0, B] \rightarrow [0, B + \mu(R(C))]$  is continuous and strictly increasing and we denote by  $\eta : [0, B + \mu(R(C))] \rightarrow [0, B]$  its strictly increasing inverse. For  $s < t$ , we have

$$t - s = \psi(\eta(t)) - \psi(\eta(s)) = \eta(t) - \eta(s) + \mu((\eta(s), \eta(t))) \geq \eta(t) - \eta(s),$$

i.e.  $\eta$  is 1-Lipschitz continuous.

We define  $w : [0, B + \mu(R(C))] \rightarrow \mathbb{R}$ ,

$$w(s) := v(\eta(s)).$$

The curve  $w$  is Lipschitz continuous and  $\eta'(s) = 1$  for all  $s \in \psi([0, B] \setminus R(C))$ . Moreover, it holds that

$$\{s \in [0, B + \mu(R(C))] : g(w(s)) = 0\} = \{\psi(t) : t \in [0, B], g(v(t)) = 0\} = \psi(R(C)).$$

From this we can infer

$$w'(s) = g(w(s)) \text{ for all } s \in (0, B + \mu(R(C)))$$

(in particular,  $w$  is of class  $C^1$ ).

The set  $\psi(R(C))$  has Lebesgue measure  $\mu(R(C)) > 0$ . So, the gradient flow  $w$  is not minimal but along the curve the energy  $-G \circ w : [0, B + \mu(R(C))] \rightarrow \mathbb{R}$  is strictly decreasing.

The example could be set in a more general way, starting from a Cantor-like set and an ordinary differential equation with non-uniqueness at the end points of a reference interval.

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Zentrum Mathematik der Technischen  
Universität München  
Boltzmannstraße 3  
85748 Garching bei München, Germany  
fleissne@ma.tum.de

Giuseppe Savaré  
Dipartimento di Matematica “F. Casorati”  
Università di Pavia  
Via Ferrata, 1  
27100 Pavia, Italia  
giuseppe.savare@unipv.it