Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XX (2020), 721-750

Pointwise estimates of solutions to nonlinear equations for nonlocal operators

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Abstract. We study pointwise behavior of positive solutions to nonlinear integral equations, and related inequalities, of the type

$$u(x) - \int_{\Omega} G(x, y)\, g(u(y)) d\sigma(y) = h,$$

where (Ω, σ) is a locally compact measure space, $G(x, y)\colon \Omega \times \Omega \to [0, +\infty]$ is a kernel that satisfies a weak form of the maximum principle, $h \geq 0$ is a measurable function, and $g\colon [0, \infty) \to [0, \infty)$ is a monotone increasing function.

In the special case where G is Green's function of the Laplacian (or fractional Laplacian) that satisfies the maximum principle and $h = 1$, a typical global pointwise bound for any supersolution $u > 0$ is given by

$$u(x) \geq F^{-1}(G\sigma(x)), \quad x \in \Omega,$$

where $F(t) := \int_1^t \frac{ds}{g(s)}$, $t \geq 1$, and necessarily

$$G\sigma(x) < F(\infty) = \int_1^{+\infty} \frac{ds}{g(s)},$$

for every $x \in \Omega$ such that $u(x) < \infty$.

This problem is motivated by the semilinear fractional Laplace equation

$$(-\Delta)^{\frac{\alpha}{2}} u - g(u)\sigma = \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

with measure coefficients σ, μ, where $g(u) = u^q$, $q > 0$, and $0 < \alpha < n$, in domains $\Omega \subseteq \mathbb{R}^n$, or Riemannian manifolds, with positive Green's function G.

In a similar way, we treat positive solutions to the equation

$$u(x) + \int_{\Omega} G(x, y)\, g(u(y)) d\sigma(y) = h,$$

and the corresponding fractional Laplace equation $(-\Delta)^{\frac{\alpha}{2}} u + g(u)\sigma = \mu$, with a monotone decreasing function g, in particular $g(u) = u^q$, $q < 0$.

Mathematics Subject Classification (2010): 35J61 (primary); 31B10, 42B37 (secondary).

The work is supported by German Research Council through SFB 1283.

Received February 13, 2018; accepted in revised form June 28, 2018.
Published online May 2020.

1. Introduction

We study pointwise behavior of non-negative solutions u to nonlinear integral equations (and related inequalities) of the type

$$u(x) = \int_\Omega G(x, y)\, g(u(y)) d\sigma(y) + h(x), \quad x \in \Omega. \tag{1.1}$$

Here Ω is a locally compact Hausdorff space, σ is a Radon measure on Ω, $G(x, y) \colon \Omega \times \Omega \to [0, +\infty]$ is a lower semicontinuous function, $g \colon [0, \infty) \to [0, \infty)$ is a monotone increasing continuous function, and $h \ge 0$ is a given σ-measurable function.

We also treat positive solutions to more general equations and inequalities with measure-valued kernels G, or equivalently, operator equations of the type

$$u = T(g(u)) + h \quad \text{in } \Omega \tag{1.2}$$

on a measurable space (Ω, Ξ), where T is a positivity preserving linear operator acting in the cone of measurable functions $\Omega \to [0, +\infty]$, and continuous with respect to pointwise limits of monotone increasing sequences of functions (see Section 2, Examples 2.2-2.4).

Our main applications are to positive weak solutions of the semilinear fractional Laplace problem with measure coefficients,

$$(-\Delta)^{\frac{\alpha}{2}} u - u^q \sigma = \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c, \tag{1.3}$$

where $q > 0$, $0 < \alpha < n$, and μ, σ are Radon measures in a domain $\Omega \subseteq \mathbb{R}^n$ (or a Riemannian manifold) with positive Green's function G. Equation (1.3) is equivalent to the integral equation (1.1) with

$$h = G\mu = \int_\Omega G(\cdot, y)\, d\mu(y).$$

Similar results are obtained for the integral equation

$$u(x) = -\int_\Omega G(x, y)\, g(u(y)) d\sigma(y) + h(x), \quad x \in \Omega, \tag{1.4}$$

where g is a monotone decreasing function, and the corresponding fractional Laplace problem

$$(-\Delta)^{\frac{\alpha}{2}} u + u^q \sigma = \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c, \tag{1.5}$$

in the case $q < 0$.

More precisely, we obtain sharp global lower bounds for non-negative measurable functions u satisfying the integral inequality

$$u(x) \ge G(g(u)\, d\sigma)(x) + h(x) \quad \text{in } \Omega \tag{1.6}$$

in the case of monotone increasing g, and upper bounds for solutions of

$$u(x) \le -G(g(u)\, d\sigma)(x) + h(x) \quad \text{in } \Omega \tag{1.7}$$

in the case of monotone decreasing g.

We assume that the kernel G satisfies the following form of the *weak maximum principle*:

For any Radon measure ν *in* Ω *with compact support* $K = \text{supp}(\nu)$*,*

$$G\nu \le 1 \text{ in } K \implies G\nu \le \mathfrak{b} \text{ in } \Omega, \tag{1.8}$$

with some constant $\mathfrak{b} \ge 1$.

This property of G is sometimes referred to as the generalized, or rough maximum principle (see [1, 17, 19]), and is known for many local and non-local operators. It is used extensively in this paper, together with a weak form of the domination principle (1.11) considered below.

These basic principles make it possible to extend the results of [13], where classical solutions to (1.3) and (1.5), with continuous σ, μ (not necessarily positive), were treated for the (weighted) Laplace operator on Euclidean domains and weighted Riemannian manifolds, to certain nonlocal problems for integral operators with kernel G, and measures σ, μ. The exact value of the constant $\mathfrak{b}$ in (1.8) is not essential, although it appears in various constants in our estimates, which are sharp if $\mathfrak{b} = 1$ (in particular, in the case of local operators).

As suggested by the referee, we list some examples of operators L whose associated kernel G satisfies (1.8). Notice that not all of these examples are independent, and some of them contain others as special cases. Related examples and various extensions can be found in [2,7,8,11,19].

(1) The Laplace operator $L = -\Delta$ on a domain $\Omega \subseteq \mathbb{R}^n$ $(n \ge 1)$ which has a nontrivial positive Green's function G (see [13,19]);
(2) The Laplace-Beltrami operator $L = -\Delta$ on a Riemannian manifold M_n which has a nontrivial positive Green's function G, and more generally its weighted analogue $L = -\frac{1}{\omega}\text{div}(\omega\nabla)$ on a weighted Riemannian manifold (see [12,13]);
(3) The fractional Laplace or Laplace-Beltrami operator $L = (-\Delta)^{\frac{\alpha}{2}}$ $(0 < \alpha \le 2)$ on a domain $\Omega \subseteq \mathbb{R}^n$, or a Riemannian manifold M_n, respectively, which has a nontrivial positive Green's function G ([2,19]);
(4) The fractional Laplace operator $L = (-\Delta)^{\frac{\alpha}{2}}$ $(0 < \alpha < n)$, or $L = (1 - \Delta)^{\frac{\alpha}{2}}$ $(\alpha > 0)$ on $\mathbb{R}^n$, or a ball, or half-space ([1,8]);
(5) Convolution operators $Lf = k \star f$ on $\mathbb{R}^n$ with radial monotone non-increasing kernel $k : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ discussed below after Theorem 1.1 (see also [1, Theorem 2.6.2]);
(6) Integral operators with quasi-metric kernels $G : \Omega \times \Omega \to (0, +\infty]$ on a locally compact Hausdorff space Ω such that $d(x, y) = \frac{1}{G(x,y)}$ satisfies the quasi-triangle inequality with constant $\varkappa$ (see Section 4 below, and also [8,15, 16]);
(7) Second-order differential operators $L = -\text{div}(a\nabla\cdot) - b \cdot \nabla + c$ in divergence form, with real-valued measurable coefficients $a = (a_{ij})$, $b = (b_i)$ and c, where b and $c \ge 0$ are bounded, and a is symmetric and uniformly elliptic, with Green's function G on an open subset $\Omega \subseteq \mathbb{R}^n$ ([2,11]);

(8) Second-order differential operators with real-valued continuous coefficients in non-divergence form $L = -\sum_{i,j} a_{ij}\partial_i\partial_j - b\cdot\nabla + c$, where $c \geq 0$, and a is symmetric and positive definite, with Green's function G on an open subset $\Omega \subseteq \mathbb{R}^n$ ([2,11]);

(9) More generally, Green's functions G associated with connected Brelot spaces ([2,7,11]);

(10) More generally, $\mathfrak{P}$-harmonic Bauer spaces with Green's function G; in particular, the heat kernel associated with the operator $\frac{\partial}{\partial t} + L$, where L is either one of the operators in Example 7 or 8 ([2,7,11]).

We remark that in Examples 1-3 and 7-10 we have $\mathfrak{b} = 1$ in (1.8). In Examples 4 and 5, the constant $\mathfrak{b}$ depends on n, whereas in Example 6 we have $\mathfrak{b} \leq 8\varkappa^3$ (see Lemma 4.2 below).

There are well-known cases where Green's function G of the operator L is known explicitly, for instance, $G(x,y) = \min(x,y)$ for $Lu = -u''$ on the positive half-axis $\mathbb{R}^+$ with zero boundary condition $u(0) = 0$, or $G(x,y) = c_{\alpha,n}|x-y|^{\alpha-n}$ for $L = (-\Delta)^{\frac{\alpha}{2}}$ $(0 < \alpha < n)$ on $\mathbb{R}^n$. For bounded $C^{1,1}$-domains $\Omega \subset \mathbb{R}^n$, there are explicit two-sided estimates of Green's function of $L = (-\Delta)^{\frac{\alpha}{2}}$ $(0 < \alpha \leq 2$, $\alpha < n)$:

$$G(x,y) \approx \left[\frac{\delta(x)\,\delta(y)}{|x-y|^2 + \delta(x)\,\delta(y)}\right]^{\frac{\alpha}{2}} |x-y|^{\alpha-n}, \quad x, y \in \Omega,$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. These estimates can be used in combination with our results to provide more explicit pointwise estimates of solutions (see, for instance, [8,14]). Examples providing explicit estimates of Green's functions for $L = -\Delta$ on Riemannian manifolds can be found in [11].

In the case $h = 1$ our main result is the following theorem.

Theorem 1.1. *Let G satisfy in Ω the weak maximum principle* (1.8) *with constant $\mathfrak{b} \geq 1$. Assume that g is a monotone non-decreasing positive continuous function in* $[1, +\infty]$, *and set*

$$F(t) = \int_1^t \frac{ds}{g(s)}, \quad t \geq 1.$$

If u is a positive measurable function on Ω that satisfies

$$u(x) \geq G(g(u)\,d\sigma)(x) + 1 \tag{1.9}$$

for σ-almost all $x \in \Omega$, then, at any point $x \in \Omega$ where (1.9) *is satisfied and $u(x) < +\infty$, we have*

$$G\sigma(x) < \mathfrak{b}\,F(\infty) = \mathfrak{b}\int_1^{+\infty} \frac{ds}{g(s)},$$

and

$$u(x) \geq 1 + \mathfrak{b}\left[F^{-1}\Big(\mathfrak{b}^{-1}G\sigma(x)\Big) - 1\right].$$

For a similar result in a more general setup of measure-valued kernels (Example 2.2), see Theorem 3.2; non-increasing nonlinearities g are treated in Theorem 3.3.

For example, for $g(s) = s^q$ with $q > 0, q \neq 1$, we obtain

$$u(x) \geq 1 + \mathfrak{b}\left[\left(1 + \frac{(1-q)\, G\sigma(x)}{\mathfrak{b}}\right)^{\frac{1}{1-q}} - 1\right],$$

where in the case $q > 1$ necessarily

$$G\sigma(x) < \frac{\mathfrak{b}}{q-1}.$$

In the case $q = 1$ we have

$$u(x) \geq 1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1} G\sigma(x)} - 1\right).$$

Theorem 1.1 and its corollaries yield global pointwise estimates for positive solutions to a broad class of elliptic and parabolic PDE, as well as non-local problems on Euclidean domains and manifolds. In particular, they are applicable to convolution equations on $\mathbb{R}^n$ of the type

$$u = k \star (g(u) d\sigma) + 1. \tag{1.10}$$

Here

$$k \star (f d\sigma)(x) = \int_{\mathbb{R}^n} k(x-y) f(y) d\sigma(y),$$

where $k = k(|x|) > 0$ is an arbitrary lower semi-continuous, radially non-increasing function on $\mathbb{R}^n$. Such kernels are known to satisfy the weak maximum principle (1.8) with constant $\mathfrak{b}$ which depends only on the dimension n of the underlying space (see [1, Theorem 2.6.2]).

To treat more general right-hand sides $h \geq 0$, we invoke a weak form of the *domination principle* for kernels G with respect to a Radon measure σ on Ω and a lower semi-continuous function h:

For any bounded measurable function $f \geq 0$ with compact support,

$$G(f d\sigma)(x) \leq h(x) \text{ in } \operatorname{supp}(f) \implies G(f d\sigma)(x) \leq \mathfrak{b}\, h(x) \text{ in } \Omega, \tag{1.11}$$

provided $G(f d\sigma)$ is bounded on $\operatorname{supp}(f)$.

Closely related properties are sometimes referred to as the dilated domination principle, the complete maximum principle, or the second maximum principle in the case $\mathfrak{b} = 1$ (see [2, 18, 19], and Sections 4 and 5 below).

We observe that the weak domination principle holds for Green's kernels associated with a large class of elliptic and parabolic problems, along with many non-local operators, and certain classes of functions h. In particular, (1.11) holds with $\mathfrak{b} = 1$ for nonnegative superharmonic functions h in Examples 1-3 and 7-9 considered above, but generally fails in Example 10 (some additional assumptions are needed). It is still true for $L = \frac{\partial}{\partial t} - \Delta$ and the classical heat kernel G with $h = G\nu + c$ for nonnegative measures ν and constants c; see [2,7].

Our main estimates in the case of general h and $g(t) = t^q$, $q \in \mathbb{R} \setminus \{0\}$, are contained in the following theorem. A more general statement in the context of measure-valued kernels G can be found in Theorem 5.1 below.

Theorem 1.2. *Let $h > 0$ be a lower semi-continuous function in Ω. Let G be a non-negative kernel in $\Omega \times \Omega$ which satisfies the domination principle* (1.11) *with respect to h with constant $\mathfrak{b} \geq 1$. Suppose that u is a non-negative function such that $u^q \in L^1_{\mathrm{loc}}(\sigma)$, which satisfies* (1.6) *if $q > 0$, and* (1.7) *if $q < 0$.*
Then if $q > 0$ ($q \neq 1$), we have

$$u(x) \geq h(x)\left\{1 + \mathfrak{b}\left[\left(1 + \frac{(1-q)\,G(h^q d\sigma)(x)}{\mathfrak{b}\,h(x)}\right)^{\frac{1}{1-q}} - 1\right]\right\}, \quad x \in \Omega, \quad (1.12)$$

where in the case $q > 1$ necessarily

$$G(h^q d\sigma)(x) < \frac{\mathfrak{b}}{q-1}\,h(x), \quad (1.13)$$

for all $x \in \Omega$ such that $0 \leq u(x) < +\infty$, and (1.6) *holds.*
In the case $q = 1$,

$$u(x) \geq h(x)\left[1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1}\frac{G(hd\sigma)(x)}{h(x)}} - 1\right)\right], \quad x \in \Omega. \quad (1.14)$$

If $q < 0$, then

$$u(x) \leq h(x)\left\{1 - \mathfrak{b}\left[1 - \left(1 - \frac{(1-q)\,G(h^q d\sigma)(x)}{\mathfrak{b}\,h(x)}\right)^{\frac{1}{1-q}}\right]\right\}, \quad x \in \Omega, \quad (1.15)$$

and necessarily

$$G(h^q d\sigma)(x) < \frac{\mathfrak{b}}{1-q}\left[1 - \left(1 - \mathfrak{b}^{-1}\right)^{1-q}\right] h(x), \quad (1.16)$$

for all $x \in \Omega$ such that $u(x) > 0$, $h(x) < +\infty$, and (1.7) *holds.*

These estimates with $h = G\mu$ yield the corresponding lower bounds for positive weak solutions $u \in L^q_{\mathrm{loc}}(\sigma)$ to the inequality

$$(-\Delta)^{\frac{\alpha}{2}} u - u^q \sigma \geq \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

for $q > 0$, and the upper bounds for positive weak solutions to the inequality

$$(-\Delta)^{\frac{\alpha}{2}} u + u^q \sigma \leq \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

for $q < 0$, where μ, σ are Radon measures in an arbitrary domain $\Omega \subseteq \mathbb{R}^n$ with positive Green's function G in the case $0 < \alpha \le 2$, when the domination principle holds with $\mathfrak{b} = 1$. They also hold in the case $0 < \alpha < n$ (with some constant $\mathfrak{b} \ge 1$) provided Green's function G is quasi-metric, or quasi-metrically modifiable, for instance, if Ω is the entire space, a ball, or half-space (see [8]).

For classical solutions and local elliptic differential operators, such estimates were obtained earlier in [13] in the case where $\sigma \in C(\Omega)$ is a continuous function which may change sign, and $\mu \ge 0$ is a locally Hölder continuous function in $\Omega \subseteq \mathbb{R}^n$, or a smooth Riemannian manifold.

In the linear case $q = 1$, estimate (1.14) in Theorem 1.2 obviously yields

$$u(x) \ge h(x)\, e^{\mathfrak{b}^{-1} \frac{G(h d\sigma)(x)}{h(x)}}, \quad x \in \Omega.$$

This is a refinement (with sharp constant in the case $\mathfrak{b} = 1$) of the lower bound obtained in [8] along with a matching upper bound, for quasi-metric kernels G. See Sections 4 and 5 where these and more general classes of kernels are treated.

In the superlinear case $q > 1$, the necessary condition (1.13) was found in [16] for quasi-metric kernels G (without the explicit constant), and in [3] for the Laplace operator $-\Delta$ with Dirichlet boundary conditions, with sharp constant $\frac{1}{q-1}$. In the latter case, the domination principle holds with $\mathfrak{b} = 1$, so that this constant is the same as in (1.13).

There are certain upper bounds for u in the case $q > 1$, and lower bounds in the case $q < 0$, which are true for general non-negative kernels G (without the weak domination principle) provided conditions (1.13) and (1.16) respectively hold with smaller constants depending on q, which ensure the existence of positive solutions ([13, Theorem 3.5]; see also [3,12,16] for $q > 1$).

For the *homogeneous* problem (1.6) with $h = 0$ in the sublinear case $0 < q < 1$, we have a similar lower estimate for non-trivial solutions u (see Corollary 3.6 for a more general setup of measure-valued kernels). No such estimates of positive solutions to (1.6) or (1.7) with $h = 0$ are available for $q \ge 1$ or $q < 0$, respectively.

Theorem 1.3. *Let* $0 < q < 1$. *Suppose* G *is a non-negative kernel on* $\Omega \times \Omega$ *satisfying the weak maximum principle* (1.8) *with constant* $\mathfrak{b} \ge 1$. *If* $u \in L^q_{\mathrm{loc}}(\sigma)$, $u > 0$, *is a solution to the integral inequality* (1.6) *with* $h = 0$, *that is,* $u \ge G(u^q d\sigma)$ *in* Ω, *then*

$$u(x) \ge (1-q)^{\frac{1}{1-q}}\, \mathfrak{b}^{-\frac{q}{1-q}} \big[G\sigma(x)\big]^{\frac{1}{1-q}}, \quad x \in \Omega. \tag{1.17}$$

If $\mathfrak{b} = 1$, then the constant $(1-q)^{\frac{1}{1-q}}$ in (1.17) is the same as in the local case ([13, Theorem 3.3]), and is sharp. This estimate was first deduced in [4] for solutions to the equation $-\Delta u = u^q \sigma$ on $\mathbb{R}^n$ (without the sharp constant). See also [5,6] for matching lower and upper bounds of solutions to the equation $(-\Delta)^{\frac{\alpha}{2}} u = u^q \sigma$ $(0 < \alpha < n)$ on $\mathbb{R}^n$ in the case $0 < q < 1$.

We remark that necessary and sufficient conditions for the existence of a positive solution in the case $0 < q < 1$ to the homogeneous equation $u = G(u^q d\sigma)$

in Ω for quasi-symmetric kernels G which satisfy the weak maximum principle are given in [22] (see also [21]).

ACKNOWLEDGEMENTS. The authors are grateful to Alexander Bendikov and Wolfhard Hansen for stimulating discussions. The second author wishes to thank the Mathematics Department at Bielefeld University for the hospitality during his visits.

2. The weak maximum principle and iterated estimates

We first prove a series of lemmas.

Lemma 2.1. *Let* (Ω, ω) *be a* σ*-finite measure space, and let* $a = \omega(\Omega) \leq +\infty$. *Let* $f : \Omega \to [0, +\infty]$ *be a measurable function. Let* $\phi : [0, a) \to [0, +\infty)$ *be a continuous, monotone non-decreasing function, and set* $\phi(a) := \lim_{t \to a^-} \phi(t) \in (0, +\infty]$. *Then the following inequality holds:*

$$\int_0^{\omega(\Omega)} \phi(t)\, dt \leq \int_\Omega \phi\left(\omega\left(\{z \in \Omega : f(z) \leq f(y)\}\right)\right) d\omega(y). \tag{2.1}$$

Proof. If $\omega(\Omega) = \infty$ then consider an exhausting sequence $\{\Omega_k\}_{k=1}^\infty$ of measurable subsets of Ω with $\omega(\Omega_k) < +\infty$. Since

$$\begin{aligned} &\int_\Omega \phi\left(\omega\left(\{z \in \Omega : f(z) \leq f(y)\}\right)\right) d\omega(y) \\ \geq &\int_{\Omega_k} \phi\left(\omega\left(\{z \in \Omega_k : f(z) \leq f(y)\}\right)\right) d\omega(y), \end{aligned}$$

it suffices to prove (2.1) for Ω_k instead of Ω and then let $k \to \infty$. Hence, we can assume without loss of generality that $a = \omega(\Omega) < \infty$.

If ϕ is unbounded (as $t \to a^-$) then replace ϕ by a bounded function $\phi_j = \min(\phi, j)$, prove (2.1) for ϕ_j and then let $j \to \infty$. Therefore, we assume in what follows that ϕ is continuous on $[0, a]$.

Assume first that the function f is simple, that is, f takes on only a finite set of values $\{b_i\}_{i=1}^N$ where the sequence $\{b_i\}$ is arranged in the increasing order. Set

$$a_i = \omega\{z \in \Omega : f(z) \leq b_i\} \text{ for } i = 1, ..., N$$

and $a_0 = 0$. If $y \in \Omega$ is such that $f(y) = b_i$, then

$$\phi\left(\omega\{z \in \Omega : f(z) \leq f(y)\}\right) = \phi(a_i).$$

It follows that

$$\begin{aligned}
&\int_\Omega \phi\left(\omega\{z\in\Omega\colon\ f(z)\le f(y)\}\right)d\omega(y)\\
&=\sum_{i=1}^N\int_{\{y\in\Omega:f(y)=b_i\}}\phi\left(\omega\{z\in\Omega\colon\ f(z)\le f(y)\}\right)d\omega(y)\\
&=\sum_{i=1}^N\phi(a_i)\,\omega\{y\in\Omega: f(y)=b_i\}\\
&=\sum_{i=1}^N\phi(a_i)\,(a_i-a_{i-1}).
\end{aligned}$$

Since the function ϕ is non-decreasing, we obtain

$$\sum_{i=1}^N\phi(a_i)\,(a_i-a_{i-1})\ge\sum_{i=1}^N\int_{a_{i-1}}^{a_i}\phi(t)\,dt=\int_{a_0}^{a_N}\phi(t)\,dt=\int_0^{\omega(\Omega)}\phi(t)\,dt,$$

whence (2.1) follows.

Let f be an arbitrary measurable function. By a standard argument, there is an increasing sequence $\{f_n\}$ of simple functions such that $f_n\uparrow f$ as $n\to\infty$. Applying the first part of the proof to f_n, we obtain

$$\begin{aligned}
\int_\Omega\phi\left(\omega\{z\in\Omega\colon\ f_n(z)\le f(y)\}\right)d\omega(y)&\ge\int_\Omega\phi\left(\omega\{z\in\Omega\colon\ f_n(z)\le f_n(y)\}\right)d\omega(y)\\
&\ge\int_0^{\omega(\Omega)}\phi(t)\,dt.
\end{aligned}$$

The sequence $F_n(y):=\omega\{z\in\Omega\colon\ f_n(z)\le f(y)\}$ of functions of $y\in\Omega$ is decreasing in n and converges to $\omega\{z\in\Omega\colon\ f(z)\le f(y)\}$ as $n\to\infty$. Since ϕ is bounded and continuous and $\omega(\Omega)<\infty$, we obtain by the bounded convergence theorem that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\phi\left(\omega\{z\in\Omega\colon\ f_n(z)\le f(y)\}\right)d\omega(y)\\
&=\int_\Omega\phi\left(\omega\{z\in\Omega\colon\ f(z)\le f(y)\}\right)d\omega(y),
\end{aligned}$$

whence (2.1) follows. □

In the rest of the section we assume that (Ω,Ξ) is a measurable space, and that $G(x,dy)$ is a *σ-finite kernel* in Ω, which means that, for any $x\in\Omega$, $G(x,dy)$ is a σ-finite measure on (Ω,Ξ), and this measure depends on x measurably. The latter means that, for any Ξ-measurable function $f:\Omega\to[0,\infty]$, the function

$$Gf(x)=\int_\Omega f(y)\,G(x,dy)$$

is also Ξ-measurable. We assume here that G satisfies the *weak maximum principle* in the following form:

There is a constant $\mathfrak{b} \geq 1$ *such that, for any non-negative bounded measurable function* f *on* Ω*,*

$$Gf \leq 1 \text{ in } \{f > 0\} \implies Gf \leq \mathfrak{b} \text{ in } \Omega. \tag{2.2}$$

Clearly, (2.2) implies the following: for any $\varepsilon \geq 0$,

$$Gf \leq \varepsilon \text{ in } \{f > 0\} \implies Gf \leq \varepsilon\mathfrak{b} \text{ in } \Omega. \tag{2.3}$$

Let us consider some examples where this framework is applicable.

Example 2.2. Let Ω be a locally compact Hausdorff space with countable base. Assume that $G(x, dy)$ is a *Radon kernel* in Ω, that is, for any $x \in \Omega$, $G(x, dy)$ is a Radon measure on Ω (in particular, σ-finite). The weak maximum principle for a Radon kernel G can be stated as follows:

For any bounded measurable function f *with compact support,*

$$Gf \leq 1 \text{ in } \operatorname{supp}(f) \implies Gf \leq \mathfrak{b} \text{ in } \Omega. \tag{2.4}$$

Then the weak maximum principle holds also in the form (2.2) by approximating an arbitrary function f by $f 1_F$ with compact F.

Example 2.3. Let Ω again be a locally compact Hausdorff space with countable base, and let ω be a Radon measure on Ω. Suppose $K : \Omega \times \Omega \to [0, +\infty]$ is a measurable function such that $K(x, \cdot) \in L^1_{\text{loc}}(\Omega, \omega)$ for any $x \in \Omega$. Then set

$$G(x, E) := \int_E K(x, y) d\omega(y), \tag{2.5}$$

so that G is a Radon kernel in the above sense. In this case we have

$$Gf(x) = \int_\Omega K(x, y) f(y) d\omega(y).$$

Example 2.4. Let (Ω, Ξ) be a measurable space. Let T be a positivity preserving linear operator acting in the cone of measurable functions $\Omega \to [0, +\infty]$. Assume also that T is continuous with respect to pointwise limits of monotone increasing sequences of functions.

Fix some $x \in \Omega$ and define the measure

$$\omega_x(E) = T1_E(x), \quad E \in \Xi.$$

Then, for any non-negative measurable function f on Ω, we have

$$Tf(x) = \int_\Omega f d\omega_x. \tag{2.6}$$

Indeed, if u is a simple function of the form

$$f = \sum_{i=1}^{\infty} f_i 1_{E_i},$$

where f_i are non-negative constants, then

$$Tf(x) = \sum_{i=1}^{\infty} f_i T 1_{E_i}(x) = \sum_{i=1}^{\infty} u_i \omega_x(E_i) = \int_\Omega f d\omega_x.$$

For a general measurable function f one proves (2.6) by passing to the limit using an increasing sequence of simple functions.

The measure ω_x is clearly σ-additive. The σ-finiteness of ω_x has to be assumed in addition (for example, if $T1(x) < \infty$ then the measure ω_x is finite). Assuming that ω_x is σ-finite, set $G(x, dy) := d\omega_x(y)$, or equivalently,

$$G(x, E) = \int_E G(x, dy) = T 1_E(x), \quad E \in \Xi. \tag{2.7}$$

The following is the key lemma used repeatedly throughout this paper.

Lemma 2.5. *Let* (Ω, Ξ) *be a measurable space, and let* $G(x, dy)$ *be a* σ*-finite kernel in* Ω. *Assume that* G *satisfies the weak maximum principle* (2.2). *Fix* $x \in \Omega$ *and set* $a = G1(x) \le +\infty$. *If a function* $\phi\colon [0, a) \to [0, +\infty)$ *satisfies the conditions of Lemma* 2.1 *and* $\phi(t) = \phi(a)$ *for* $t \ge a$, *then*

$$\int_0^{G1(x)} \phi(t)dt \le G\left[\phi(\mathfrak{b}\, G1)\right](x) \quad \text{for all } x \in \Omega. \tag{2.8}$$

Proof. For any $y \in \Omega$, set

$$E_y = \{z \in \Omega\colon\ G1(z) \le G1(y)\}.$$

Clearly,

$$G1_{E_y}(z) \le G1(z) \le G1(y) \quad \text{for all } z \in E_y.$$

Hence, by the weak maximum principle (2.3) applied to $f = 1_{E_y}$, we obtain

$$G1_{E_y}(z) \le \mathfrak{b}\, G1(y) \quad \text{for all } z \in \Omega,$$

which implies, for $z = x$,

$$G1_{E_y}(x) \le \mathfrak{b}\, G1(y). \tag{2.9}$$

Applying Lemma 2.1 to the σ-finite measure $\omega(dy) = G(x, dy)$ and the function $f = G1$, noticing that $\omega(E) = G1_E(x)$, and using (2.9), we obtain

$$\begin{aligned}\int_0^{G1(x)} \phi(t)\,dt &\le \int_\Omega \phi\left(\omega\{z \in \Omega\colon\ G1(z) \le G1(y)\}\right) \omega(dy) \\ &= \int_\Omega \phi\left(\omega\left(E_y\right)\right) \omega(dy) \\ &\le \int_\Omega \phi\left(\mathfrak{b}\, G1(y)\right) \omega(dy) \\ &= G\left[\phi\left(\mathfrak{b}\, G1\right)\right](x),\end{aligned}$$

which proves (2.8). □

Remark 2.6. In the particular case $\phi(t) = t^{r-1}$ $(t \ge 0)$ where $r \ge 1$, Lemma 2.5 gives the following estimate:

$$[G1(x)]^r \le r\, \mathfrak{b}^{r-1} G\left[(G1)^{r-1}\right](x) \quad \text{for all } x \in \Omega. \tag{2.10}$$

In the case $\phi(t) = t^{r-1}$ $(t > 0)$ where $0 < r \le 1$, it is easy to see that the converse inequality to (2.10) holds, that is,

$$[G1(x)]^r \ge r\, \mathfrak{b}^{r-1} G\left[(G1)^{r-1}\right](x) \quad \text{for all } x \in \Omega. \tag{2.11}$$

Lemma 2.7. *In the setting of Lemma* 2.5 *define a sequence* $\{f_k\}_{k=0}^\infty$ *of functions on* Ω *by*

$$f_0 = G1, \quad f_{k+1} = G\left(\phi\left(f_k\right)\right), \tag{2.12}$$

for all $k \ge 0$. *Set*

$$\psi(t) = \phi\left(\mathfrak{b}^{-1} t\right), \tag{2.13}$$

and define also the sequence $\{\psi_k\}_{k=0}^\infty$ *of functions on* $[0, \infty)$ *by* $\psi_0(t) = t$ *and*

$$\psi_{k+1}(t) = \int_0^t \psi \circ \psi_k(s) ds, \tag{2.14}$$

for all $k \ge 0$. *Then, for all* $x \in \Omega$ *and* $k \ge 0$,

$$\psi_k\left(f_0(x)\right) \le f_k(x). \tag{2.15}$$

For example, we have

$$f_1 = G\left(\phi\left(f_0\right)\right), \quad f_2 = G\left(\phi\left(f_1\right)\right),$$

and

$$\psi_1(t) = \int_0^t \psi(s)\, ds, \quad \psi_2(t) = \int_0^t \psi\left(\psi_1(s)\right) ds.$$

Proof. For $k = 0$ estimate (2.15) is trivial. For $k = 1$ estimate (2.15) follows from Lemma 2.5 with ψ in place of ϕ, since by (2.8)

$$\psi_1(f_0(x)) = \int_0^{G1(x)} \psi(s)ds \le G(\psi(\mathfrak{b}G1))(x) = G(\phi(G1))(x) = f_1(x).$$

Let us make an inductive step from k to $k+1$, where $k \ge 1$. Fix $y \in \Omega$ and define the set

$$\Omega_y = \{z \in \Omega\colon\ f_k(z) \le f_k(y)\}.$$

Then by (2.12) we have

$$G\left(1_{\Omega_y}\phi(f_{k-1})\right)(z) \le f_k(z) \le f_k(y) \quad \text{for all } z \in \Omega_y,$$

which implies by the weak maximum principle that

$$G\left(1_{\Omega_y}\phi(f_{k-1})\right)(x) \le \mathfrak{b}\, f_k(y) \ \text{ for all } x \in \Omega. \tag{2.16}$$

Consider now the kernel

$$\hat{G}(x, dz) = G\left(x, 1_{\Omega_y}dz\right),$$

and define a sequence of functions $\{\hat{f}_k\}$ similarly to (2.12):

$$\hat{f}_0 = \hat{G}1 = G1_{\Omega_y}, \quad \hat{f}_{k+1} = \hat{G}(\phi(\hat{f}_k)) = G\left(1_{\Omega_y}\phi(\hat{f}_k)\right),\ k \ge 0.$$

It follows from (2.16) that

$$\hat{f}_k(x) \le \mathfrak{b}\, f_k(y) \ \text{ for all } x \in \Omega.$$

By the inductive hypothesis, we have, for all $x \in \Omega$,

$$\psi_k\left(\hat{f}_0(x)\right) \le \hat{f}_k(x).$$

It follows that

$$\psi \circ \psi_k\left(\hat{f}_0(x)\right) \le \psi(\hat{f}_k(x)) \le \psi(\mathfrak{b}\, f_k(y)) = \phi(f_k(y)),$$

that is,

$$\psi \circ \psi_k\left(G1_{\Omega_y}\right)(x) \le \phi(f_k(y)) \ \text{ for all } x \in \Omega. \tag{2.17}$$

Fix now also $x \in \Omega$ and apply Lemma 2.1 with the σ-finite measure $\omega(dy) = G(x, dy)$. We obtain, using (2.14) and (2.17), that

$$\begin{aligned}
\psi_{k+1}(f_0(x)) &= \int_0^{f_0(x)} \psi \circ \psi_k(s)\, ds = \int_0^{\omega(\Omega)} \psi \circ \psi_k(s)\, ds \\
&\le \int_\Omega \psi \circ \psi_k(\omega(z \in \Omega : f_k(z) \le f(y)))\, \omega(dy) \\
&= \int_\Omega \psi \circ \psi_k\left(\omega\left(\Omega_y\right)\right) \omega(dy) \\
&= \int_\Omega \psi \circ \psi_k\left(G 1_{\Omega_y}(x)\right) \omega(dy) \\
&\le \int_\Omega \phi(f_k(y))\, \omega(dy) \\
&= G(\phi(f_k))(x) = f_{k+1}(x),
\end{aligned}$$

which finishes the inductive step. □

Setting in Lemma 2.7 $\phi(t) = t^q$, $q > 0$, we obtain the following.

Corollary 2.8. *Under the hypotheses of Lemma* 2.7, *we have, for any* $q > 0$, $k \ge 0$, *and all* $x \in \Omega$,

$$[f_0(x)]^{1+q+\cdots+q^k} \le c(q,k)\, \mathfrak{b}^{q+q^2+\cdots+q^k} f_k(x), \tag{2.18}$$

where f_k *are defined by* (2.12) *with* $\phi(t) = t^q$, *and*

$$c(q,k) = \prod_{j=1}^{k} \left(1 + q + \cdots + q^j\right)^{q^{k-j}}. \tag{2.19}$$

In particular, in the case $q = 1$, *for all* $x \in \Omega$ *we have*

$$[f_0(x)]^{k+1} \le (k+1)!\, \mathfrak{b}^k f_k(x). \tag{2.20}$$

Remark 2.9. A direct proof by induction using Lemma 2.5 gives a constant that grows in $\mathfrak{b}$ much faster than $\mathfrak{b}^{q+q^2+\cdots+q^k}$ in (2.18).

3. Monotone nonlinearities

In this section we will apply estimates of Section 2 to the following nonlinear problem. Let Ω and G be as above, that is (Ω, Ξ) is a measurable space and $G(x, dy)$ be a σ-finite kernel in Ω, satisfying the weak maximum principle (2.2).

Let $g\colon [1,+\infty) \to [0,+\infty)$ be a continuous, monotone non-decreasing function; set $g(+\infty) = \lim_{t\to+\infty} g(t)$. We consider the non-linear integral inequality

$$u(x) \geq G(g(u))(x) + 1 \text{ for all } x \in \Omega, \tag{3.1}$$

where $u : \Omega \to [1,\infty)$ is a measurable function. Our goal is to obtain sharp pointwise lower estimates of $u(x)$ which are better than the trivial estimate $u(x) \geq 1$. In what follows we always assume that $g(1) \geq 1$.

Remark 3.1. Indeed, if $g(1) = 0$, then simple examples, for instance, $g(t) = \log t$ and $u \equiv 1$, show that we cannot expect any non-trivial estimates for u. If $g(1) > 0$, then by renaming $\frac{g}{g(1)}$ back to g and changing G appropriately, we can assume that $g(1) = 1$. Hence, the assumption $g(1) \geq 1$ is natural in this setting.

The next theorem is our main result.

Theorem 3.2. *Let G be a σ-finite kernel on Ω satisfying the weak maximum principle* (2.2) *with $\mathfrak{b} \geq 1$. Let $g\colon [1,+\infty) \to [1,+\infty)$ be a continuous monotone non-decreasing function. Set*

$$F(t) = \int_1^t \frac{ds}{g(s)}, \quad t \geq 1. \tag{3.2}$$

If u satisfies (3.1) *then, for all $x \in \Omega$ such that $u(x) < +\infty$, the following inequalities hold:*

$$u(x) \geq 1 + \mathfrak{b}\left[F^{-1}\left(\mathfrak{b}^{-1}G1(x)\right) - 1\right] \tag{3.3}$$

and

$$\mathfrak{b}^{-1}G1(x) < a := \int_1^{+\infty} \frac{ds}{g(s)}. \tag{3.4}$$

Note that the function F is defined on $[1,\infty)$. Hence, the inverse function F^{-1} is defined on $[0,a)$, and takes values in $[1,\infty)$. Hence, the condition (3.4) is necessary for the right-hand side of (3.3) to be well-defined.

Proof. Set for any $t \geq 0$

$$\phi(t) = g(t+1) \quad \text{and} \quad \psi(t) = \phi(\mathfrak{b}^{-1}t) = g(\mathfrak{b}^{-1}t + 1). \tag{3.5}$$

Define the sequence $\{f_k\}$ of functions on Ω by (2.12), that is,

$$f_0 = G1, \quad f_{k+1} = G(\phi(f_k)).$$

We claim that, for all $k \geq 0$,

$$u \geq f_k + 1 \text{ in } \Omega. \tag{3.6}$$

Indeed, it follows from (3.1) that $u \geq 1$, and one more application of (3.1) yields

$$u \geq G(g(1)) + 1 \geq G1 + 1 = f_0 + 1,$$

that is, (3.6) for $k = 0$. If (3.6) is already proved for some $k \geq 0$, then substituting (3.6) into (3.1) yields

$$u \geq G\left(g\left(f_k + 1\right)\right) + 1 = G\left(\phi\left(f_k\right)\right) + 1 = f_{k+1} + 1,$$

which finishes the proof of (3.6).

Consider now the sequence $\{\psi_k\}_{k=0}^{\infty}$ of functions on $[0, \infty)$ defined by (2.14), that is, $\psi_0(t) = t$ and

$$\psi_{k+1}(t) = \int_0^t \psi \circ \psi_k(s) ds. \tag{3.7}$$

By Lemma 2.7, we have, for all $x \in \Omega$ and $k \geq 0$,

$$f_k(x) \geq \psi_k\left(f_0(x)\right),$$

which together with (3.6) imply

$$u(x) \geq \psi_k\left(G1(x)\right) + 1 \text{ for all } x \in \Omega.$$

By (3.5) the function ψ is monotone non-decreasing and $\psi \geq 1$, which implies that the sequence $\{\psi_k\}_{k=0}^{\infty}$ is non-decreasing, that is, $\psi_{k+1}(t) \geq \psi_k(t)$ for all $t \geq 0$. Indeed, for $k = 0$ it follows from

$$\psi_1(t) = \int_0^t \psi(t)\, dt \geq t = \psi_0(t),$$

and if $\psi_k \geq \psi_{k-1}$ is already proved then $\psi_{k+1} \geq \psi_k$ follows from (3.7) and the monotonicity of ψ.

Set

$$\psi_\infty(t) = \lim_{k\to\infty} \psi_k(t)$$

so that

$$u(x) \geq \psi_\infty\left(G1(x)\right) + 1 \text{ for all } x \in \Omega. \tag{3.8}$$

Let us fix $x \in \Omega$ such that $u(x) < +\infty$. It follows from (3.8) that

$$t_0 := G1(x) < +\infty \text{ and } \psi_\infty(t_0) < \infty.$$

Without loss of generality we may assume that $t_0 > 0$ since in the case $G1(x) = 0$ the estimates (3.3), (3.4) are obvious. Then we see that the function ψ_∞ is finite on $[0, t_0]$, positive on $(0, t_0]$ and satisfies the integral equation

$$\psi_\infty(t) = \int_0^t \psi \circ \psi_\infty(s)\, ds, \quad 0 < t \leq t_0. \tag{3.9}$$

It follows that ψ_∞ is continuously differentiable on $[0, t_0]$ and satisfies the differential equation

$$\frac{d\psi_\infty}{dt} = \psi(\psi_\infty(t)), \quad \psi_\infty(0) = 0. \tag{3.10}$$

Setting

$$\Psi(\xi) = \int_0^\xi \frac{ds}{\psi(s)} = \mathfrak{b}\, F\big(1 + \mathfrak{b}^{-1}\xi\big)$$

and observing that by (3.10)

$$\frac{d\Psi(\psi_\infty)(t)}{dt} = 1,$$

we obtain, for any $t \in [0, t_0]$,

$$\Psi\left(\psi_\infty(t)\right) = t. \tag{3.11}$$

It follows that, for $t = t_0$,

$$F(1 + \mathfrak{b}^{-1}\psi_\infty(t_0)) = \mathfrak{b}^{-1}t_0. \tag{3.12}$$

Since all values of F are located in $[0, a)$, we obtain that

$$\mathfrak{b}^{-1}t_0 < a,$$

which is equivalent to (3.4). Next, we obtain from (3.12) that

$$\psi_\infty(t_0) = \mathfrak{b}F^{-1}\left(\mathfrak{b}^{-1}t_0\right) - 1.$$

Substituting this into (3.8) yields (3.3), which finishes the proof. □

Our methods are also applicable to non-linear integral inequalities of the type

$$u(x) + G(g(u)) \le 1 \ \text{ in } \Omega, \tag{3.13}$$

defined on measurable functions $0 \le u \le 1$. Here $g\colon (0, 1] \to (0, +\infty)$ is a continuous, monotone non-increasing function and $g(0) = \lim_{t\to 0^+} g(t) \le +\infty$.

We exclude the case $g(1) = 0$, since otherwise we cannot expect an upper estimate for u better than the trivial estimate $u \le 1$. In fact, without loss of generality we may assume $g(1) \ge 1$ by using $\frac{g}{g(1)}$ and changing G appropriately as above (see Remark 3.1). Thus, we assume that $g\colon [0, 1] \to [1, +\infty]$.

Theorem 3.3. *Let G be a σ-finite kernel on Ω satisfying the weak maximum principle* (2.2) *with $\mathfrak{b} \ge 1$. Let $g : [0, 1] \to [1, +\infty]$ be a continuous monotone non-increasing function. Set*

$$F(t) = \int_t^1 \frac{ds}{g(s)}, \quad 0 \le t \le 1. \tag{3.14}$$

If $u \ge 0$ satisfies (3.13) *then, for all $x \in \Omega$ such that $u(x) > 0$, the following inequalities hold:*

$$u(x) \le 1 - \mathfrak{b}\left[1 - F^{-1}\Big(\mathfrak{b}^{-1}G1(x)\Big)\right], \tag{3.15}$$

and

$$\mathfrak{b}^{-1}G1(x) < F(1-\mathfrak{b}^{-1}) = \int_{1-\mathfrak{b}^{-1}}^{1} \frac{ds}{g(s)}, \tag{3.16}$$

so that the right-hand side of (3.15) *is well-defined and positive.*

Proof. For any $t \in [0, 1]$ set

$$\phi(t) = g(1-t) \quad \text{and} \quad \psi(t) = \phi(\mathfrak{b}^{-1}t) = g(1-\mathfrak{b}^{-1}t), \tag{3.17}$$

so that ϕ and ψ are non-decreasing. As in the proof of Theorem 3.2, define the sequence $\{f_k\}_{k=0}^{\infty}$ of functions on Ω by (2.12),

$$f_0 = G1, \quad f_{k+1} = G\left(\phi\left(f_k\right)\right).$$

We claim that $0 \le f_k \le 1$ for all $k \ge 0$, so that $\phi\left(f_k\right)$ is well-defined, and moreover

$$u \le 1 - f_k \text{ in } \Omega. \tag{3.18}$$

Indeed, obviously $u \le 1$, and hence $g(u) \ge g(1) \ge 1$. Consequently by (3.13),

$$u \le 1 - G\left(g\left(1\right)\right) \le 1 - G1 = 1 - f_0,$$

so that $f_0 \le 1$, and (3.18) holds for $k = 0$. If (3.18) has been proved for some $k \ge 0$, then substituting (3.18) into (3.13) yields

$$u \le 1 - G\left(g\left(1 - f_k\right)\right) = 1 - G\left(\phi\left(f_k\right)\right) = 1 - f_{k+1},$$

which proves (3.18). In particular, $0 \le f_k \le 1$ for all $k \ge 0$.

Set $t_0 := G1(x) = f_0(x) \in [0, 1]$. Consider now the sequence $\{\psi_k\}_{k=0}^{\infty}$ of functions on $[0, t_0]$ defined by (2.14),

$$\psi_0\left(t\right) = t, \quad \psi_{k+1}(t) = \int_0^t \psi \circ \psi_k(s)ds.$$

We need to show that the functions $\psi_k(t)$ are well-defined, that is, $\psi_k(t) \in [0, 1]$ for all $t \in [0, t_0]$ and $k \ge 1$, and

$$f_k(x) \ge \psi_k(f_0(x)). \tag{3.19}$$

By (3.18), we deduce

$$G(g(u)) \ge G(g(1 - f_k)) = G(\phi(f_k)) = f_{k+1}. \tag{3.20}$$

Clearly, ψ_1 is well-defined. Hence, by Lemma 2.7 and the preceding inequality with $k = 0$,

$$1 \ge 1 - u(x) \ge G(g(u))(x) \ge f_1(x) \ge \psi_1(f_0(x)) = \psi_1(t_0).$$

Since ψ_1 is non-decreasing, we see that $\psi_1(t) \in [0, 1]$ for all $t \in [0, t_0]$.

If the inequalities $\psi_k(t_0) \le 1$ and (3.19) have been proved for some $k \ge 1$, then $\psi_{k+1}(t)$ is well-defined on $[0, t_0]$. By Lemma 2.7 and (3.20),

$$1 \ge 1 - u(x) \ge G(g(u))(x) \ge f_{k+1}(x) \ge \psi_{k+1}(f_0(x)) = \psi_{k+1}(t_0).$$

Since $\psi_{k+1}(t)$ is a non-decreasing function, it follows that $\psi_{k+1}(t) \le 1$ for all $t \in [0, t_0]$, and (3.19) holds for all $k \ge 1$. Consequently, for all $k \ge 0$,

$$1 \ge u(x) + f_k(x) \ge u(x) + \psi_k(f_0(x)). \tag{3.21}$$

The rest of the proof is similar to that of Theorem 3.2. Passing to the limit as $k \to \infty$ in (3.21) yields

$$1 \ge u(x) + \psi_\infty(f_0(x)),$$

where $\psi_\infty(t)$ is the unique solution of the integral equation (3.9) in the interval $0 \le t \le t_0$. From this we deduce that

$$\Psi\big(\psi_\infty(t)\big) = t, \quad 0 \le t \le t_0,$$

where, for $\xi \in [0, 1]$,

$$\Psi(\xi) = \int_0^\xi \frac{ds}{\psi(t)} = \mathfrak{b}\, F\big(1 - \mathfrak{b}^{-1}\xi\big).$$

Here F is defined by (3.14). Hence,

$$\psi_\infty(t) = \Psi^{-1}(t) = \mathfrak{b}\left(1 - F^{-1}(\mathfrak{b}^{-1}t)\right), \quad 0 \le t \le t_0, \quad t_0 \in \text{Range}(\Psi).$$

Thus,

$$G1(x) = t_0 \le \int_0^1 \frac{dt}{\psi(t)} = \mathfrak{b} \int_{1-\mathfrak{b}^{-1}}^1 \frac{dt}{g(t)},$$

so that (3.15) holds, where its right-hand side is well defined. Clearly, (3.15) implies (3.16) provided $u(x) > 0$. □

We now consider some special cases of the integral inequalities (3.1) and (3.13). Let $g(t) = t^q$ $(q > 0)$ in the case of (3.1), that is,

$$u \ge G(u^q) + 1 \quad \text{in } \Omega. \tag{3.22}$$

Corollary 3.4. *Let $q > 0$. Under the assumptions of Theorem 3.2, suppose that u satisfies (3.22).*

If $q \ne 1$, then

$$u(x) \ge 1 + \mathfrak{b}\left[\Big(1 + (1-q)\mathfrak{b}^{-1} G1(x)\Big)^{\frac{1}{1-q}} - 1\right], \tag{3.23}$$

where in the case $q > 1$, for all $x \in \Omega$ such that $u(x) < +\infty$,

$$\mathfrak{b}^{-1} G1(x) < \frac{1}{q-1}. \tag{3.24}$$

In the case $q = 1$,

$$u(x) \geq 1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1} G1(x)} - 1\right), \quad x \in \Omega. \tag{3.25}$$

Proof. We apply Theorem 3.2 with $g(t) = t^q$, $t \geq 1$. If $q > 0$ ($q \neq 1$), we have

$$F(t) = \int_1^t s^{-q} ds = \frac{1}{1-q}\left(t^{1-q} - 1\right), \quad t \geq 1,$$

and consequently

$$F^{-1}(\tau) = [1 + (1-q)\,\tau]^{\frac{1}{1-q}}, \ \ 0 \leq \tau \leq \frac{1}{q-1} \text{ if } q > 1; \ \ \tau \geq 0 \text{ if } 0 < q < 1.$$

Therefore, for all $x \in \Omega$ such that $u(x) < +\infty$, we deduce (3.23), where $G1(x) < \frac{\mathfrak{b}}{1-q}$, so that the right-hand side of (3.23) is well defined, that is, (3.24) holds.

In the case $q = 1$, we have $F(t) = \log t$ for $t \geq 1$, and $F^{-1}(\tau) = e^{\tau}$ for $\tau \geq 0$, which gives (3.25). □

We now consider inequalities of the type (3.13) with $g(t) = t^q$ for $q < 0$, that is,

$$u + G(u^q) \leq 1 \quad \text{in } \Omega. \tag{3.26}$$

Corollary 3.5. *Let $q < 0$. Under the assumptions of Theorem 3.3, suppose that $u \geq 0$ satisfies (3.26). Then, for all $x \in \Omega$ such that $u(x) > 0$,*

$$G1(x) < \frac{\mathfrak{b}}{1-q}\left[1 - \left(1 - \mathfrak{b}^{-1}\right)^{1-q}\right], \tag{3.27}$$

and

$$u(x) \leq 1 - \mathfrak{b}\left[1 - \left(1 - (1-q)\mathfrak{b}^{-1} G1(x)\right)^{\frac{1}{1-q}}\right]. \tag{3.28}$$

Proof. We apply Theorem 3.3 with $g(t) = t^q$, $q < 0$, where $t \in (0, 1]$. In this case,

$$F(t) = \int_t^1 s^{-q} ds = \frac{1}{1-q}\left(1 - t^{1-q}\right), \quad 0 \leq t \leq 1.$$

Then

$$F^{-1}(\tau) = [1 - (1-q)\,\tau]^{\frac{1}{1-q}}, \quad 0 \leq \tau \leq \frac{1}{1-q}.$$

Therefore, for all $x \in \Omega$ such that $u(x) > 0$, we deduce (3.28), where $G1(x) \leq \frac{\mathfrak{b}}{1-q}$ so that the right-hand side of (3.28) is well defined. Moreover, (3.27) holds since $u(x) > 0$ in (3.28). □

In the following corollary we give pointwise estimates for super-solutions to *homogeneous* equations in the sublinear case.

Corollary 3.6. *Let G be a σ-finite kernel on Ω satisfying the weak maximum principle* (2.2) *with $\mathfrak{b} \geq 1$. Let $0 < q < 1$. If $u > 0$ in Ω satisfies*

$$u \geq G(u^q) \quad \text{in } \Omega, \tag{3.29}$$

then

$$u(x) \geq (1-q)^{\frac{1}{1-q}} \mathfrak{b}^{-\frac{q}{1-q}} \left(G1(x)\right)^{\frac{1}{1-q}}, \quad x \in \Omega. \tag{3.30}$$

Remark 3.7. The constant $(1-q)^{\frac{1}{1-q}}$ in (3.30) coincides with that in [13, Theorem 3.3], if $\mathfrak{b} = 1$, and is sharp.

Proof. For $a > 0$, set

$$E_a = \{y \in \Omega\colon\ u(y) \geq a\}.$$

Then

$$u \geq G(u^q) \geq a^q \left(G(1_{E_a} u^q)\right) \quad \text{in } E_a.$$

Iterating this inequality, we obtain

$$u \geq a^{q^{k+1}} G f_k \quad \text{in } E_a,$$

where f_k is defined by (2.12) with $\phi(t) = t^q$. Hence, by Corollary 2.8,

$$u \geq c(q,k)^{-1}\, a^{q^{k+1}} \mathfrak{b}^{-q(1+q+\cdots+q^{k-1})} \left(G1_{E_a}(x)\right)^{1+q+\cdots+q^k} \quad \text{in } E_a.$$

Notice that

$$\begin{aligned} c(q,k) &= \prod_{j=1}^{k} \left(1+q+\cdots+q^j\right)^{q^{k-j}} \\ &< \prod_{j=1}^{k} (1-q)^{-q^{k-j}} \\ &< (1-q)^{-(1-q)^{-1}}. \end{aligned}$$

It follows

$$u \geq (1-q)^{(1-q)^{-1}} a^{q^{k+1}} \mathfrak{b}^{-q(1+q+\cdots+q^{k-1})} \left(G1_{E_a}(x)\right)^{1+q+\cdots+q^k} \quad \text{in } E_a.$$

Letting $k \to +\infty$, we obtain

$$u \geq (1-q)^{\frac{1}{1-q}} \mathfrak{b}^{-\frac{q}{1-q}} (G1_{E_a})^{\frac{1}{1-q}} \quad \text{in } E_a.$$

Finally, letting $a \to 0^+$ yields (3.30) in Ω. □

4. Quasi-metric kernels

Consider the setting of Example 2.3, that is, Ω is a locally compact Hausdorff space with countable base, ω is a Radon measure on Ω. Let $K : \Omega \times \Omega \to (0, +\infty]$ be a lower semi-continuous function. Assume that for any $x \in \Omega$, $K(x, \cdot) \in L^1_{\mathrm{loc}}(\Omega, \omega)$. Set

$$Gf(x) = \int_\Omega K(x, y) f(y) d\omega(y).$$

The kernel K is called *quasi-metric* (see [8, 14, 16]) if K is symmetric, that is, $K(x, y) = K(y, x)$ and $d(x, y) = \frac{1}{K(x,y)}$ is quasi-metric, that is, there exists a *quasi-metric constant* $\varkappa > 0$ such that the quasi-triangle inequality holds:

$$d(x, y) \leq \varkappa [d(x, z) + d(y, z)], \quad \forall x, y, z \in \Omega.$$

Without loss of generality we may assume that $d(x, y) \neq 0$ for some $x, y \in \Omega$, so that $\varkappa \geq \frac{1}{2}$.

The following lemma is proved in [22, Lemma 3.5].

Lemma 4.1. *Suppose G is a quasi-metric kernel in Ω with quasi-metric constant $\varkappa$. Then G satisfies the weak maximum principle* (1.8) *with constant $\mathfrak{b} = 2\varkappa$.*

In the next lemma we consider a certain modification of a quasi-metric kernel.

Lemma 4.2. *Suppose K is a quasi-metric kernel in Ω with constant $\varkappa$. For $w \in \Omega$, let $\Omega_w = \{x \in \Omega : \ K(x, w) < +\infty\}$. Then*

$$K_w(x, y) = \frac{K(x, y)}{K(x, w)\, K(y, w)}, \quad x, y \in \Omega_w, \tag{4.1}$$

is a quasi-metric kernel on Ω_w with quasi-metric constant $4\varkappa^2$.

In particular, K_w satisfies the weak maximum principle (1.8) *in Ω_w with constant $\mathfrak{b} = 8\varkappa^3$.*

Proof. This is immediate from the so-called Ptolemy inequality for quasi-metric spaces, [8, Lemma 2.2] (see also [15, Proposition 8.1]),

$$d(x, y)\, d(z, w) \leq 4\varkappa^2 [d(x, w)\, d(y, z) + d(x, z)\, d(y, w)], \ \forall x, y, z, w \in \Omega. \tag{4.2}$$

Dividing both sides of the preceding inequality by $d(x, w)\, d(y, w)\, d(z, w)$, we deduce

$$\frac{d(x, y)}{d(x, w)\, d(y, w)} \leq 4\varkappa^2 \left[\frac{d(x, z)}{d(x, w)\, d(z, w)} + \frac{d(y, z)}{d(y, w)\, d(z, w)} \right],$$

for all $x, y, z \in \Omega_w$. $\square$

Let $h : \Omega \to (0, +\infty)$ be a lower semicontinuous function on Ω. For a general kernel $K : \Omega \times \Omega \to (0, +\infty]$, consider a modified kernel

$$K^h(x, y) = \frac{K(x, y)}{h(x)\, h(y)} \quad \text{for } x, y \in \Omega. \tag{4.3}$$

Here we discuss the question how to verify the weak maximum principle for K^h.

Remark 4.3. As we will demonstrate below (see Lemma 4.5), K^h satisfies the weak maximum principle (1.8) provided K satisfies the following form of the *weak domination principle*:

Given a positive lower semicontinuous function h in Ω,

$$K\mu(x) \le M\, h(x) \quad \forall x \in \text{supp}(\mu) \implies K\mu(x) \le \mathfrak{b}\, M\, h(x) \quad \forall x \in \Omega, \tag{4.4}$$

for any compactly supported Radon measure μ *with finite energy in* Ω, i.e., $\int_\Omega K\mu\, d\mu < +\infty$, *and any constant* $M > 0$.

This property is sometimes called a *dilated* domination principle (see, *e.g.*, [17]). In the case where (4.4) holds with $\mathfrak{b} = 1$ for any $h = K\nu + a$, where ν is a Radon measure and $a \ge 0$ is a constant, it is called the *complete* maximum principle (see *e.g.*, [2,14]).

The weak domination principle holds for Green's kernels associated with a large class of local and non-local operators, and super-harmonic h.

Remark 4.4. It is easy to see that, for a quasi-metric kernel K, the modified kernel K^h with $h = K\nu > 0$, where ν is a Radon measure, is generally not quasi-metric. However, it does satisfy the weak maximum principle (1.8) under some mild assumptions. See Lemma 4.6 below.

The modified kernel K^h in this case is essentially quasi-metric if ν is a measure supported at a single point $w \in \Omega$, *i.e.*, when $h(x) = c\, K(x, w)$, $c > 0$, by Lemma 4.2.

Let us denote by $M^+(\Omega)$ the class of Radon measures in Ω.

Lemma 4.5. *Suppose K is a non-negative lower semicontinuous kernel in* Ω *which satisfies the domination principle* (4.4). *Suppose h is a positive lower semicontinuous function in* Ω. *Then the modified kernel* K^h *defined by* (4.3) *satisfies the weak maximum principle* (1.8) *in* $\Omega' = \Omega \setminus \{x : h(x) < +\infty\}$ *with the same constant* $\mathfrak{b}$.

In particular, if (4.4) *holds for K and h with* $\mathfrak{b} = 1$, *then* K^h *satisfies the strong maximum principle in* Ω'.

Proof. For $\nu \in M^+(\Omega')$ with compact support, let $d\tilde{\nu} = \frac{1}{h(x)} d\nu$. Notice that h is bounded below by a positive constant on any compact set so that $\tilde{\nu} \in M^+(\Omega)$, and $\text{supp}(\tilde{\nu}) \subseteq \text{supp}(\nu)$.

Let

$$\Omega_m := \{x \in \Omega \colon h(x) \le m\}, \quad m = 1, 2, \dots \tag{4.5}$$

Clearly, each Ω_m is a closed set, and $\Omega' = \bigcup_{m=1}^{+\infty} \Omega_m$. Let $d\nu_m = \chi_{\Omega_m} d\nu$, so that $\text{supp}(\nu_m) \subseteq \Omega_m$.

Suppose that, for a positive constant M,

$$K^h\nu(x) \le M \text{ for all } x \in \mathrm{supp}(\nu).$$

Then obviously

$$K^h\nu_m(x) \le M \text{ for all } x \in \mathrm{supp}(\nu_m).$$

It follows that

$$K\tilde{\nu}_m(x) \le M\,h(x) \text{ for all } x \in \mathrm{supp}(\tilde{\nu}_m), \quad \text{where} \quad d\tilde{\nu}_m(x) = \frac{1}{h\,(x)}d\nu_m.$$

Clearly, $\tilde{\nu}_m$ has finite energy with respect to K, since

$$\int_\Omega K\tilde{\nu}_m\,d\tilde{\nu}_m \le M\int_\Omega hd\tilde{\nu}_m = M\int_\Omega d\nu_m < +\infty.$$

Hence, by (4.4)

$$K\tilde{\nu}_m(x) \le \mathfrak{b}\,M\,h(x) \quad \forall x \in \Omega.$$

Consequently, by the monotone convergence theorem

$$K\tilde{\nu}(x) \le \mathfrak{b}\,M\,h(x) \quad \forall x \in \Omega'.$$

Equivalently,

$$K^h\nu(x) \le \mathfrak{b}\,M \quad \forall x \in \Omega'.$$

Thus, K^h satisfies the weak maximum principle (respectively, the strong maximum principle if $\mathfrak{b} = 1$) in Ω'. □

Lemma 4.6. *Let K be a quasi-metric kernel on $\Omega \times \Omega$, continuous in the extended sense. Let $h = K\nu$ where $\nu \in M^+(\Omega)$, $h \not\equiv +\infty$. Then the modified kernel K^h defined by* (4.3) *satisfies the weak maximum principle in $\Omega' = \{x \in \Omega : \ h(x) < +\infty\}$.*

Proof. If $\nu = \delta_w$ for some $w \in \Omega$ then by Lemma 4.2 the modified kernel K_w given by (4.1) is a quasi-metric kernel on $\Omega_w = \{x \in \Omega\colon \ K(x,w) < +\infty\}$. By Lemma 4.1, K_w satisfies the weak maximum principle with constant $\mathfrak{b} = 8\kappa^3$ in $\Omega' = \Omega_w$.

To show that, for general $h = K\nu$, the modified kernel K^h defined by (4.3) satisfies the weak maximum principle in Ω', we invoke the idea used in [20, Theorem 7], (see also [17,18]) which reduces it to the *elementary domination principle* in the case $\nu = \delta_w$.

Suppose first that $\mu \in M^+(\Omega)$ is a measure with compact support and of finite energy, and $h = K\nu$. Let us show that K satisfies (4.4) for $\mu \in M^+(\Omega)$. To this end, we argue by contradiction. Assume that

$$K\mu \le K\nu \quad \text{on } F = \mathrm{supp}\,(\mu), \tag{4.6}$$

but there exists $w \in \Omega \setminus F$ such that

$$K\mu(w) > \mathfrak{b}\, K\nu(w), \tag{4.7}$$

where without loss of generality we may let $K\nu(w) < +\infty$.

Notice that quasi-metric kernels are symmetric, and strictly positive. Hence, $\mathrm{cap}(F) < +\infty$ for any compact set $F \subset \Omega$ (see [9]), and there exists an extremal measure $\mu_F \in M^+(\Omega)$ of finite energy, with $\mathrm{supp}\,(\mu_F) \subseteq F$, such that by [20, Lemma 1*], (see also [17,18]),

$$K\mu_F(z) \le K(z, w), \quad \forall z \in \mathrm{supp}\,(\mu_F), \tag{4.8}$$

and

$$K\mu_F(z) \ge K(z, w) \quad \text{on } F. \tag{4.9}$$

Since K_w is a quasi-metric kernel in Ω_w, it satisfies the weak maximum principle with constant $\mathfrak{b} = 8\varkappa^3$, and consequently G satisfies the domination principle (4.4) with $\nu = \delta_w$ and the same constant $\mathfrak{b}$ in Ω_w. In fact, the domination principle for μ_F and $\nu = \delta_w$ holds in Ω, *i.e.*,

$$K\mu_F(x) \le \mathfrak{b}\, K(x, w), \quad \forall x \in \Omega, \tag{4.10}$$

where $\mathfrak{b} = 8\kappa^3$, since the right-hand side of (4.10) is infinite on $\Omega \setminus \Omega_w$, and for all measures of finite energy $\mu(\Omega \setminus \Omega_w) = 0$. Indeed, by the quasi-triangle inequality $K(x, y) = \frac{1}{d(x,y)} = +\infty$ if $x, y \in \Omega \setminus \Omega_w$, and so $K\mu = +\infty$ on $\Omega \setminus \Omega_w$, unless $\mu(\Omega \setminus \Omega_w) = 0$.

We denote by $\mathcal{E}(\mu, \nu)$ the mutual energy of the measures $\mu, \nu \in M^+(\Omega)$:

$$\mathcal{E}(\mu, \nu) := \int_\Omega K\nu \, d\mu = \int_\Omega K\mu \, d\nu. \tag{4.11}$$

Let us estimate the mutual energy $\mathcal{E}(\mu_F, \nu)$. Integrating both sides of (4.10) against $d\nu$ we deduce

$$\begin{aligned} \mathcal{E}(\mu_F, \nu) &= \int_\Omega K\mu_F \, d\nu \\ &\le \mathfrak{b} \int_\Omega K(x, w)\, d\nu(x) = \mathfrak{b}\, G\nu(w). \end{aligned}$$

On the other hand, it follows from (4.9) and (4.7) that

$$\begin{aligned} \mathcal{E}(\mu_F, \mu) &= \int_F K\mu_F \, d\mu \\ &\ge \int_F K(x, w)\, d\mu(x) \\ &= K\mu(w) > \mathfrak{b}\, K\nu(w). \end{aligned}$$

Since $\mathcal{E}(\mu_F, \nu) \ge \mathcal{E}(\mu_F, \mu)$ by (4.6), we arrive at a contradiction.

Suppose now that $\mu \in M^+(\Omega)$ has compact support $F \subset \Omega'$, and $h = K\nu$. Then for $\Omega_m \subset \Omega'$ defined by (4.5) and $d\mu_m = \chi_{\Omega_m} d\mu$ we have

$$K\mu_m \le h \le m \quad \text{in } \Omega_m.$$

Consequently, $\mu_m \in M^+(\Omega)$ has finite energy, $\operatorname{supp}(\mu_m) \subset F \cap \Omega_m$ is a compact set, and by the previous case

$$K\mu_m(x) \le \mathfrak{b}\, h(x), \quad x \in \Omega',$$

for m large enough. Passing to the limit as $m \to +\infty$ we obtain by the monotone convergence theorem

$$K\mu(x) \le \mathfrak{b}\, h(x), \quad x \in \Omega'. \qquad \square$$

5. The weak domination principle and nonlinear integral inequalities

In the setting of Example 2.2, let Ω be a locally compact Hausdorff space with countable base, and let $G(x, dy)$ be a Radon kernel in Ω. Let $h : \Omega \to (0, +\infty)$ be a given positive lower semi-continuous function in Ω. In particular, $\inf_F h > 0$ for every compact set $F \subset \Omega$.

In this section we consider super-solutions $u : \Omega \to [0, +\infty)$ of

$$u(x) \ge G(u^q)(x) + h(x) \text{ in } \Omega, \tag{5.1}$$

in the case $q > 0$, and sub-solutions $u : \Omega \to (0, +\infty)$ of

$$u(x) \le -G(u^q)(x) + h(x) \text{ in } \Omega, \tag{5.2}$$

in the case $q < 0$.

We will assume that G satisfies the *weak domination principle* in the following form:

For any bounded measurable function $f \ge 0$ with compact support,

$$Gf(x) \le h(x) \text{ in } \operatorname{supp}(f) \implies Gf(x) \le \mathfrak{b}\, h(x) \text{ in } \Omega, \tag{5.3}$$

provided Gf is bounded on $\operatorname{supp}(f)$.

Our main result in this setup is as follows.

Theorem 5.1. *In the above setting, for a given function h, assume that G satisfies the weak domination principle* (5.3) *in Ω. Suppose that $u \ge 0$ satisfies* (5.1) *if $q > 0$, or $u > 0$ and satisfies* (5.2) *if $q < 0$. Then $u(x)$ satisfies the following estimates for all $x \in \Omega$:*

(i) *If* $q > 0, q \neq 1$, *then*

$$u(x) \geq h(x) \left\{ 1 + \mathfrak{b} \left[\left(1 + \frac{(1-q)\, G(h^q)(x)}{\mathfrak{b}\, h(x)} \right)^{\frac{1}{1-q}} - 1 \right] \right\}, \tag{5.4}$$

where in the case $q > 1$ *necessarily*

$$G(h^q)(x) < \frac{\mathfrak{b}}{q-1}\, h(x). \tag{5.5}$$

(ii) *If* $q = 1$, *then*

$$u(x) \geq h(x) \left[1 + \mathfrak{b} \left(e^{\mathfrak{b}^{-1} \frac{G(h)(x)}{h(x)}} - 1 \right) \right]. \tag{5.6}$$

(iii) *If* $q < 0$, *then*

$$u(x) \leq h(x) \left\{ 1 - \mathfrak{b} \left[1 - \left(1 - \frac{(1-q)\, G(h^q)(x)}{\mathfrak{b}\, h(x)} \right)^{\frac{1}{1-q}} \right] \right\} \tag{5.7}$$

and necessarily

$$G(h^q)(x) < \frac{\mathfrak{b}}{1-q} \left[1 - \left(1 - \mathfrak{b}^{-1} \right)^{1-q} \right] h(x). \tag{5.8}$$

Proof. Suppose first that $q > 0$. Let us consider a modified kernel

$$G^h(x, dy) = \frac{h(y)^q}{h(x)} G(x, dy).$$

Clearly, G^h is also a Radon kernel on any subset

$$\Omega_m = \{ x \in \Omega : \; h(x) \leq m \}, \quad m \geq 1. \tag{5.9}$$

Notice that each Ω_m is closed, $\Omega_m \subseteq \Omega_{m+1}$, and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$.

Setting $G_m^h = G^h(x, 1_{\Omega_m} dy)$ and

$$v(x) := \frac{u(x)}{h(x)}, \quad x \in \Omega, \tag{5.10}$$

we see that v satisfies the inequality

$$v(x) \geq G_m^h(v^q)(x) + 1 \; \text{ for all } x \in \Omega. \tag{5.11}$$

Moreover, G^h_m satisfies the weak maximum principle (2.4) in Ω with the same constant $\mathfrak{b}$, that is, for any bounded measurable function f with compact support in Ω,

$$\frac{1}{h}\,G(1_{\Omega_m} h^q f) \le 1 \text{ in } \operatorname{supp}(f) \implies \frac{1}{h}\,G(1_{\Omega_m} h^q f) \le \mathfrak{b} \text{ in } \Omega. \tag{5.12}$$

Indeed, this follows from the weak domination principle (5.3) applied to $1_{\Omega_m} h^q f$ in place of f, which yields

$$G(1_{\Omega_m} h^q f) \le h \text{ in } \operatorname{supp}(f) \cap \Omega_m \implies G(1_{\Omega_m} h^q f) \le \mathfrak{b}\, h \text{ in } \Omega. \tag{5.13}$$

This proves (5.12).

Hence, by Corollary 3.4 with G^h_m in place of G, it follows from (5.11) that v satisfies the following estimates for all $x \in \Omega$:

$$v(x) \ge 1 + \mathfrak{b}\left[\left(1 + \mathfrak{b}^{-1}(1-q)\, G^h_m(1)(x)\right)^{\frac{1}{1-q}} - 1\right], \tag{5.14}$$

if $q > 0, q \ne 1$, where in the case $q > 1$, we have

$$G^h_m(1)(x) < \frac{\mathfrak{b}}{q-1}. \tag{5.15}$$

If $q = 1$, then

$$v(x) \ge 1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1} G^h_m(1)(x)} - 1\right). \tag{5.16}$$

Passing to the limit as $m \to \infty$ we deduce by the monotone convergence theorem that, for $q > 0, q \ne 1$,

$$v(x) \ge 1 + \mathfrak{b}\left[\left(1 + \mathfrak{b}^{-1}(1-q)\, G^h(1)(x)\right)^{\frac{1}{1-q}} - 1\right],$$

where the strict inequality holds for $q > 1$ in

$$G^h(1)(x) < \frac{\mathfrak{b}}{q-1},$$

since in the preceding estimate we have $v(x) = u(x)\, h(x) < +\infty$. If $q = 1$, then

$$v(x) \ge 1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1} G^h(1)(x)} - 1\right).$$

Going back from v, G^h to u, G in these estimates yields that (5.4) or (5.6) hold at every $x \in \Omega$, and in the case $q > 1$ the necessary condition (5.5) holds.

In the case $q < 0$, estimates (5.7) and (5.8) are deduced in a similar way from Corollary 3.5, provided $u(x) > 0$. □

Remark 5.2. The results of Section 4 show that, in that setup, the estimates of Theorem 5.1 hold for quasi-metric kernels K and $h = K\nu$ in $\Omega' = \{x \in \Omega : h(x) < +\infty\}$, for all Radon measures ν in Ω such that $K\nu \not\equiv +\infty$.

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