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# Blow-up analysis for nodal radial solutions in Moser-Trudinger critical equations in $\mathbb{R}^2$

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**Abstract.** In this paper we consider sign-changing radial solutions $u_\varepsilon$ to the problem

$$\begin{cases} -\Delta u = \lambda u e^{u^2+|u|^{1+\varepsilon}} & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

and we study their asymptotic behaviour as $\varepsilon \searrow 0$.

We show that when $u_\varepsilon = u_\varepsilon(r)$ has $k$ interior zeros, it exhibits a multiple blow-up behaviour in the first $k$ nodal sets while it converges to the least energy solution of the problem with $\varepsilon = 0$ in the $(k+1)$-th one. We also prove that in each concentration set, with an appropriate scaling, $u_\varepsilon$ converges to the solution of the classical Liouville problem in $\mathbb{R}^2$.

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## 1. Introduction

The classical Moser-Trudinger inequality [23,26,29]

$$\sup_{\int_\Omega |\nabla u|^2 \le 1} \int_\Omega e^{4\pi u^2} \le C|\Omega|\,, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $C$ is a positive constant and $u$ belongs to the Sobolev space $W_0^{1,2}(\Omega)$ has been the subject of much research in recent years.

In the pioneering paper [11] it was proved that the supremum in (1.1) is achieved at a positive function $u \in W_0^{1,2}(\Omega)$ and the corresponding Euler-Lagrange equation

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satisfied by $u$ is given by

$$\begin{cases} -\Delta u = \dfrac{ue^{u^2}}{\int_\Omega u^2 e^{u^2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This result was the starting point for many problems involving the Moser-Trudinger inequality. From now on we will focus our interest when $\Omega$ is the unit ball $B$, although some of the results hold for more general domains. Extensions of (1.2) to a more general setting like

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.3}$$

where $\lambda$ is a positive parameter and $u$ is a positive solution inspired several people. In [25] there is an interesting discussion on relationship between the maximizer to (1.1) and solutions of (1.3).

Note that by Gidas, Ni, Nirenberg's Theorem [16] we have that all solutions to (1.3) are radial.

In [1] it was proved the existence of solutions $u_\lambda$ to (1.3) for any $\lambda \in (0, \lambda_1)$ where $\lambda_1$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions (see also [2,15]). The behavior of the solution $u_\lambda$ as $\lambda \to 0$ is very interesting because a *concentration phenomenon* appears. This means that $||u_\lambda||_\infty = u_\lambda(0) \to +\infty$ and $u_\lambda(x) \to 0$ for any $x \neq 0$ (see [3,25]).

These kind of results hold also for more general problems like

$$\begin{cases} -\Delta u = \lambda f(u) e^{u^2} & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.4}$$

We refer to [1,2,13–15] for the precise assumptions on $f$ and the statements of the results.

Next let us consider sign changing *radial* solutions. Here we find some interesting differences towards the case of positive solutions. Indeed, in [5] the authors showed that it is not possible to have sign changing *radial* solutions to (1.3) for any $\lambda \in (0, \lambda_1)$. Actually, in order to have analogous existence results we need to add some perturbation terms in (1.3). A particular case is the following:

**Theorem 1.1.** (*See* [5] *and* [6]) *Let us consider the problem*

$$\begin{cases} -\Delta u = \lambda u e^{u^2 + |u|^\beta} & \textit{in } B \\ u = 0 & \textit{on } \partial B. \end{cases} \tag{1.5}$$

*Then we have that*

i) *if* $1 < \beta < 2$ *there exists a radial solution with k interior nodal zones for any integer* $k \geq 1$ *and for any* $\lambda \in (0, \lambda_1)$*;*
ii) *if* $0 \leq \beta \leq 1$ *there exists* $\lambda_{AY} = \lambda_{AY}(\beta) > 0$ *such that for any* $0 < \lambda < \lambda_{AY}$ *there exist no sign-changing solution to* (1.5).

From this results we get that the nonlinearity $g(s) = se^{s^2+|s|}$ is the *threshold* which separates the existence and nonexistence of nodal solutions as $\lambda$ is small. Hence it becomes interesting to study the asymptotic behavior of the solution $u$ in (1.5) as $\beta = 1 + \varepsilon, 0 < \lambda < \lambda_{AY}$ and $\varepsilon \searrow 0$.

In order to state our main result we need to introduce some notations. First let us denote by $u_0$ the solution of

$$\begin{cases} -\Delta u = \lambda u e^{u^2+u} & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.6}$$

Next, for $u \in H_0^1(B)$ and $\varepsilon \geq 0$ let us consider the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_B |\nabla u|^2 - \int_B F_\varepsilon(u)\,, \tag{1.7}$$

where $F_\varepsilon(s) = \lambda \int_0^s te^{t^2+|t|^{1+\varepsilon}}dt$. We have two main theorems:

- A *global* result where we describe the behavior of the solution in $B \setminus \{0\}$;
- A *local* one where we characterize the behavior of the solution in a ball and annuli shrinking near the origin.

Finally let us denote by $S(r) = \left\{x \in \mathbb{R}^2 : |x| = r\right\}$.

**Theorem 1.2 (Global behavior).** *We choose* $\lambda_{AY}$ *for* $\beta = 1$ *from Theorem* 1.1 *and assume* $0 < \lambda < \min\{\lambda_{AY}, \lambda_1\}$. *Let* $u_\varepsilon$ *be a nodal radial solution given by* [6, *Theorem* 1.3] *which verifies*

$$\begin{cases} -\Delta u = \lambda u e^{u^2+|u|^{1+\varepsilon}} & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.8}$$

*with k interior nodal zones,* i.e. *there are k values* $0 < r_{1,\varepsilon} < r_{2,\varepsilon} < \cdots < r_{k,\varepsilon} < 1$ *such that* $u_{\varepsilon|S(r_{i,\varepsilon})} = 0$ *for* $i = 1, \cdots, k$. *Moreover set* $r_{0,\varepsilon} = 0$ *and* $r_{k+1,\varepsilon} = 1$ *and assume that* $u_\varepsilon(0) > 0$ *and* $(-1)^{i-1}u(x) > 0$ *for any* $|x| \in (r_{i-1,\varepsilon}, r_{i,\varepsilon})$ *and* $i = 1, \cdots, k+1$. *Then we have that, as* $\varepsilon \to 0$,

$$u_\varepsilon(x) \to (-1)^k u_0(x) \quad \text{in } C^2_{\mathrm{loc}}(B \setminus \{0\}) \tag{1.9}$$

$$r_{i,\varepsilon} \to 0 \quad \text{for any } i = 1, \ldots, k, \tag{1.10}$$

$$\int_B |\nabla u_\varepsilon|^2 \to \int_B |\nabla u_0|^2 + 4k\pi, \tag{1.11}$$

$$I_\varepsilon(u_\varepsilon) \to I_0(u_0) + 2k\pi. \tag{1.12}$$

**Theorem 1.3 (Local behavior).** *For $r \in [0,1]$ let $u_\varepsilon = u_\varepsilon(r)$ be the solution considered in the previous theorem. Then for $i = 1, \dots, k$ let $A_{i,\varepsilon} = (r_{i-1,\varepsilon}, r_{i,\varepsilon})$, $u_{i,\varepsilon} = u_{\varepsilon|A_{i,\varepsilon}}$ and $M_{i,\varepsilon} \in [0,1)$ be such that $u_{i,\varepsilon}(M_{i,\varepsilon} r_{i,\varepsilon}) = ||u_{i,\varepsilon}||_{L^\infty(A_{i,\varepsilon})}$ (we have that $M_{1,\varepsilon} = 0$). Then if $\delta_{i,\varepsilon}$ is defined as $\delta_{i,\varepsilon} = r_{i,\varepsilon}\gamma_{i,\varepsilon}$ with*

$$2\lambda r_{i,\varepsilon}^2 e^{||u_{i,\varepsilon}||^2_{L^\infty(A_{i,\varepsilon})} + ||u_{i,\varepsilon}||^{1+\varepsilon}_{L^\infty(A_{i,\varepsilon})}} ||u_{i,\varepsilon}||^2_{L^\infty(A_{i,\varepsilon})} \gamma_{i,\varepsilon}^2 = 1 \tag{1.13}$$

*we have that $\delta_{i,\varepsilon} \to 0$ and*

$$\begin{aligned} &2||u_{i,\varepsilon}||_{L^\infty(A_{i,\varepsilon})} \left( \left| u_{i,\varepsilon}\left(M_{i,\varepsilon} r_{i,\varepsilon} + \delta_{i,\varepsilon} r\right)\right| - ||u_{i,\varepsilon}||_{L^\infty(A_{i,\varepsilon})}\right) \\ &\to \log \frac{1}{\left(1 + \frac{r^2}{8}\right)^2} \quad \text{in } C^1_{\mathrm{loc}}(0, +\infty). \end{aligned} \tag{1.14}$$

**Remark 1.4.** Another interesting problem with similar behavior is given by

$$\begin{cases} -\Delta u = \lambda u e^{u^{2-\varepsilon}} & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.15}$$

As for (1.8) it is possible to show that there exists a family of nodal solutions $u_\varepsilon$ for any $\varepsilon > 0$. Despite the nonlinearity is not covered by the assumptions in [6, Theorem 1.3] we can still repeat the proof in order to get the existence result. Moreover the result in [5] applies and so there exists a constant $\bar{\lambda}$ such that for any $\lambda \in (0, \bar{\lambda})$ there exists no sign changing solution.

It is possible to show that analogous results like in Theorems 1.2 and 1.3 hold. The interest in this type of nonlinearity is given by the similarity with the analogous in higher dimension (see problem (1.21) and the comments below).

**Remark 1.5.** Similar phenomena to Theorem 1.1, 1.2 and 1.3 appear in higher dimensions for the problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}} u + \lambda u & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.16}$$

where $N \geq 3$ and $B$ is the unit ball of $\mathbb{R}^N$.

In [8] it was proved that if $N = 4, 5, 6$ there exists $\lambda^* > 0$ such that there is no nodal radial solution for $0 < \lambda < \lambda^*$. The asymptotic behavior of the solution $u_\lambda$ as $\lambda \to \bar{\lambda}$ for a limit value $\bar{\lambda} > 0$ and $N = 4, 5, 6$ was studied in [20]. Note that the case $N = 6$ has strong similarities with our results when $k = 1$. Other existence results for $N = 4, 5$ can be founded in [21].

It is interesting to compare the previous results with other similar problems like

$$\begin{cases} -\Delta u = |u|^{p-1} u & \text{in } B \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.17}$$

(see [17]) and

$$\begin{cases} -\Delta u = \lambda \sinh & \text{in } B \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.18}$$

(see [18]).

Both this problems share the feature that suitable transformations of positive solutions converge to the limit problem

$$\begin{cases} -\Delta u = e^u & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < +\infty. \end{cases} \tag{1.19}$$

We want to compare Theorems 1.2 and 1.3 with the analogous ones for (1.17) and (1.18).

The global behavior is different: indeed solutions studied in [17] tend towards 0 in $C^2(B \setminus \{0\})$ and solutions founded in [18] converge to suitable multiples of the Green function which does not belong to $W_0^{1.2}(B)$.

However more striking differences appear if we look at the local behavior. Indeed, suitable rescaling of solutions to (1.17) and (1.18) converge to solutions of the singular Liouville problem

$$\begin{cases} -\Delta u = |x|^\alpha e^u & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |x|^\alpha e^u < +\infty, \end{cases} \tag{1.20}$$

for some suitable positive number $\alpha$. We refer to [17] and [18] for more precises statements. In our case the local behavior of the solution is again related to the problem (1.19). In some sense our problem is more similar to the "almost critical" problem in higher dimensions $N \geq 3$ given by

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}-\varepsilon} u & \text{in } B \subset \mathbb{R}^N \\ u = 0 \text{ on } \partial B. \end{cases} \tag{1.21}$$

In this case the local behavior of nodal solutions is given by the (unique) positive smooth solution of the limit problem (see [10,12,27])

$$-\Delta u = u^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N. \tag{1.22}$$

In our opinion this similarity is due to the effect of the nonlinearity which is very close to those in Moser-Trudinger inequality.

The paper is organized as follows: in Section 2 we prove some energy estimates for the solution $u_\varepsilon$. In Section 3 we study the behavior of $u_\varepsilon$ in the ball $B_{r_{1,\varepsilon}}$ where $r_{1,\varepsilon}$ is the first zero of $u_\varepsilon$. In Section 4 and 5 we consider the behavior of $u_\varepsilon$ in the other annular regions and in Section 6 we give the proof of Theorems 1.2 and 1.3. Finally in Appendix A we prove some technical lemmas.

For all $u \in H_0^1(B)$, we define $\|u\| := \left(\int_B |\nabla u|^2 dx\right)^{1/2}$. In addition, let $B(0,r) := B_r$ and $B(r,s) := B_s \setminus B_r$ for $r, s > 0$.

## 2. Energy estimates for $u_\varepsilon$

In the following, we always assume $0 < \lambda < \min\{\lambda_1, \lambda_{\mathrm{AY}}\}$ and we consider the least energy nodal solution $u_\varepsilon$ of (1.8) obtained in [6, Theorem 1.3]. More precisely, we define $H^1_{\mathrm{r},0}(B)$ as a subspace of $H^1_0(B)$ which consists of all the radial functions and by the Nehari manifold

$$\mathcal{N}_\varepsilon = \left\{ u \in H^1_{\mathrm{r},0}(B) \setminus \{0\} \mid \int_B |\nabla u|^2 dx = \int_B f_\varepsilon(u) u \, dx \right\},$$

where $f_\varepsilon(t) = \lambda t e^{t^2+|t|^{1+\varepsilon}}$ and for $k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{N}_{k,\varepsilon} := \Big\{ & u \in H^1_{\mathrm{r},0}(B) \mid \exists r_i \in (0,1);\ 0 = r_0 < r_1 < \cdots < r_{k+1} = 1, \\ & u(r_i) = 0,\ u_i := u|_{B(r_{i-1}, r_i)},\ (-1)^{i-1} u_i > 0,\ u_i \in \mathcal{N}_\varepsilon,\ 1 \le i \le k+1 \Big\}.\end{aligned}$$

Then let $u_\varepsilon \in \mathcal{N}_{k,\varepsilon}$ be a solution to (1.8) such that

$$I_\varepsilon(u_\varepsilon) = \inf_{u \in \mathcal{N}_{k,\varepsilon}} I_\varepsilon(u).$$

We choose constants $0 = r_{0,\varepsilon} < r_{1,\varepsilon} < \cdots < r_{k,\varepsilon} < r_{k+1,\varepsilon} = 1$ so that $u_\varepsilon(r_{i,\varepsilon}) = 0$ for $i = 1, 2, \cdots, k$. Moreover, for each $i = 1, 2, \cdots, k+1$, define $u_{i,\varepsilon} := u_\varepsilon|_{B(r_{i-1,\varepsilon}, r_i)}$ with zero extension to whole $B$.

First let us show a suitable upper bound for $I_\varepsilon(u_\varepsilon)$. To this end, we use the Moser function defined in [2]. For $0 < l < R \le 1$, we define

$$m_{l,R}(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \left(\log \dfrac{R}{l}\right)^{\frac12} & 0 \le |x| < l \\ \dfrac{\log \frac{R}{|x|}}{\left(\log \dfrac{R}{l}\right)^{\frac12}} & l \le |x| \le R \\ 0 & |x| > R. \end{cases}$$

Then it satisfies $m_{l,R} \in H^1_0(B)$ and $\|m_{l,R}\| = 1$. In addition let us define a cut off function,

$$\phi_{l,R}(x) = 1 - \frac{m_{l,R}(x)}{\sqrt{2\pi}^{-1} \left(\log \frac{R}{l}\right)^{\frac12}} \in H^1(B).$$

Then we have $0 \le \phi_{l,R} \le 1$, $\phi_{l,R} = 0$ on $B_l$ and $\phi_{l,R} = 1$ on $B \setminus B_R$. For $0 < l_1 = l_{1,\varepsilon} < R_1 = R_{1,\varepsilon} < p_1 = p_{1,\varepsilon} < l_2 = l_{2,\varepsilon} < R_2 = R_{2,\varepsilon} < p_2 = p_{2,\varepsilon} < \cdots < l_k = l_{k,\varepsilon} < R_k = R_{k,\varepsilon} < 1$, we set

$$\begin{cases} w_{1,\varepsilon} := m_{l_1,R_1} \\ w_{i,\varepsilon} := (-1)^{i-1} \phi_{R_{i-1},p_{i-1}} m_{l_i,R_i} \text{ for } i = 2, \cdots, k, \text{ and} \\ w_{k+1,\varepsilon} := (-1)^k \phi_{R_k,1} u_0, \end{cases}$$

where $u_0$ is the least energy solution of (1.6) obtained in [2] and thus, it satisfies

$$I_0(u_0) = \inf_{u \in \mathcal{N}_0} I_0(u) \in (0, 2\pi).$$

We choose $l_1, R_1, p_1, \cdots, l_k, R_k$ so that $R_k \to 0$ and

$$\begin{cases} \dfrac{\log \frac{1}{R_i}}{\log \frac{1}{l_i}} \to 0 \ (i = 1, 2, \cdots, k) \\ \dfrac{\log \frac{R_i}{l_i}}{\log \frac{p_{i-1}}{R_{i-1}}} \to 0, \quad \frac{p_{i-1}}{l_i} \to 0 \ (i = 2, \cdots, k), \end{cases} \tag{2.1}$$

as $\varepsilon \to 0$. For example, take any $R_k > 0$ such that $R_k \to 0$ as $\varepsilon \to 0$ and then, choose $l_k = e^{-1/R_k}$, $p_{k-1} = l_k^2$, and $R_{k-1} = p_{k-1}e^{-1/l_k}$. Similarly, set $l_{k-1} = e^{-1/R_{k-1}}$, $p_{k-2} = l_{k-1}^2$, $R_{k-2} = p_{k-2}e^{-1/l_{k-1}}$ and so on. We note that, for every $i = 1, 2, \cdots, k+1$ and $\varepsilon \in (0, 1)$, there exists a constant $t_{i,\varepsilon} > 0$ such that $t_{i,\varepsilon} w_{i,\varepsilon} \in \mathcal{N}_\varepsilon$. (See Step 2 in the [2, proof of Lemma 3.4].) We define a test function

$$w_\varepsilon(x) := \sum_{i=1}^{k+1} t_{i,\varepsilon} w_{i,\varepsilon}.$$

Then we have $w_\varepsilon \in \mathcal{N}_{k,\varepsilon}$. We obtain the following:

**Lemma 2.1.** *We get*

$$\limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \le 2\pi k + I_0(u_0).$$

*Proof.* First observe that since $w_\varepsilon \in \mathcal{N}_{k,\varepsilon}$, we have

$$I(u_\varepsilon) \le I_\varepsilon(w_\varepsilon) = \sum_{i=1}^{k+1} I_\varepsilon(t_{i,\varepsilon} w_{i,\varepsilon}).$$

Then it suffices to show:

(I) $\limsup_{\varepsilon \to 0} I_\varepsilon(t_{1,\varepsilon} w_{1,\varepsilon}) \le 2\pi$;
(II) $\limsup_{\varepsilon \to 0} I_\varepsilon(t_{i,\varepsilon} w_{i,\varepsilon}) \le 2\pi$, for $i = 2, \cdots, k$;
(III) $\limsup_{\varepsilon \to 0} I_\varepsilon(t_{k+1,\varepsilon} w_{k+1,\varepsilon}) \le I_0(u_0)$.

(I) We claim

$$\limsup_{\varepsilon \to 0} t_{1,\varepsilon}^2 \le 4\pi. \tag{2.2}$$

If not, there exist a sequence $(\varepsilon_n)$ and a constant $\delta > 0$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $t_{1,\varepsilon_n}^2 \ge 4\pi(1+\delta)$ for all $n$. Set $t_n := t_{1,\varepsilon_n}$, $w_n := w_{1,\varepsilon_n}$, $l_n := l_{1,\varepsilon_n}$ and $R_n := R_{1,\varepsilon_n}$ for simplicity. Since $t_n w_n \in \mathcal{N}_{\varepsilon_n}$, we get

$$t_n^2 \|w_n\|^2 = \lambda \int_B (t_n w_n)^2 e^{|t_n w_n|^2 + |t_n w_n|^{1+\varepsilon_n}} dx.$$

Then we have

$$
\begin{aligned}
t_n^2 &\geq \lambda \int_{B_{l_n}} (t_n w_n)^2 e^{|t_n w_n|^2 + |t_n w_n|^{1+\varepsilon_n}} dx \\
&\geq \frac{\lambda}{2} t_n^2 l_n^2 \log \frac{R_n}{l_n} e^{\frac{t_n^2}{2\pi} \log \frac{R_n}{l_n}} \\
&= \frac{1}{2} t_n^2 \log \frac{R_n}{l_n} \exp\left\{ \frac{t_n^2}{2\pi} \left( \log \frac{1}{l_n} - \log \frac{1}{R_n} \right) - 2 \log \frac{1}{l_n} \right\}.
\end{aligned}
$$

By (2.1) we know that

$$\log \frac{R_n}{l_n} \to \infty \ (n \to \infty),$$

and equivalently

$$\log \frac{1}{R_n} = o\left( \log \frac{1}{l_n} \right).$$

As a consequence, we find a constant $\delta' > 0$ such that

$$2 \geq \exp\left\{ \delta' \log \frac{1}{l_n} \right\}$$

for large $n$. Taking $n \to \infty$, we have a contradiction. Now, since $t_{1,\varepsilon} w_{1,\varepsilon} \in \mathcal{N}_\varepsilon$, $\|w_{1,\varepsilon}\| = 1$ and $\limsup_{\varepsilon \to 0} t_{1,\varepsilon}^2 \leq 4\pi$, we get

$$\left| \int_B f_\varepsilon(t_{1,\varepsilon} w_{1,\varepsilon}) t_{1,\varepsilon} w_{1,\varepsilon} dx \right| \leq C$$

for some constant $C > 0$ uniformly for $\varepsilon > 0$. Furthermore, note $t_{1,\varepsilon} w_{1,\varepsilon} \to 0$ a.e. on $B$. Then by Lemma A.1 in Appendix A, we find

$$\lim_{\varepsilon \to 0} \int_B F_\varepsilon(t_{1,\varepsilon} w_{1,\varepsilon}) dx = \int_B F_0(0) dx = 0.$$

As a consequence, we get

$$\limsup_{\varepsilon \to 0} I_\varepsilon(t_{1,\varepsilon} w_{1,\varepsilon}) = \limsup_{\varepsilon \to 0} \frac{t_{1,\varepsilon}^2}{2} \leq 2\pi.$$

This finishes the proof of (I).

(II) Fix $i = 2, 3, \cdots, k$. We first claim that $\lim_{\varepsilon \to 0} \int_B |\nabla w_{i,\varepsilon}|^2 dx = 1$. In fact, putting $l_i = l_{i,\varepsilon}$ $(i = 2, \cdots, k)$, $R_i = R_{i,\varepsilon}$ $(i = 1, \cdots, k)$ and $p_i = p_{i,\varepsilon}$ $(i =$

$1, \cdots, k-1$) for simplicity, we get

$$
\begin{aligned}
\int_B |\nabla w_{i,\varepsilon}|^2 dx = & \int_B \left|\nabla \phi_{R_{i-1},p_{i-1}}\right|^2 m_{l_i,R_i}^2 dx \\
& + 2\int_B \phi_{R_{i-1},p_{i-1}} m_{l_i,R_i} \nabla \phi_{R_{i-1},p_{i-1}} \nabla m_{l_i,R_i} dx \\
& + \int_B \left|\nabla m_{l_i,R_i}\right|^2 \phi_{R_{i-1},p_{i-1}}^2 dx \\
= & I_1 + I_2 + I_3.
\end{aligned}
$$

It follows from (2.1) that

$$
I_1 = \int_{B(R_{i-1},p_{i-1})} \left|\nabla \phi_{R_{i-1},p_{i-1}}\right|^2 m_{l_i,R_i}^2 dx = \frac{\log \frac{R_i}{l_i}}{\log \frac{p_{i-1}}{R_{i-1}}} \to 0
$$

as $\varepsilon \to 0$. Since $\phi_{R_{i-1},p_{i-1}} m_{l_i,R_i} \nabla \phi_{R_{i-1},p_{i-1}} \nabla m_{l_i,R_i} = 0$ on $B$, we get $I_2 = 0$. Furthermore, as $\phi_{R_{i-1},p_{i-1}} = 1$ on $B(l_i, R_i)$ and $\nabla m_{l_i,R_i} = 0$ on $B_{l_i}$, we clearly have

$$
I_3 = \int_B \left|\nabla m_{l_i,R_i}\right|^2 dx = 1.
$$

This shows the claim. Now we shall show $\limsup_{\varepsilon\to 0} t_{i,\varepsilon}^2 \le 4\pi$. If not, there exists a constant $\delta > 0$ such that $t_{i,\varepsilon}^2 \ge 4\pi(1+\delta)$ for all small $\varepsilon > 0$ by extracting a sequence if necessary. Then noting $t_{i,\varepsilon} w_{i,\varepsilon} \in \mathcal{N}_\varepsilon$ and (2.1), we get

$$
\begin{aligned}
1 + o(1) = & \lambda \int_B \left(\phi_{R_{i-1},p_{i-1}} m_{l_i,R_i}\right)^2 \exp\Big\{ \left(t_{i,\varepsilon}\phi_{R_{i-1},p_{i-1}} m_{l_i,R_i}\right)^2 \\
& \qquad + \left|t_{i,\varepsilon}\phi_{R_{i-1},p_{i-1}} m_{l_i,R_i}\right|^{1+\varepsilon} \Big\} dx \\
\ge & \lambda \int_{B(p_{i-1},l_i)} m_{l_i,R_i}^2 \exp\left\{\left(t_{i,\varepsilon} m_{l_i,R_i}\right)^2\right\} dx \\
= & \frac{\lambda}{2}\log\frac{R_i}{l_i}\exp\left\{\frac{t_{i,\varepsilon}^2}{2\pi}\left(\log\frac{1}{l_i} - \log\frac{1}{R_i}\right) - 2\log\frac{1}{l_i} - 2\log\frac{1}{1-(p_{i-1}/l_i)^2}\right\} \\
\ge & C\exp\left(\delta'\log\frac{1}{l_i}\right),
\end{aligned}
$$

for some constants $C, \delta' > 0$ if $\varepsilon$ is small enough. Taking $\varepsilon \to 0$, we get a contradiction. Then, analogously with the conclusion for (I), we obtain

$$
\limsup_{\varepsilon\to 0} I_\varepsilon(t_{i,\varepsilon} w_{i,\varepsilon}) = \limsup_{\varepsilon\to 0} \frac{\|t_{i,\varepsilon} w_{i,\varepsilon}\|^2}{2} \le 2\pi.
$$

This proves (II).

(III) We claim that $t_{k+1,\varepsilon}$ is bounded. To see this, we follow the argument in [6, pages 493-494]. We assume on the contrary, for a sequence $(\varepsilon_n)$, we have $\varepsilon_n \to 0$ and $t_{k+1,\varepsilon_n} \to \infty$ as $n \to \infty$. Then we let

$$v_n := \frac{t_{k+1,\varepsilon_n} w_{k+1,\varepsilon_n}}{\|t_{k+1,\varepsilon_n} w_{k+1,\varepsilon_n}\|} = \frac{w_{k+1,\varepsilon_n}}{\|w_{k+1,\varepsilon_n}\|}.$$

Then using (2.1), we get $v_n \to v_0 = (-1)^k u_0/\|u_0\| \neq 0$ in $H_0^1(B)$. Furthermore, noting $t_{i,\varepsilon_n}$ is bounded for all $i = 1, 2, \cdots, k$ as proved in (I) and (II), we obtain

$$\|w_{\varepsilon_n}\|^2 = \sum_{i=1}^{k} t_{i,\varepsilon_n}^2 + t_{k+1,\varepsilon_n}^2 \|w_{k+1,\varepsilon_n}\|^2 = t_{k+1,\varepsilon_n}^2 \|w_{k+1,\varepsilon_n}\|^2 (1+\eta_n),$$

for a sequence $(\eta_n) \subset \mathbb{R}^+$ with $\eta_n \to 0$ as $n \to \infty$. Therefore, we get

$$\frac{w_{\varepsilon_n}}{\|w_{\varepsilon_n}\|} = \frac{1}{(1+\eta_n)^{\frac{1}{2}}} \left( v_n + \sum_{i=1}^{k} \frac{t_{i,\varepsilon_n}}{t_{k+1,\varepsilon_n}\|w_{k+1,\varepsilon_n}\|} w_{i,\varepsilon_n} \right) \to v_0 \neq 0 \text{ in } H_0^1(B).$$

Finally using $w_{\varepsilon_n} \in \mathcal{N}_{\varepsilon_n}$ and the Fatou lemma, we have

$$\begin{aligned} 1 &= \liminf_{n\to\infty} \frac{1}{\|w_{\varepsilon_n}\|^2} \int_B f_{\varepsilon_n}(w_n) w_n dx \\ &\geq \int_B \liminf_{n\to\infty} \frac{f_{\varepsilon_n}(w_{\varepsilon_n})}{w_{\varepsilon_n}} \left( \frac{w_{\varepsilon_n}}{\|w_{\varepsilon_n}\|} \right)^2 dx \\ &= \infty, \end{aligned}$$

a contradiction. This proves the claim. Finally let us end the proof. We suppose the conclusion of (III) does not hold on the contrary. Then, we have a sequence $(\varepsilon_n)$ and a constant $\delta > 0$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $I_{\varepsilon_n}(t_{k+1,\varepsilon_n} w_{k+1,\varepsilon_n}) \geq I_0(u_0) + \delta$ for all $n$. On the other hand, as $t_{k+1,\varepsilon_n}$ is bounded, there exists a constant $t_0 \geq 0$ such that $t_{k+1,\varepsilon_n} \to t_0$ as $n \to \infty$ up to subsequences. This implies $t_{k+1,\varepsilon_n} w_{k+1,\varepsilon_n} \to t_0 u_0$ in $H_0^1(\Omega)$ as $n \to \infty$ and then, we get $t_0 u_0 \in \mathcal{N}_0$. It follows that $t_0 = 0$ or $1$. (See Step 2 in the [2, proof of Lemma 3.4].) Consequently, we deduce

$$\lim_{n\to\infty} I_{\varepsilon_n}(t_{k+1,\varepsilon_n} w_{k+1,\varepsilon_n}) \leq I_0(u_0),$$

which implies a contradiction. This completes (III). $\square$

**Lemma 2.2.** *There exist constants* $0 < K < K'$ *such that*

$$K \leq \|u\|^2 \leq K'$$

*for all* $u \in \mathcal{N}_\varepsilon$ *and small* $\varepsilon > 0$.

*Proof*. The lower bound is clearly confirmed by Lemma A.2 in Appendix A. On the other hand, the upper bound is proved similarly in [2, Claim 1 on page 404]. This finishes the proof. □

Next we study the behavior of $r_{i,\varepsilon}$. To this end we recall the next lemma.

**Lemma 2.3 (Radial lemma [28]).** *Let $B^N \subset \mathbb{R}$ be a $N$-dimensional unit ball and $H_{\mathrm{rad}}(B^N)$ be a subspace of $H^1(B^N)$ which consists of all the radial functions. Then, there exists a constant $c_N > 0$ such that*

$$|u(r)| \le c_N \|u\|/r^{\frac{N-1}{2}} \quad \Big(u \in H_{\mathrm{rad}}\big(B^N\big) \text{ and } r \in (0,1)\Big).$$

*In particular, for $N = 2$ we have $|u(r)| \le c_2\|u\|/\sqrt{r}$.*

We deduce the following:

**Lemma 2.4.** *We see*

$$r_{i,\varepsilon} \to 0 \text{ as } \varepsilon \to 0$$

*for all $i = 1, 2, \cdots, k$.*

*Proof*. By Lemma 2.2, we may assume $u_\varepsilon$ is bounded in $H_0^1(B)$ and $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(B)$ as $\varepsilon \to 0$ where $u$ is a radial solution $u$ to (1.8) with $\varepsilon = 0$. Moreover we recall that $u_{i,\varepsilon} = u_\varepsilon|_{B(r_{i-1,\varepsilon}, r_{i,\varepsilon})}$ satisfies $(-1)^{i-1}u_{i,\varepsilon} \ge 0$ for all $i = 1, 2, \cdots, k+1$. Then, we can suppose there exists a function $u_i \in H_0^1(B)$ such that $u_{i,\varepsilon} \rightharpoonup u_i$ weakly in $H_0^1(B)$ and $(-1)^{i-1}u_i \ge 0$ for all $i = 1, 2, \cdots, k+1$ and further, $u = \sum_{i=1}^{k+1} u_i$. Now, let us show $r_{k,\varepsilon} \to 0$ which also implies $r_{i,\varepsilon} \to 0$ for all $i = 1, 2, \cdots, k-1$ as $\varepsilon \to 0$. If not, we may suppose that there exists a constant $r_k \in (0, 1]$ such that $r_{k,\varepsilon} \to r_k$ as $\varepsilon \to 0$. We then claim $u_{k+1} \ne 0$. Indeed, if $u_{k+1} = 0$, on the contrary, we have $\int_B u_{k+1,\varepsilon}^2 dx \to 0$ as $\varepsilon \to 0$. It follows that $\|u_{k+1,\varepsilon}\|_\infty = \sup_{r \in (r_{k,\varepsilon}, 1)} u_{k+1,\varepsilon}(r) \to \infty$ as $\varepsilon \to 0$. Otherwise, from Lemma 2.2, we get

$$\begin{aligned} 0 < K \le \|u_{k+1,\varepsilon}\|^2 &= \lambda \int_B u_{k+1,\varepsilon}^2 e^{u_{k+1,\varepsilon}^2 + |u_{k+1,\varepsilon}|^{1+\varepsilon}} dx \\ &\le \lambda e^{\|u_{k+1,\varepsilon}\|_\infty^2 + \|u_{k+1,\varepsilon}\|_\infty^{1+\varepsilon}} \int_B u_{k+1,\varepsilon}^2 dx \to 0 \end{aligned}$$

as $\varepsilon \to 0$, a contradiction. As a consequence, setting $\|u_{k+1,\varepsilon}\|_\infty = u_{k,\varepsilon}(r_{k,\varepsilon}^*)$ with a value $r_{k,\varepsilon}^* \in (r_{k,\varepsilon}, 1)$, we get from Lemma 2.3 that

$$\|u_{k,\varepsilon}\| \ge c_2^{-1}\big|u_{k,\varepsilon}\big(r_{k,\varepsilon}^*\big)\big|\big(r_{k,\varepsilon}^*\big)^{\frac{1}{2}} \ge c_2^{-1}\big|u_{k,\varepsilon}\big(r_{k,\varepsilon}^*\big)\big| r_{k,\varepsilon}^{\frac{1}{2}} \to \infty$$

as $\varepsilon \to 0$ since $r_k > 0$, which contradicts Lemma 2.2. This shows the claim. Especially we get $0 \le r_1 \le \cdots \le r_k \in (0, 1)$. Now recalling that $u$ is a radial solution and $\lambda < \lambda_{\mathrm{AY}}$ and then, noting $(-1)^k u_{k+1} \geq 0$ is nontrivial and $(-1)^{k-1} u_k \geq 0$, we must have $u_k = 0$. Then, the maximum principle yields $r_k = r_{k+1}$. Finally, repeating the argument above, we get $\sup_{r\in(r_{k-1,\varepsilon},r_{k,\varepsilon})} u_{k,\varepsilon}(r) \to \infty$ as $\varepsilon \to 0$ and then Lemmas 2.2 and 2.3 lead us to the contradiction. This finishes the proof. $\square$

Finally, let us investigate the limit value of the energy $I_\varepsilon(u_\varepsilon)$ more precisely.

**Lemma 2.5.** *We get*

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_{i,\varepsilon}) = 2\pi$$

*for all* $i = 1, 2, \cdots, k$. *Furthermore, we obtain*

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = I_0(u_0).$$

*Proof.* Choose $i = 1, 2, \cdots, k$. We first claim

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_{i,\varepsilon}) \geq 2\pi. \tag{2.3}$$

Indeed, let $\tilde{u}_{0,\varepsilon} \in H_0^1(B)$ be a positive solution of (1.8) with $B$ replaced by $B_{r_{i,\varepsilon}}$ which satisfies

$$I_\varepsilon(\tilde{u}_{0,\varepsilon}) = \inf\left\{ I_\varepsilon(u) \;\middle|\; u \in H_0^1\big(B_{r_{i,\varepsilon}}\big),\ \int_{B_{r_{i,\varepsilon}}} |\nabla u|^2 dx = \int_{B_{r_{i,\varepsilon}}} f_\varepsilon(u) u dx. \right\}.$$

The existence of $\tilde{u}_{0,\varepsilon}$ is ensured by [2]. Then we have $I_\varepsilon(u_{i,\varepsilon}) \geq I_\varepsilon(\tilde{u}_{0,\varepsilon})$. Hence it suffices to show $\liminf_{\varepsilon\to 0} I_\varepsilon(\tilde{u}_{0,\varepsilon}) \geq 2\pi$. Now we assume, on the contrary, $\liminf_{\varepsilon\to 0} I_\varepsilon(\tilde{u}_{0,\varepsilon}) < 2\pi$. Set $v_{i,\varepsilon}(x) = \tilde{u}_{0,\varepsilon}(r_{i,\varepsilon} x)$. Then $v = v_{i,\varepsilon}$ satisfies

$$\begin{cases} -\Delta v = \lambda r_{i,\varepsilon}^2 v e^{v^2 + v^{1+\varepsilon}},\ v > 0 & \text{in } B \\ v = 0 & \text{on } \partial B. \end{cases} \tag{2.4}$$

We define the energy associated to (2.4).

$$J_\varepsilon(v) = \int_B |\nabla v|^2 dx - r_{i,\varepsilon}^2 \int_B F_\varepsilon(v) dx \ \Big(v \in H_0^1(B)\Big).$$

Then we have $I_\varepsilon(\tilde{u}_{0,\varepsilon}) = J_\varepsilon(v_{i,\varepsilon})$ and thus, $\liminf_{\varepsilon\to 0} J_\varepsilon(v_{i,\varepsilon}) < 2\pi$. In particular, we have a sequence $(\varepsilon_n)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $c := \lim_{n\to\infty} J_{\varepsilon_n}(v_{i,\varepsilon_n}) < 2\pi$. Notice that Lemma 2.2 ensures $c > 0$. Then, noting $J'_{\varepsilon_n}(v_{i,\varepsilon_n}) = 0$ and Lemma A.3 in Appendix A, we can find a function $v_0 \in H_0^1(\Omega)$ such that $v_{i,\varepsilon_n} \to v_0$ in $H_0^1(\Omega)$ as $n \to \infty$ up to subsequences. Lastly, using (2.4), we get

$$\begin{cases} -\Delta v_0 = 0,\ v_0 \geq 0 & \text{in } B \\ v_0 = 0 & \text{on } \partial B. \end{cases}$$

Then, the maximum principle shows $v_0 = 0$. But this contradicts $c > 0$. Next let us show

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_{i,\varepsilon}) \le 2\pi, \quad \text{and} \quad \limsup_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = I_0(u_0). \tag{2.5}$$

In fact, we get by Lemma 2.1 and (2.3) that

$$2\pi k + I_0(u_0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \ge 2\pi k + \limsup_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon})$$

which implies $I_0(u_0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon})$. Furthermore, let $u_{0,\varepsilon}$ be the least energy solution of (1.8) obtained by [2]. It follows that $I_0(u_0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_{0,\varepsilon})$. We claim $\limsup_{\varepsilon\to 0} I_\varepsilon(u_{0,\varepsilon}) \ge I_0(u_0)$. If not, we have a sequence $(\varepsilon_n)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} I_{\varepsilon_n}(u_{0,\varepsilon}) < I_0(u_0)$. Note $I_0(u_0) \in (0, 2\pi)$. Then from Lemma A.3, we deduce, by subtracting a subsequence if necessary, $u_{0,\varepsilon_n} \to \tilde{u}_0$ in $H_0^1(B)$ as $n \to \infty$ and further, $\tilde{u}_0$ is a nontrivial solution of (1.6) with $I_0(\tilde{u}_0) \in (0, I_0(u_0))$. But as $\tilde{u}_0 \in \mathcal{N}_0$, we obtain a contradiction by the definition of $u_0$. This proves the claim. Now again arguing as the beginning, we get

$$2\pi k + I_0(u_0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \ge 2\pi(k-1) + \limsup_{\varepsilon\to 0} I_\varepsilon(u_{i,\varepsilon}) + I_0(u_0).$$

This completes (2.5). As a consequence, (2.3) and (2.5) finish the proof. □

**Lemma 2.6.** *We have*

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = I_0(u_0).$$

*Proof.* Since $\liminf_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) \le I_0(u_0)$, arguing as in the previous proof, we can get

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = I_0(u_0).$$

Then combining this together with the final assertion in the previous lemma, we complete the proof. □

## 3. Behavior of $u_\varepsilon$ in the ball $B_{r_{1,\varepsilon}}$

Let us start our main argument with studying the behavior on a ball. To this end, we first observe that $u_{1,\varepsilon} = u_\varepsilon|_{B_{r_{1,\varepsilon}}}$ is a solution to

$$\begin{cases} -\Delta u = \lambda u e^{u^2+|u|^{1+\varepsilon}}, \ u > 0 & \text{in } B_{r_{1,\varepsilon}} \\ u = 0 & \text{on } \partial B_{r_{1,\varepsilon}}, \end{cases} \tag{3.1}$$

for $\varepsilon > 0$. Then the results in [16] shows that $u_{1,\varepsilon}$ is radial and $\|u_{1,\varepsilon}\|_{L^\infty(B_{r_{1,\varepsilon}})} = u_{1,\varepsilon}(0)$. Next we see that $v_{1,\varepsilon}(x) := u_\varepsilon(r_{1,\varepsilon}x)$ $(x \in B_1)$ is a solution of

$$\begin{cases} -\Delta v = \lambda r_{1,\varepsilon}^2 v e^{v^2+|v|^{1+\varepsilon}}, \ v > 0 & \text{in } B \\ v = 0 & \text{on } \partial B, \end{cases} \tag{3.2}$$

for $\varepsilon > 0$ and $\|v_{1,\varepsilon}\|_{L^\infty(B)} = v_\varepsilon(0)$. Notice $\lambda r_{1,\varepsilon}^2 \to 0$ as $\varepsilon \to 0$ by Lemma 2.4. Furthermore, by Lemma 2.5, we get

$$J_\varepsilon(v_\varepsilon) := \frac{1}{2}\int_B |\nabla v_\varepsilon|^2 dx - r_{1,\varepsilon}^2 \int_B F_\varepsilon(v_\varepsilon) dx \to 2\pi,$$

as $\varepsilon \to 0$. We have the following:

**Proposition 3.1.** *We get $v_{1,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(B)$, $v_{1,\varepsilon}(0) \to \infty$ and*

$$\int_B |\nabla v_{1,\varepsilon}|^2 dx \to 4\pi,$$

*as $\varepsilon \to 0$. Furthermore, let $\gamma_{1,\varepsilon} > 0$ be such that*

$$2\lambda r_{1,\varepsilon}^2 v_{1,\varepsilon}(0)^2 e^{v_{1,\varepsilon}(0)^2 + v_{1,\varepsilon}(0)^{1+\varepsilon}} \gamma_{1,\varepsilon}^2 = 1.$$

*Then we have $\gamma_{1,\varepsilon} \to 0$ and*

$$2v_{1,\varepsilon}(0)(v_{1,\varepsilon}(\gamma_{1,\varepsilon}x) - v_{1,\varepsilon}(0)) \to \log \frac{1}{(1+|x|^2/8)^2} \ \textit{in } C^2_{\mathrm{loc}}(\mathbb{R}^2),$$

*as $\varepsilon \to 0$.*

*Proof.* It is a direct consequence in [3, Theorem 2]. □

**Corollary 3.2.** *We obtain $u_{1,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(B)$, $u_{1,\varepsilon}(0) \to \infty$ and*

$$\int_{B_{r_{1,\varepsilon}}} |\nabla u_{1,\varepsilon}|^2 dx \to 4\pi,$$

*and*

$$I_\varepsilon(u_{1,\varepsilon}) \to 2\pi,$$

*as $\varepsilon \to 0$. Furthermore, let $\delta_{1,\varepsilon} = r_{1,\varepsilon}\gamma_{1,\varepsilon} > 0$. Then we have $\delta_{1,\varepsilon} \to 0$ and*

$$2u_\varepsilon(0)(u_\varepsilon(\delta_{1,\varepsilon}x) - u_\varepsilon(0)) \to \log \frac{1}{(1+|x|^2/8)^2} \ \textit{in } C^2_{\mathrm{loc}}\big(\mathbb{R}^2\big),$$

*as $\varepsilon \to 0$.*

*Proof*. The proof follows from Proposition 3.1 and Lemma 2.5. ☐

## 4. Behavior of $u_\varepsilon$ on annuli

We next investigate the behavior of $u_\varepsilon$ on annuli. Fix $i \in \{2, \cdots, k\}$ and set $u_{i,\varepsilon} := u_\varepsilon|_{B(r_{i-1,\varepsilon}, r_{i,\varepsilon})}$. Then $u_{i,\varepsilon} \in H_0^1(B)$ by zero extension. Since $u_{i,\varepsilon}$ is radial, we may assume it satisfies

$$\begin{cases} -u''_{i,\varepsilon} - \frac{1}{r} u'_{i,\varepsilon} = \lambda u_{i,\varepsilon} e^{u_{i,\varepsilon}^2 + u_{i,\varepsilon}^{1+\varepsilon}} & \text{in } (r_{i-1,\varepsilon}, r_{i,\varepsilon}) \\ u_{i,\varepsilon} > 0 & \text{in } (r_{i-1,\varepsilon}, r_{i,\varepsilon}) \\ u_{i,\varepsilon}(r_{i-1,\varepsilon}) = u_{i,\varepsilon}(r_{i,\varepsilon}) = 0. \end{cases} \tag{4.1}$$

Now we have the following result:

**Proposition 4.1.** *We get $u_{i,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(B)$,*

$$\int_{B(r_{i-1,\varepsilon}, r_{i,\varepsilon})} |\nabla u_{i,\varepsilon}|^2 dx \to 4\pi,$$

*and*

$$I_\varepsilon(u_{i,\varepsilon}) \to 2\pi,$$

*as $\varepsilon \to 0$. Moreover, let us denote by $M_{i,\varepsilon} r_{i,\varepsilon} \in (r_{i-1,\varepsilon}, r_{i,\varepsilon})$ with $M_{i,\varepsilon} < 1$, the point such that $||u_{i,\varepsilon}||_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})} = u_{i,\varepsilon}(M_{i,\varepsilon} r_{i,\varepsilon})$. Then if we set $\delta_{i,\varepsilon} = \gamma_{i,\varepsilon} r_{i,\varepsilon} > 0$ with*

$$2\lambda ||u_{i,\varepsilon}||^2_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})} e^{||u_{i,\varepsilon}||^2_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})} + ||u_{i,\varepsilon}||^{1+\varepsilon}_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})}} r_{i,\varepsilon}^2 \gamma_{i,\varepsilon}^2 = 1,$$

*we get $\delta_{i,\varepsilon} \to 0$ and further,*

$$\begin{aligned} &2||u_{i,\varepsilon}||_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})} \left( u_{i,\varepsilon}(M_\varepsilon r_{i,\varepsilon} + \delta_{i,\varepsilon} r) - ||u_{i,\varepsilon}||_{L^\infty(r_{i-1,\varepsilon}, r_{i,\varepsilon})} \right) \\ &\to \log \frac{1}{\left(1 + r^2/8\right)^2} \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^+), \end{aligned}$$

*as $\varepsilon \to 0$.*

In the following, we set $M_\varepsilon := M_{i,\varepsilon}$ for simplicity. We get the following:

**Lemma 4.2.** $u_{i,\varepsilon}(M_\varepsilon r_{i,\varepsilon}) \to +\infty$ *as* $\varepsilon \to 0$.

*Proof*. Integrating (4.1) we get

$$\begin{aligned} \int_{r_{i-1,\varepsilon}}^{r_{i,\varepsilon}} (u'_{i,\varepsilon})^2 r dr &= \lambda \int_{r_{i-1,\varepsilon}}^{r_{i,\varepsilon}} u_{i,\varepsilon}^2 e^{u_{i,\varepsilon}^2 + u_{i,\varepsilon}^{1+\varepsilon}} r\, dr \\ &\le \lambda e^{u_{i,\varepsilon}^2(M_\varepsilon r_{i,\varepsilon}) + u_{i,\varepsilon}^{1+\varepsilon}(M_\varepsilon r_{i,\varepsilon})} \int_{r_{i-1,\varepsilon}}^{r_{i,\varepsilon}} u_{i,\varepsilon}^2 r\, dr \\ \text{(using the Poincaré inequality)} &\le \lambda \frac{e^{u_{i,\varepsilon}^2(M_\varepsilon r_{i,\varepsilon}) + u_{i,\varepsilon}^{1+\varepsilon}(M_\varepsilon r_{i,\varepsilon})}}{\lambda_1(r_{i-1,\varepsilon}, r_{i,\varepsilon})} \int_{r_{i-1,\varepsilon}}^{r_{i,\varepsilon}} \left(u'_{i,\varepsilon}\right)^2 r dr, \end{aligned}$$

where $\lambda_1(r_{i-1,\varepsilon}, r_{i,\varepsilon})$ is the first eigenvalue of the operator $-u''-\frac{1}{r}u'$ in $(r_{i-1,\varepsilon}, r_{i,\varepsilon})$. Since $r_{i-1,\varepsilon}, r_{i,\varepsilon} \to 0$ we get that $\lambda_1(r_{i-1,\varepsilon}, r_{i,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$. This gives the claim. □

Now, let us consider the scaled function, $v_\varepsilon : \left(\frac{r_{i-1,\varepsilon}}{r_{i,\varepsilon}}, 1\right) \to \mathbb{R}$ defined as

$$v_\varepsilon(r) = u_{i,\varepsilon}(r_{i,\varepsilon} r)$$

which satisfies

$$\begin{cases} -v_\varepsilon'' - \frac{1}{r}v_\varepsilon' = \lambda r_{i,\varepsilon}^2 v_\varepsilon e^{v_\varepsilon^2 + v_\varepsilon^{1+\varepsilon}} & \text{in } \left(\dfrac{r_{i-1,\varepsilon}}{r_{i,\varepsilon}}, 1\right) \\ v_\varepsilon > 0 & \text{in } \left(\dfrac{r_{i-1,\varepsilon}}{r_{i,\varepsilon}}, 1\right) \\ v_\varepsilon\left(\dfrac{r_{i-1,\varepsilon}}{r_{i,\varepsilon}}\right) = v_\varepsilon(1) = 0. \end{cases} \tag{4.2}$$

Set

$$r_\varepsilon = \frac{r_{i-1,\varepsilon}}{r_{i,\varepsilon}}.$$

Then we have the following local behavior:

**Lemma 4.3.** *Choose $M_\varepsilon \in (r_\varepsilon, 1)$ as in Proposition 4.1. Then if we set $\gamma_{i,\varepsilon} > 0$ so that*

$$2\lambda \|v_\varepsilon\|_{L^\infty(r_\varepsilon,1)}^2 e^{\|v_\varepsilon\|_{L^\infty(r_\varepsilon,1)}^2 + \|v_\varepsilon\|_{L^\infty(r_\varepsilon,1)}^{1+\varepsilon}} r_{i,\varepsilon}^2 \gamma_{i,\varepsilon}^2 = 1,$$

*we get $\gamma_{i,\varepsilon} \to 0$ and*

$$\begin{aligned} &2\|v_\varepsilon\|_{L^\infty(r_\varepsilon,1)} \left(v_\varepsilon(M_\varepsilon + \gamma_{i,\varepsilon} r) - \|v_\varepsilon\|_{L^\infty(r_\varepsilon,1)}\right) \to z(r) \\ &= \log \frac{1}{\left(1 + r^2/8\right)^2} \text{ in } C^2_{\mathrm{loc}}(0, +\infty), \end{aligned}$$

*as $\varepsilon \to 0$.*

*Proof.* Let $v_\varepsilon$, $r_\varepsilon$ and $M_\varepsilon \in (r_\varepsilon, 1)$ as above. For $\gamma_\varepsilon > 0$, which will be chosen later, we define the scaled function

$$z_\varepsilon(r) = 2v_\varepsilon(M_\varepsilon)(v_\varepsilon(M_\varepsilon + \gamma_\varepsilon r) - v_\varepsilon(M_\varepsilon)). \tag{4.3}$$

We have that $z_\varepsilon$ solves the equation,

$$\begin{cases} -z_\varepsilon'' - \dfrac{1}{\frac{M_\varepsilon}{\gamma_\varepsilon}+r} z_\varepsilon' = 2\lambda\gamma_\varepsilon^2 r_{i,\varepsilon}^2 e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} v_\varepsilon^2(M_\varepsilon)\left(\dfrac{z_\varepsilon}{2v_\varepsilon^2(M_\varepsilon)}+1\right) \\ \quad \times \exp^{\left\{z_\varepsilon\left(\frac{z_\varepsilon}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)\right\}} \text{ in } \left(\dfrac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}, \dfrac{1-M_\varepsilon}{\gamma_\varepsilon}\right) \\ z_\varepsilon(r) \le 0,\ z_\varepsilon(0) = z_\varepsilon'(0) = 0 \\ z_\varepsilon\left(\dfrac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}\right) = z_\varepsilon\left(\dfrac{1-M_\varepsilon}{\gamma_\varepsilon}\right) = -2v_\varepsilon^2(M_\varepsilon) \to -\infty\ (\varepsilon\to 0). \end{cases}$$

So setting

$$2\lambda\gamma_\varepsilon^2 r_{i,\varepsilon}^2 e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} v_\varepsilon^2(M_\varepsilon) = 1$$

we get

$$\begin{cases} -z_\varepsilon'' - \dfrac{1}{\frac{M_\varepsilon}{\gamma_\varepsilon}+r} z_\varepsilon' = \left(\dfrac{z_\varepsilon}{2v_\varepsilon^2(M_\varepsilon)}+1\right) \\ \quad \times \exp^{\left\{z_\varepsilon\left(\frac{z_\varepsilon}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)\right\}} \text{ in } \left(\dfrac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}, \dfrac{1-M_\varepsilon}{\gamma_\varepsilon}\right) \\ z_\varepsilon(r) \le 0,\ z_\varepsilon(0) = z_\varepsilon'(0) = 0 \\ z_\varepsilon\left(\dfrac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}\right) = z_\varepsilon\left(\dfrac{1-M_\varepsilon}{\gamma_\varepsilon}\right) = -2v_\varepsilon^2(M_\varepsilon) \to -\infty\ (\varepsilon\to 0). \end{cases} \tag{4.4}$$

Note that $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. Actually, multiplying (4.2) by $v_\varepsilon r$ and integrating over $(0, 1)$, we get

$$\begin{aligned} \int_0^1 (v_\varepsilon')^2 r\,dr &= \lambda r_{i,\varepsilon}^2 \int_0^1 v_\varepsilon^2 e^{v_\varepsilon^2+v_\varepsilon^{1+\varepsilon}} r\,dr \\ &\le \lambda r_{i,\varepsilon}^2 e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} \int_0^1 v_\varepsilon^2 r\,dr \\ \text{(applying the Poincaré inequality )} &\le \frac{\lambda}{\lambda_1} r_{i,\varepsilon}^2 e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} \int_0^1 (v_\varepsilon')^2 r\,dr. \end{aligned}$$

This shows

$$r_{i,\varepsilon}^2 e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} \ge C > 0$$

for some constant $C > 0$ and small $\varepsilon > 0$ . Then noting our choice of $\gamma_\varepsilon$ and Lemma 4.2, we prove the claim. Moreover we clearly have that $\lim_{\varepsilon\to 0} \frac{1-M_\varepsilon}{\gamma_\varepsilon} \to \infty$, $\lim_{\varepsilon\to 0} \frac{M_\varepsilon - r_\varepsilon}{\gamma_\varepsilon} = l \in [0, \infty]$ and $\lim_{\varepsilon\to 0} \frac{M_\varepsilon}{\gamma_\varepsilon} = m \in [l, \infty]$. Now let us show that for any compact subset $K \Subset (-l, \infty)$ ($[0, \infty)$ if $l = 0$), there exists a constant $C > 0$ which is independent of $\varepsilon$ such that

$$\|z_\varepsilon\|_{C^1(K)} \le C.$$

Indeed, from (4.4), we get that $-z_\varepsilon'' - \frac{1}{\frac{M_\varepsilon}{\gamma_\varepsilon}+r} z_\varepsilon' \le 1$. First assume $l > 0$ and choose any $K \Subset (-l, 0]$. We may suppose $K \Subset (\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon}, 0]$ for small $\varepsilon > 0$. Define $a = \min K < 0$ and set $C_\varepsilon = \frac{M_\varepsilon}{\gamma_\varepsilon}$. Then, for any $r \in K$, we derive,

$$-\left[z_\varepsilon'(r)(C_\varepsilon + r)\right]' \le C_\varepsilon + r.$$

Integrating between $r$ and 0 we obtain

$$z_\varepsilon'(r)(C_\varepsilon + r) \le -\left(C_\varepsilon r + \frac{1}{2} r^2\right).$$

Since $C_\varepsilon + r > 0$ for small $\varepsilon > 0$, we show

$$z_\varepsilon'(r) \le -\frac{C_\varepsilon r + \frac{1}{2}r^2}{C_\varepsilon + r} \text{ and thus, } z_\varepsilon(r) \ge \int_r^0 \frac{C_\varepsilon s + \frac{1}{2}s^2}{C_\varepsilon + s} ds$$

for small $\varepsilon > 0$. If we set $G_\varepsilon(s) = \frac{C_\varepsilon s + \frac{1}{2}s^2}{C_\varepsilon + s}$, we get that $G_\varepsilon'(s) \ge 0$ for all $s \in K$. So we find that $G_\varepsilon(s) \ge G_\varepsilon(a)$ for all $s \in K$. Now, if $C_\varepsilon \to \infty$ as $\varepsilon \to 0$, we get $G_\varepsilon(a) \ge -2|a|$ for small $\varepsilon > 0$. If $C_\varepsilon$ is bounded, we get a constant $c_0 > 0$ such that $G_\varepsilon(a) \ge -c_0$ for small $\varepsilon > 0$. This implies that there exists a constant $c_1 > 0$ such that

$$z_\varepsilon'(r) \le c_1 \text{ and thus, } z_\varepsilon \ge c_1 a \text{ on } K, \tag{4.5}$$

for all small $\varepsilon$. Hence we have a constant $C > 0$ such that $\|z_\varepsilon\|_{C^1(K)} \le C$ uniformly for small $\varepsilon > 0$. On the other hand, for any compact subset $K \Subset [0, \infty)$, repeating the same argument as above, we get the desired uniform bound for $\|z_\varepsilon\|_{C^1(K)}$. This proves the claim. Consequently, we may pass to the limit in the equation (4.4). Now let us discuss the "limit domain". We have three possibilities,

1. $\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon} \to -\infty$;
2. $\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon} \to -l < 0$;
3. $\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon} \to 0$.

We will show that only case 3 occurs.

**Case 1: $\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon} \to -\infty$ cannot occur**

First we note that in this case we have that $\frac{M_\varepsilon}{\gamma_\varepsilon} \to +\infty$. Then, passing to the limit in (4.4), we get that there exists a function $z$ which satisfies $z_\varepsilon \to z$ in $C^2_{\text{loc}}(\mathbb{R})$ and

$$\begin{cases} -z'' = e^z & \text{in } \mathbb{R} \\ z(0) = z'(0) = 0. \end{cases} \tag{4.6}$$

Hence $z(s) = \log \frac{4e^{\sqrt{2}s}}{\left(1+e^{\sqrt{2}s}\right)^2}$. So we have that

$$
\begin{aligned}
&\int_{r_\varepsilon}^{1} |v_\varepsilon'|^2 r\,dr = \lambda r_{i,\varepsilon}^2 \int_{r_\varepsilon}^{1} v_\varepsilon^2 e^{v_\varepsilon^2+v_\varepsilon^{1+\varepsilon}} r\,dr\\
&= \lambda r_{i,\varepsilon}^2 \gamma_\varepsilon e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} v_\varepsilon^2(M_\varepsilon)\\
&\quad \times \int_{\frac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}}^{\frac{1-M_\varepsilon}{\gamma_\varepsilon}} \left(\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right)^2 \exp^{z_\varepsilon(r)\left(\frac{z_\varepsilon(r)}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)}\\
&\quad \times (M_\varepsilon+\gamma_\varepsilon r)\,dr\\
&\geq \lambda r_{i,\varepsilon}^2 \gamma_\varepsilon M_\varepsilon e^{v_\varepsilon^2(M_\varepsilon)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)} v_\varepsilon^2(M_\varepsilon)\\
&\quad \times \int_{0}^{\frac{1-M_\varepsilon}{\gamma_\varepsilon}} \left(\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right)^2 \exp^{z_\varepsilon(r)\left(\frac{z_\varepsilon(r)}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)} dr\\
&= \frac{M_\varepsilon}{2\gamma_\varepsilon} \int_{0}^{\frac{1-M_\varepsilon}{\gamma_\varepsilon}} \left(\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right)^2 \exp^{z_\varepsilon(r)\left(\frac{z_\varepsilon(r)}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)} dr.
\end{aligned}
$$

Here Fatou's lemma implies that

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{0}^{\frac{1-M_\varepsilon}{\gamma_\varepsilon}} \left(\frac{z_\varepsilon(r)}{2v_\varepsilon(M_\varepsilon)}+1\right)^2 \exp^{z_\varepsilon(r)\left(\frac{z_\varepsilon(r)}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)} dr\\
&\geq \int_0^{+\infty} e^{z(s)}\,dr > 0.
\end{aligned}
$$

Therefore by Lemma 2.2, we deduce a contradiction since $M_\varepsilon/\gamma_\varepsilon \to \infty$ as $\varepsilon \to 0$. This ends Case 1.

**Case 2: $\frac{r_\varepsilon - M_\varepsilon}{\gamma_\varepsilon} \to -l < 0$ cannot occur**

Noting $m := \lim\limits_{\varepsilon\to 0} \frac{M_\varepsilon}{\gamma_\varepsilon}$ and $m \geq l$, we get, passing to the limit in (4.4), that the weak limit $z$ satisfies

$$
\begin{cases}
-z'' - \dfrac{1}{m+r} z' = e^z & \text{in } (-l, +\infty)\\
z(r) \leq 0,\ z(0) = z'(0) = 0.
\end{cases}
$$

Then, setting $Z(s) = z(s-m)$ we derive that $Z$ satisfies

$$
\begin{cases}
-Z'' - \frac{1}{r} Z' = e^Z & \text{in } (m-l, +\infty)\\
Z(r) \leq 0,\ Z(m) = Z'(m) = 0.
\end{cases}
$$

This Cauchy problem admits the unique solution (see [17])

$$Z(s)=\log\frac{4\alpha^2m^{\alpha+2}s^{\alpha-2}}{((\alpha+2)m^\alpha+(\alpha-2)s^\alpha)^2},$$

where $\alpha=\sqrt{2m^2+4}$. Let us show $m=l$. To this end, we can proceed as in [17, Lemma 3.5]. For the sake of the completeness, we sketch it. We shall show that $z_\varepsilon((r_\varepsilon-M_\varepsilon)/\gamma_\varepsilon)\to-\infty$ implies that $m=l$. Indeed, arguing as above, we have that for any $r\in[(r_\varepsilon-M_\varepsilon)/\gamma_\varepsilon,0]$,

$$z_\varepsilon'(r)\left(\frac{M_\varepsilon}{\gamma_\varepsilon}+r\right)\le-\left(\frac{M_\varepsilon}{\gamma_\varepsilon}r+\frac{1}{2}r^2\right).$$

If by contradiction we have that $m>l$, we deduce that $\frac{M_\varepsilon}{\gamma_\varepsilon}+r\ge m-l+o(1)$ where $o(1)\to0$ as $\varepsilon\to0$ and then we get that

$$z_\varepsilon'(r)\le C \text{ in } [(r_\varepsilon-M_\varepsilon)/\gamma_\varepsilon,0]$$

for a constant $C>0$ which is independent of small $\varepsilon>0$. On the other hand, by the mean value theorem, since $z_\varepsilon((r_\varepsilon-M_\varepsilon)/\gamma_\varepsilon)\to-\infty$ and $z_\varepsilon(0)=0$ we deduce the existence of $\xi_\varepsilon\in\left(\frac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon},0\right)$ such that $z_\varepsilon'(\xi_\varepsilon)\to-\infty$ which gives a contradiction. So $m=l$. Now, from Lemmas 2.5, A.1 and the blow-up procedure as above, we get

$$\begin{aligned}2&=\lambda r_{i,\varepsilon}^2\int_{r_\varepsilon}^1 v_\varepsilon^2e^{v_\varepsilon^2+v_\varepsilon^{1+\varepsilon}}r\,dr+o(1)\\&=\frac12\int_{\frac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}}^{\frac{1-M_\varepsilon}{\gamma_\varepsilon}}\left(\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right)^2\\&\quad\times\exp^{z_\varepsilon(r)\left(\frac{z_\varepsilon(r)}{4v_\varepsilon^2(M_\varepsilon)}+1\right)+v_\varepsilon^{1+\varepsilon}(M_\varepsilon)\left(\left|\frac{z_\varepsilon(r)}{2v_\varepsilon^2(M_\varepsilon)}+1\right|^{1+\varepsilon}-1\right)}\left(\frac{M_\varepsilon}{\gamma_\varepsilon}+r\right)dr+o(1),\end{aligned}$$

where $o(1)\to0$ as $\varepsilon\to0$. Then using $m=l>0$ and Fatou's Lemma, we obtain

$$2\ge\frac12\int_0^\infty e^{Z(s)}s\,ds=\sqrt{2m^2+4}>2,$$

a contradiction. This finishes Case 2.

**Case 3: $\frac{r_\varepsilon-M_\varepsilon}{\gamma_\varepsilon}\to0$ occurs.**

Repeating the procedure in Case 2 we can show $m=l=0$. As a consequence, we deduce

$$z_\varepsilon\to z \text{ in } C_{\mathrm{loc}}([0,\infty))\cap C^2_{\mathrm{loc}}((0,\infty))$$

and then, $z$ satisfies

$$\begin{cases} -z'' - \frac{1}{r}z' = e^z & \text{in } (0, +\infty) \\ z(r) \le 0, \ z(0) = 0. \end{cases}$$

The previous equation can be integrated giving the solutions (see [17, pages 744-745])

$$z(r) = \log\left(\frac{4}{\delta^2}\frac{e^{\sqrt{2}\frac{\log r - y}{\delta}}}{\left(1 + e^{\sqrt{2}\frac{\log r - y}{\delta}}\right)^2}\right) - 2\log r \tag{4.7}$$

for some constants $\delta \neq 0$, $y \in \mathbb{R}$. Moreover a direct calculation shows

$$z(r) = 2\log\frac{2}{\delta} - \frac{\sqrt{2}}{\delta}y + \left(\frac{\sqrt{2}}{\delta} - 2\right)\log r - 2\log\left(1 + e^{\sqrt{2}\frac{\log r - y}{\delta}}\right).$$

Since $z(0) = 0$, we must have $\delta = 1/\sqrt{2}$. Then we clearly deduce $y = \log 2\sqrt{2}$. This completes the proof. $\square$

*Proof of Proposition* 4.1. The proposition follows from Lemmas 2.5, A.1 and 4.3. $\square$

**Remark 4.4.** If we consider a radial nodal solution $u_p$ to the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{4.8}$$

then in [17, Proposition 3.1] it was proved that Case 2 occurs for some suitable $m < 0$. This shows that the shape of the nonlinearity plays a crucial role.

## 5. Behavior of $u_\varepsilon$ in $B \setminus B_{r_{k,\varepsilon}}$

Next we show the behavior on $B \setminus B_{r_{k,\varepsilon}}$. We set $u_{k+1,\varepsilon} := u_\varepsilon|_{B\setminus B_{r_{k,\varepsilon}}} \in H_0^1(B)$ by zero extension. Then we have the following:

**Proposition 5.1.** *We get*

$$u_{k+1,\varepsilon} \to u_0 \ \textit{in} \ H_0^1(B),$$

*as* $\varepsilon \to 0$ *where* $u_0$ *is the least energy solution of* (1.6).

First observe that we have already proved

$$0 < \lim_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = \inf_{u\in\mathcal{N}_0} I_0(u) < 2\pi$$

by Lemma 2.6. This means that the energy of $u_{k+1,\varepsilon}$ belongs to the suitable compactness region for Palais-Smale sequences [1]. Although we do not ensure $\lim_{\varepsilon\to 0} I'_\varepsilon(u_{k+1,\varepsilon}) = 0$, we can accomplish the proof by the argument based on Lions' concentration compactness result [22]. We refer to the proof in [1] (and also [15]).

*Proof of Proposition* 5.1. Since $u_\varepsilon$ is bounded, we can assume, by choosing a sequence if necessary, that there exists a function $u_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(B),\\ &u_\varepsilon \to u_0 \text{ in } L^p(B) \text{ for all } p \geq 1,\\ &u_\varepsilon \to u_0 \text{ a.e. on } B \end{aligned} \tag{5.1}$$

as $\varepsilon \to 0$. Then, since $u_{i,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(B)$ for all $i = 1, 2, \cdots, k$, we also have

$$\begin{aligned} &u_{k+1,\varepsilon} \rightharpoonup u_0 \text{ weakly in } H_0^1(B),\\ &u_{k+1,\varepsilon} \to u_0 \text{ in } L^p(B) \text{ for all } p \geq 1,\\ &u_{k+1,\varepsilon} \to u_0 \text{ a.e. on } B, \end{aligned} \tag{5.2}$$

as $\varepsilon \to 0$. Furthermore, since $\langle I'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = 0$, we get $\int_B f_\varepsilon(u_\varepsilon)u_\varepsilon dx$ is bounded. Then Lemma A.1 implies $f_\varepsilon(u_\varepsilon) \to f_0(u_0)$ in $L^1(B)$. We claim that $u_0$ is a weak solution of (1.6) with $\varepsilon = 0$. In fact, for all $\psi \in C_0^\infty(B)$, we get by the weak convergence of $u_\varepsilon$ and $L^1(B)$ convergence of $f_\varepsilon(u_\varepsilon)$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\{ \int_B \nabla u_\varepsilon \nabla\psi dx - \int_B f_\varepsilon(u_\varepsilon)\psi dx \right\}\\ &= \int_B \nabla u_0 \nabla\psi dx - \int_B f_0(u_0)\psi dx. \end{aligned}$$

By a density argument we prove the claim. Next we shall show that there exists a constant $q > 1$ such that

$$\int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx \text{ is bounded.} \tag{5.3}$$

To see this, we observe that for a constant $\beta > 1$, which will be determined later, there exists $C > 0$ such that $|f_\varepsilon(t)| \leq Ce^{\beta t^2}$ for all $t \in \mathbb{R}$ and small $\varepsilon > 0$. Then for $q > 1$, which will be also chosen later, we get

$$\int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx \leq C\int_B e^{q\beta u_{k+1,\varepsilon}^2} dx = C\int_B e^{q\beta \|u_{k+1,\varepsilon}\|^2 v_\varepsilon^2} dx\,,$$

where we set $v_\varepsilon := u_{k+1,\varepsilon}/\|u_{k+1,\varepsilon}\|$. Notice $\|v_\varepsilon\| = 1$ and $v_\varepsilon \rightharpoonup v_0$ weakly in $H_0^1(B)$ for a function $v_0$ with $0 \le \|v_0\| \le 1$. We claim that $v_0 \ne 0$. If on the contrary $v_0 = 0$ we get $u_0 = 0$. Then Lemma A.1 shows $\int_B F_\varepsilon(u_{k+1,\varepsilon})dx \to 0$ as $\varepsilon \to 0$. It follows that

$$0 < 2 \lim_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) = \lim_{\varepsilon\to 0} \|u_{k+1,\varepsilon}\|^2 < 4\pi. \tag{5.4}$$

Consequently we can choose $\beta, q > 1$ so that

$$\int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx \le C \int_B e^{q\beta\|u_{k+1,\varepsilon}\|^2 v_\varepsilon^2} dx \le C \int_B e^{4\pi v_\varepsilon^2} dx$$

for small $\varepsilon > 0$. Notice that the Trudinger-Moser inequality implies that the right-hand side is bounded uniformly for small $\varepsilon > 0$. Now setting $q' > 1$ so that $1/q + 1/q' = 1$, we get by the Hölder inequality that

$$\begin{aligned}
\|u_{k+1,\varepsilon}\|^2 &= \int_B u_{k+1,\varepsilon} f_\varepsilon(u_{k+1,\varepsilon}) dx \\
&\le \left(\int_B |u_{k+1,\varepsilon}|^{q'} dx\right)^{\frac{1}{q'}} \left(\int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx\right)^{\frac{1}{q}} \\
&\le C \left(\int_B |u_{k+1,\varepsilon}|^{q'} dx\right)^{\frac{1}{q'}},
\end{aligned}$$

for a constant $C > 0$ if $\varepsilon > 0$ is small enough. Hence, we get $u_{k+1,\varepsilon} \to 0$ in $H_0^1(B)$ by (5.2). This contradicts (5.4). Therefore, we can assume $0 < \|v_0\| < 1$. (If $\|v_0\| = 1$, we finish the proof.) Then Lions' concentration compactness lemma ([22, Theorem I.6 ]) proves

$$\int_B e^{4\pi p v_\varepsilon^2} dx \text{ is bounded for all } p < \frac{1}{1 - \|v_0\|^2}. \tag{5.5}$$

Now recalling the facts that $\lim_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) < 2\pi$, $f_0(t)t - 2F_0(t) \ge 0$ for all $t \in \mathbb{R}$ and $\langle I_0'(u_0), u_0\rangle = 0$, we get a constant $\delta \in (0, 1)$ such that

$$\begin{aligned}
4\pi(1-\delta) &= 2 \lim_{\varepsilon\to 0} I_\varepsilon(u_{k+1,\varepsilon}) \\
&= \lim_{\varepsilon\to 0} \|u_{k+1,\varepsilon}\|^2 - 2\int_B F_0(u_0)dx - \langle I_0'(u_0), u_0\rangle \\
&\ge \lim_{\varepsilon\to 0} \|u_{k+1,\varepsilon}\|^2 - \|u_0\|^2 \\
&= \lim_{\varepsilon\to 0} \|u_{k+1,\varepsilon}\|^2 \big(1 - \|v_0\|^2\big).
\end{aligned}$$

This shows

$$q\beta \lim_{\varepsilon\to 0} \|u_{k+1,\varepsilon}\|^2 \le \frac{4\pi q\beta(1-\delta)}{1 - \|v_0\|^2}.$$

Put $p := q\beta(1-\delta)/(1-\|v_0\|^2)$. Then, we can choose $\beta, q > 1$ so that $p < 1/(1-\|v_0\|^2)$ and

$$\int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx \le C \int_B e^{4\pi p v_\varepsilon^2} dx,$$

for small $\varepsilon > 0$. Then (5.5) proves (5.3). Now choose $u_{0,\varepsilon} \in H_0^1(B)$ such that $u_{0,\varepsilon} = 0$ on $\overline{B_{r_{k,\varepsilon}}}$ and $u_{0,\varepsilon} \to u_0$ in $H_0^1(B)$ as $\varepsilon \to 0$. Define for example $u_{0,\varepsilon} := \phi_{r_{k,\varepsilon},1} u_0$ where $\phi_{r_{k,\varepsilon},1}$ is a cut off function defined as in Section 2. Then integration by parts gives that

$$\begin{aligned}\int_B \nabla u_{k+1,\varepsilon} \nabla (u_{k+1,\varepsilon} - u_{0,\varepsilon}) dx &= \int_{B \setminus B_{r_{k,\varepsilon}}} (-\Delta u_{k+1,\varepsilon})(u_{k+1,\varepsilon} - u_{0,\varepsilon}) dx \\ &= \int_B f_\varepsilon(u_{k+1,\varepsilon})(u_{k+1,\varepsilon} - u_{0,\varepsilon}) dx.\end{aligned}$$

Now again let $q' > 1$ be a constant such that $q^{-1} + q'^{-1} = 1$. Then setting $o(1) \to 0$ as $\varepsilon \to 0$, and using the Hölder inequality, (5.3), and (5.2), we get

$$\begin{aligned}\|u_{k+1,\varepsilon}\|^2 - \|u_0\|^2 &= \int_B \nabla u_{k+1,\varepsilon} \nabla (u_{k+1,\varepsilon} - u_{0,\varepsilon}) dx + o(1) \\ &= \int_B f_\varepsilon(u_{k+1,\varepsilon})(u_{k+1,\varepsilon} - u_{0,\varepsilon}) dx + o(1) \\ &\le \left( \int_B |f_\varepsilon(u_{k+1,\varepsilon})|^q dx \right)^{\frac{1}{q}} \left( \int_B |u_{k+1,\varepsilon} - u_{0,\varepsilon}|^{q'} dx \right)^{\frac{1}{q'}} + o(1) \\ &\to 0\end{aligned}$$

as $\varepsilon \to \infty$. Hence we get $u_{k+1,\varepsilon} \to u_0$ in $H_0^1(B)$ as $\varepsilon \to 0$. Finally, Lemma 2.6 proves that $u_0$ is the least energy solution of (1.6). This completes the proof. □

**Remark 5.2.** From the result above, we get $\|u_{k+1,\varepsilon}\|_{L^\infty((r_{k,\varepsilon},1))}$ is bounded. To see this, observe that the strong convergence of $u_{k+1,\varepsilon}$ implies that for all $q > 1$, $e^{u_{k+1,\varepsilon}^2}$ is bounded in $L^q(B)$ uniformly for small $\varepsilon > 0$. Set $r_{k+1,\varepsilon}^* \in (r_{k,\varepsilon}, 1)$ so that $u_{r_{k+1,\varepsilon}}(r_{k+1,\varepsilon}^*) = \|u_{k+1,\varepsilon}\|_{L^\infty((r_{k,\varepsilon},1))}$. Then we get

$$\begin{aligned}|u_{k+1,\varepsilon}(r_{k+1,\varepsilon}^*)| &= \left| \int_{r_{k+1,\varepsilon}^*}^1 f_\varepsilon(u_{k+1,\varepsilon}) r \log r \, dr \right| \\ &\le \left( \int_{r_{k+1,\varepsilon}^*}^1 f_\varepsilon(u_{k+1,\varepsilon})^2 r \, dr \right)^{\frac{1}{2}} \left( \int_{r_{k+1,\varepsilon}^*}^1 r \log^2 r \, dr \right)^{\frac{1}{2}} \\ &\le C\end{aligned}$$

for a constant $C > 0$ if $\varepsilon > 0$ is sufficiently small. This proves the claim.

**Remark 5.3.** The previous remark shows

$$\lim_{\varepsilon\to 0} r_{k,\varepsilon} u_\varepsilon'(r_{k,\varepsilon}) = 0.$$

To show this, set $r^*_{k+1,\varepsilon} \in (r_{k,\varepsilon}, 1)$ as above. First observe that $r^*_{k+1,\varepsilon} \to 0$ as $\varepsilon \to 0$. If not, we have a constant $r_0 \in (0,1)$ such that $r^*_{k+1,\varepsilon} \to r_0$ as $\varepsilon \to 0$ by choosing a sequence if necessary. Then since $u_{k+1,\varepsilon} \to u_0$ a.e. on $B$, we have $u_0(r) \le u_0(r_0)$ for a.e. $r \in (0, r_0)$. But, since $u_0$ is a positive radial solution of (1.6), the result in [16] shows $u'(r) < 0$ for all $r \in (0,1)$. This is a contradiction. Finally, integrating (1.8) over $(r_{k,\varepsilon}, r^*_{k+1,\varepsilon})$, we get by the previous remark that

$$|r_{k,\varepsilon} u_\varepsilon'(r_{k,\varepsilon})| = \left| \int_{r_{k,\varepsilon}}^{r^*_{k+1,\varepsilon}} f_\varepsilon(u_\varepsilon) r dr \right| \le C r^*_{k+1,\varepsilon},$$

for some constant $C > 0$. This completes the proof.

## 6. Proof of the main theorems

We finally conclude the proof of our main theorems.

*Proof of Theorem* 1.2. The proof of (1.10) is given in Lemma 2.4, (1.11) is shown in Corollary 3.2, Proposition 4.1 and Proposition 5.1, (1.12) is shown in Corollary 3.2, Proposition 4.1 and Proposition 5.1. So we have only to show (1.9), *i.e.*

$$u_\varepsilon \to (-1)^k u_0 \text{ in } C^2_{\mathrm{loc}}((0,1]),$$

as $\varepsilon \to 0$. To prove this, we may assume that $u_\varepsilon$ satisfies

$$\begin{cases} -u_\varepsilon'' - \frac{1}{r} u_\varepsilon' = \lambda u_\varepsilon e^{u_\varepsilon^2 + u_\varepsilon^{1+\varepsilon}}, \ u_\varepsilon > 0 \ \text{ in } (r_{k,\varepsilon}, 1) \\ u_\varepsilon(r_{k,\varepsilon}) = u_\varepsilon(1) = 0. \end{cases} \tag{6.1}$$

Now choose any compact subset $K \Subset (0,1]$. For all $r \in K$, we may suppose $r_{k,\varepsilon} < r$ by Lemma 2.4. Then multiplying (6.1) by $r$ and integrating over $(r_{k,\varepsilon}, r)$ we have

$$r u_\varepsilon'(r) = r_{k,\varepsilon} u_\varepsilon'(r_{k,\varepsilon}) - \int_{r_{k,\varepsilon}}^{r} f_\varepsilon(u_\varepsilon) r dr.$$

Then Remark 5.3 and Lemma A.1 prove that $r u_\varepsilon'$ is bounded uniformly on $K$ for small $\varepsilon$. In particular, $u_\varepsilon'$ is bounded uniformly on $K$ for small $\varepsilon$. Furthermore, since for any $r \in K$, we have

$$u_\varepsilon(r) = -\int_r^1 u_\varepsilon'(s) ds,$$

we derive $\|u_\varepsilon\|_{C^1(K)}$ is bounded uniformly for small $\varepsilon$. Then the Arzelà-Ascoli theorem ensures $u_\varepsilon \to u_0$ uniformly on $K$ as $\varepsilon \to 0$. Finally, using (6.1), we show $u_\varepsilon \to u_0$ in $C^2(K)$ as $\varepsilon \to 0$. This finishes the proof. □

*Proof of Theorem* 1.3. For $i = 0$ the proof is given in Proposition 3.1. The case $i = 1, \dots, k$ is considered in Proposition 4.1. $\square$

Using the blow-up results above, we get the following remark:

**Remark 6.1.** We have

$$\lim_{\varepsilon\to 0} \frac{\|u_\varepsilon\|_{L^\infty((r_{i,\varepsilon}, r_{i+1,\varepsilon}))}}{\|u_\varepsilon\|_{L^\infty((r_{i-1,\varepsilon}, r_{i,\varepsilon}))}} = 0 \tag{6.2}$$

for all $i = 1, 2, \cdots, k$. Let us show the proof. For $i = k$, the proof is obvious by Corollary 3.2, Lemma 4.2 and Remark 5.2. Then for $i = 1, 2, \cdots, k-1$, set $M_{i,\varepsilon} \in [0, 1)$ so that $|u_\varepsilon(M_{i,\varepsilon} r_{i,\varepsilon})| = \|u_\varepsilon\|_{L^\infty((r_{i-1,\varepsilon}, r_{i,\varepsilon}))}$, $r^*_{i,\varepsilon} := M_{i,\varepsilon} r_{i,\varepsilon}$ and $\gamma_{i,\varepsilon} = 2u_\varepsilon(r^*_{i,\varepsilon}) f_\varepsilon(u_\varepsilon(r^*_{i,\varepsilon})) > 0$. Then integrating (1.8) over $(r^*_{i,\varepsilon}, r^*_{i+1,\varepsilon})$ shows

$$\int_{r^*_{i,\varepsilon}}^{r_{i,\varepsilon}} f_\varepsilon(u_\varepsilon) r\,dr = -\int_{r_{i,\varepsilon}}^{r^*_{i+1,\varepsilon}} f_\varepsilon(u_\varepsilon) r\,dr.$$

Hence putting $v_{i,\varepsilon}(r) = |u_{i,\varepsilon}(r_{i,\varepsilon} r)|$, $v_{i+1,\varepsilon}(r) = |u_{i+1,\varepsilon}(r_{i+1,\varepsilon} r)|$ and $z_{j,\varepsilon}(r) = 2|u_\varepsilon(r^*_{j,\varepsilon})|(|v_{j,\varepsilon}(\gamma_{j,\varepsilon} r + M_{j,\varepsilon})| - |u_\varepsilon(r^*_{j,\varepsilon}))|)$ for $j = i, i+1$, we get by the blow-up procedure as above,

$$\begin{aligned}
&\frac{1}{2|u_\varepsilon(r^*_{i,\varepsilon})|}\int_0^{\frac{1-M_{i,\varepsilon}}{\gamma_{i,\varepsilon}}} \left(\frac{z_{i,\varepsilon}(r)}{2v^2_{i,\varepsilon}(M_{i,\varepsilon})} + 1\right) \\
&\quad\times e^{z_{i,\varepsilon}(r)\left(\frac{z_{i,\varepsilon}(r)}{4v^2_{i,\varepsilon}(M_{i,\varepsilon})}+1\right) + v^{1+\varepsilon}_{i,\varepsilon}(M_{i,\varepsilon})\left(\left|\frac{z_{i,\varepsilon}(r)}{2v^2_{i,\varepsilon}(M_{i,\varepsilon})}+1\right|^{1+\varepsilon} - 1\right)} \left(\frac{M_{i,\varepsilon}}{\gamma_{i,\varepsilon}} + r\right) dr \\
&= \frac{1}{2|u_\varepsilon(r^*_{i+1,\varepsilon})|}\int_{\frac{\frac{r_{i,\varepsilon}}{r_{i+1,\varepsilon}} - M_{i+1,\varepsilon}}{\gamma_{i+1,\varepsilon}}}^{0} \left(\frac{z_{i+1,\varepsilon}(r)}{2v^2_{i+1,\varepsilon}(M_{i+1,\varepsilon})} + 1\right) \times \\
&\quad\times e^{z_{i+1,\varepsilon}(r)\left(\frac{z_{i+1,\varepsilon}(r)}{4v^2_{i+1,\varepsilon}(M_\varepsilon)}+1\right) + v^{1+\varepsilon}_{i+1,\varepsilon}(M_{i+1,\varepsilon})\left(\left|\frac{z_{i+1,\varepsilon}(r)}{2v^2_{i+1,\varepsilon}(M_{i+1,\varepsilon})}+1\right|^{1+\varepsilon} - 1\right)} \left(\frac{M_{i+1,\varepsilon}}{\gamma_{i+1,\varepsilon}} + r\right) dr \\
&= o\left(\frac{1}{2|u_\varepsilon(r^*_{i+1,\varepsilon})|}\right)
\end{aligned}$$

as $\varepsilon \to 0$ since $\lim_{\varepsilon\to 0} \frac{r_{i,\varepsilon}/r_{i+1,\varepsilon} - M_{i+1,\varepsilon}}{\gamma_{i+1,\varepsilon}} = 0$. Finally using our blow-up results and the Fatou lemma for the integral on the left-hand side, we get the desired conclusion.

## Appendix A. Some basic facts

In the following, let $(\varepsilon_n) \subset \mathbb{R}^+$ be any sequence such that $\varepsilon_n \to 0$ as $n \to \infty$:

**Lemma A.1.** *Let $(u_n) \subset H_0^1(B)$ be a bounded sequence such that $u_n \rightharpoonup u$ weakly in $H_0^1(B)$ and $u_n \to u$ a.e. on $B$ as $n \to \infty$ for a function $u$. Furthermore, assume*

$$\sup_n \int_B f_{\varepsilon_n}(u_n)u_n dx < \infty.$$

*Then we have*

$$\lim_{n\to\infty} \int_B f_{\varepsilon_n}(|u_n|)dx = \int_B f_0(|u|)dx$$

*and*

$$\lim_{n\to\infty} \int_B F_{\varepsilon_n}(u_n)dx = \int_B F_0(u)dx.$$

*Proof.* Similar to the proof of 4) of [2, Lemma 3.1]. □

**Lemma A.2.** *We have*

$$\liminf_{\varepsilon\to 0} \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) > 0.$$

*Proof.* If not, we have sequences $(\varepsilon_n) \subset \mathbb{R}^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ such that $\lim_{n\to\infty} I_{\varepsilon_n}(u_n) = 0$. Then since $\lambda < \lambda_1$, analogously with Step 1 in [2, proof of Lemma 3.4], we can get a contradiction. This proves the lemma. □

**Lemma A.3.** *Let $(\mu_n) \subset \mathbb{R}^+$ and $(u_n) \subset H_0^1(B)$ be sequences such that $\mu_n \le 1$ for all $n$ and further,*

$$J_n(u_n) := \int_B |\nabla u_n|^2 dx - \mu_n \int_B F_{\varepsilon_n}(u_n)dx \to c \in (0, 2\pi) \text{ and}$$
$$J_n'(u_n) \to 0 \text{ in } H^{-1}(B),$$

*as $n \to \infty$. Then $u_n \to u$ in $H_0^1(B)$ up to a subsequence.*

*Proof.* Similar to 1) in [2, page 404 ]. □

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