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Distortion in Cremona groups

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Abstract. We study the distortion of elements in two-dimensional Cremona groups over algebraically closed fields of characteristic zero. We obtain the following trichotomy: non-elliptic elements (*i.e.*, those whose powers have unbounded degree) are undistorted, and elliptic elements have a doubly exponential distortion when they are virtually unipotent or an exponential distortion otherwise.

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1. Introduction

Let $\mathbf{k}$ be an algebraically closed field. The goal of this paper is to study the distortion in the Cremona group $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. We characterize distorted elements, and study their distortion function. The three main tools are:

(1) An upper bound on the distortion which is obtained via height estimates, using basic number theory (this holds in arbitrary dimension);
(2) A result of Blanc and Déserti concerning base points of birational transformations of the plane;
(3) A non-distortion result for parabolic elements in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, obtained via Noether inequalities and the study of the action of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ on the Picard-Manin space (an infinite-dimensional hyperbolic space).

This third step sheds new light on the geometry of the action of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ on this hyperbolic space.

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1.1. Distortion

If f and g are two real-valued functions on $\mathbf{R}_+$, we write $f \preceq g$ if there exist three positive constants C, C', C'' such that $f(x) \leq Cg(C'x) + C''$ for all $x \in \mathbf{R}_+$. We write $f \simeq g$ when $f \preceq g \preceq f$.

Let G be a group. If S and T are two subsets of G containing the neutral element 1, we write $S \preceq T$ if $S \subset T^k$ for some integer $k \geq 0$, and $S \simeq T$ if $S \preceq T \preceq S$. Let c be an element of G. Let S be a finite symmetric subset of G containing 1; if the subgroup G_S generated by S contains c, we define the *distortion function*

$$\delta_{c,S}(n) = \sup\left\{m \in \mathbf{N} : c^m \in S^n\right\}.$$

Definition 1.1. By definition, $\delta_{c,S}(n) = \infty$ if and only if c has finite order. Clearly, if $S \subset T$ then $\delta_{c,S} \leq \delta_{c,T}$. Also, $\delta_{c,S^k}(n) = \delta_{c,S}(kn)$. In particular, if $S \subset T^k$, then $\delta_{c,S} \leq \delta_{c,T}(kn)$. If $S \preceq T$, it follows that $\delta_{c,S} \preceq \delta_{c,T}$, and if $S \simeq T$ then $\delta_{c,S} \simeq \delta_{c,T}$.

If S and T both generate G then $S \simeq T$ and $\delta_{c,S} \simeq \delta_{c,T}$. Thus, when G is finitely generated, the $\simeq$-equivalence class of the distortion function only depends on (G, c), not on the finite generating subset; it is called the distortion function of c in G, and is denoted δ_c^G, or simply δ_c. The element c is called *undistorted* if $\delta_c(n) \preceq n$, and *distorted* otherwise.

Example 1.2. Fix a pair of integers $k, \ell \geq 2$. In the Baumslag-Solitar group $B_k = \langle t, x \mid \ txt^{-1} = x^k\rangle$, we have $\delta_x^{B_k}(n) \simeq \exp(n)$. In the "double" Baumslag-Solitar group $B_{k,\ell}$ (see [22, Section 8]), one finds double exponential distortion.

It is natural to consider distortion in groups that are not finitely generated. We say that an element $c \in G$ is *undistorted* if $\delta_c^H(n) \simeq n$ for every finitely generated subgroup H of G containing c. Changing H may change the distortion function δ_c^H; for instance, if c is not a torsion element, it is undistorted in $H = c^{\mathbf{Z}}$ but may be distorted in larger groups. Also, there are examples of pairs (G, c) such that c becomes more and more distorted, in larger and larger subgroups of G (see Section 8). Thus, we have a good notion of distortion, but the distortion is not measured by an equivalence class of a function "δ_c^G ".

We shall say that the *distortion type* (or class) of c in G is at least f if there is a finitely generated subgroup H containing c with $f \preceq \delta_c^H$, and is at most g if $\delta_c^H \preceq g$ for all finitely generated subgroup H containing c. If the distortion type is at least f and at most f simultaneously, we shall say that f is the distortion type of c. For instance, c may be exponentially, or doubly exponentially distorted in G.

Example 1.3. Let $\mathbf{k}$ be a field. Let c be an element of the general linear group $\mathsf{GL}_d(\mathbf{K})$; we have one of the following (see [24,25, Section 3])

- c is not virtually unipotent, *i.e.*, at least one of its eigenvalues in an algebraic closure of $\mathbf{k}$ is not a root of unity, and then c is undistorted;
- c is virtually unipotent of infinite order, and then $\delta_c(n) \simeq \exp(n)$ (this occurs only if $\mathbf{k}$ has characteristic zero);
- c has finite order.

The dimension $d \geq 0$ does not intervene in this description. In contrast, the unipotent elementary matrix $e_{12}(1) = \mathrm{Id} + \delta_{1,2}$ is undistorted in $\mathsf{SL}_2(\mathbf{Z})$ but has exponential distortion in $\mathsf{SL}_d(\mathbf{Z})$ for $d \geq 3$.

1.2. Distortion in Cremona groups

Distortion in groups of homeomorphisms is an active subject (see [2,9,23,27,28]). For instance, in the group of homeomorphisms of the sphere $\mathbb{S}^d$, every element is distorted. Our goal in this paper is to study distortion in groups of birational transformations.

If M is a projective variety over a field $\mathbf{k}$, we denote by $\mathsf{Bir}(M_{\mathbf{k}})$ its group of birational transformations over $\mathbf{k}$. When M is the projective space $\mathbb{P}^m_{\mathbf{k}}$, this group is the *Cremona group* in m variables $\mathrm{Cr}_m(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}}) = \mathsf{Bir}(\mathbb{A}^m_{\mathbf{k}})$. The problem is to describe the elements of $\mathsf{Bir}(M_{\mathbf{k}})$ which are distorted in $\mathsf{Bir}(M_{\mathbf{k}})$, and to estimate their distortion functions.

1.2.1. *Degree sequences*

Let H be a hyperplane section of M, for some fixed embedding $M \subset \mathbb{P}^N_{\mathbf{k}}$. The *degree* of a birational transformation $f\colon M \dashrightarrow M$ with respect to the polarization H is the intersection product $\deg_H(f) = H^{m-1} \cdot f^*(H)$, where $m = \dim(M)$. When M is $\mathbb{P}^m_{\mathbf{k}}$ and H is a hyperplane, then $\deg_H(f)$ is the degree of the homogeneous polynomial functions f_i, without common factor of positive degree, such that $f = [f_0 : \cdots : f_m]$ in homogeneous coordinates.

The degree function is almost submultiplicative (see [17,29,32]): there is a constant $C_{M,H}$ such that for all f and g in $\mathsf{Bir}(M_{\mathbf{k}})$

$$\deg_H(f \circ g) \leq C_{M,H} \deg_H(f) \deg_H(g). \tag{1.1}$$

Thus, we can define the *dynamical degree* $\lambda_1(f)$ by

$$\lambda_1(f) = \lim_{n\to+\infty} \left(\deg_H(f^n)^{1/n}\right).$$

By definition, $\lambda_1(f) \geq 1$, and the following well-known lemma implies that $\lambda_1(f) = 1$ when f is distorted (see Section 2).

Lemma 1.4. *Let G be a group with a finite symmetric generating subset S. Let $|w|$ denote the word length of $w \in G$ with respect to the generating subset S. Then:*

(1) *$|\cdot|$ is sub-additive: $|vw| \leq |v| + |w|$;*
(2) *The* stable length $\mathrm{sl}(c) := \lim_{n\to\infty} \frac{1}{n}|c^n|$ *is a well-defined element of* $\mathbf{R}_+$*;*
(3) *c is distorted if and only if* $\mathrm{sl}(c) = 0$.

1.2.2. *Distortion in dimension* 2

Assume, for simplicity, that the field $\mathbf{k}$ is algebraically closed. Typical elements of $\mathrm{Cr}_d(\mathbf{k})$ have dynamical degree > 1. At the opposite, we have the notion of *algebraic elements*. A birational transformation $f\colon M \dashrightarrow M$ is *algebraic*, or *bounded*, if $(\deg_H(f^n))_{n\geq 0}$ is a bounded sequence of integers; by a theorem of Weil (see [34]), f is bounded if and only if there exists a projective variety M', a birational map $\varphi\colon M' \dashrightarrow M$, and an integer $m > 0$, such that $\varphi^{-1}\circ f^m \circ \varphi$ is an element of $\mathsf{Aut}(M')^0$ (the connected component of the identity in the group of automorphisms $\mathsf{Aut}(M')$). In the case of surfaces, bounded elements are also called *elliptic*; we shall explain this terminology in Section 4.

Theorem 1.5. *Let* $\mathbf{k}$ *be a field. If an element* $f \in \mathrm{Cr}_2(\mathbf{k})$ *is distorted, then* f *is elliptic. If* $\mathbf{k}$ *is algebraically closed and of characteristic* 0*, and* $f \in \mathrm{Cr}_2(\mathbf{k})$ *is elliptic and of infinite order, then:*

- *If some positive power of* f *is conjugate to a unipotent automorphism of* $\mathbb{P}^2_{\mathbf{k}}$*, then* f *has double exponential distortion;*
- *Otherwise,* f *has exponential distortion.*

The first assertion extends to $\mathsf{Bir}(X)$ for all projective surfaces (see Theorems 6.5 and 7.1), but the second does not. For instance, if X is a complex Abelian surface and $\mathsf{Aut}(X)$ has only finitely many connected components, every translation of infinite order is undistorted and elliptic.

Consider, in $\mathrm{Cr}_2(\mathbf{k})$, the element $(x, y) \overset{s}{\mapsto} (x, xy)$; it is not elliptic and by the above theorem, it is not distorted in $\mathrm{Cr}_2(\mathbf{k})$. On the other hand, the natural embedding $\mathrm{Cr}_2(\mathbf{k}) \subset \mathrm{Cr}_3(\mathbf{k})$ maps it to $(x, y, z) \mapsto (x, xy, z)$, which is exponentially distorted in $\mathsf{Bir}(\mathbb{A}^3_{\mathbf{k}})$, while its degree growth remains linear. Thus Theorem 1.5 is specific to the projective plane.

Question 1.6. (see Section 3)

(A) In Theorem 1.5, can we remove the restriction concerning the characteristic or the algebraic closedness of the field $\mathbf{k}$?
(B) Can we find an element of infinite order with more than double exponential distortion in the Cremona group $\mathrm{Cr}_m(\mathbf{C})$, for some $m \geq 3$?

1.3. Hyperbolic spaces, horoballs, and distortion

Our proof of Theorem 1.5 makes use of the action of $\mathrm{Cr}_2(\mathbf{k})$ on an infinite dimensional hyperbolic space $\mathbb{H}_\infty$, already at the heart of several articles (see [13]). There are elements f of $\mathrm{Cr}_2(\mathbf{k})$ acting as parabolic isometries on $\mathbb{H}_\infty$, with a unique fixed point ξ_f at the boundary of the hyperbolic space. We shall show that the orbit of a sufficiently small horoball centered at ξ_f under the action of $\mathrm{Cr}_2(\mathbf{k})$ is made of a family of pairwise disjoint horoballs. We refer to Theorem C in Section 6 for that result. Theorem B, proved in Section 4, is a general result for groups acting by isometries on hyperbolic spaces that provides a control of the distortion of parabolic elements.

1.4. Remark

One step towards Theorem 1.5 is to prove that the so-called Halphen twists of $\mathsf{Cr}_2(\mathbf{k})$ (a certain type of parabolic elements) are not distorted. Blanc and Furter obtained simultaneously another proof of that result; instead of looking at the geometry of horoballs, as in our Theorem 4.1, they prove a very nice result on the length of elements of $\mathsf{Cr}_2(\mathbf{k})$ in terms of the generators provided by Noether-Castelnuovo theorem (the generating sets being $\mathsf{PGL}_3(\mathbf{k})$ and transformations preserving a pencil of lines). Our proof applies directly to Halphen twists on non-rational surfaces.

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2. Degrees and upper bounds on the distortion

The following proposition shows that the degree growth may be used to control the distortion of a birational transformation.

Proposition 2.1. *Let (M, H) be a polarized projective variety, and f be a birational transformation of M.*

(1) *If $\deg_H(f^n)$ grows exponentially, then f is undistorted;*
(2) *If $\deg(f^n) \succeq n^\alpha$ for some $\alpha > 0$, the distortion of f is at most exponential.*

Proof. According to Equation (1.1), the degree function is almost submultiplicative; replace it by $\deg'_H(f) := \deg_H(f)/C_{M,H}$ to get a submultiplicative function.

If S is a finite symmetric subset of $\mathsf{Bir}(M)$, and D is the maximum of $\deg'_H(g)$ for g in S, then D^n is an upper bound for $\deg'_H$ on the ball S^n. Hence if $\deg'(f^m) \geq Cq^m$ for some constants $C>0$ and $q>1$, and if $f^m \in S^n$ we have $Cq^m \leq D^n$. Taking logarithm, we get $m\log(q)+C \leq n\log(D)$, and then $m \leq \log(q)^{-1}(n\log(D)-C)$. Thus

$$\delta_{f,S}(n) \leq \frac{(n\log(D)-C)}{\log(q)} \preceq n$$

and the first assertion is proved. Now, assume that $\deg'_H(f^m) \geq cm^\alpha$ for some positive constants c and α. Then $cm^\alpha \leq D^n$, so $m \leq c^{-1/\alpha}D^{n/\alpha}$. Thus $\delta_{f,S}(n) \leq c^{-1/\alpha}D^{n/\alpha} \preceq \exp(n)$ and the second assertion follows. □

Remark 2.2. More generally, consider an increasing function α such that $\alpha(m) \leq \log\deg'_H(f^m)$ for all $m \geq 1$. Let β be a decreasing inverse of α, *i.e.* a function $\beta\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that $\beta(\alpha(m)) = m$ for all m. We have

$$\alpha(m) \leq \log\left(\deg'_H(f^m)\right) \leq n\log(D)$$

if f^m is in S^n, hence $\delta_{f,S}(n) \leq \beta(n\log(D))$. However, we do not know any example of birational transformation with intermediate (neither exponential nor polynomially bounded) degree growth. See [33] for a lower bound on the degree growth when $f \in \mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$.

3. Heights and distortion

In this section we study the distortion of automorphisms of $\mathbb{P}^m_{\mathbf{k}}$ in the groups $\mathsf{Aut}(\mathbb{P}^m_{\mathbf{k}})$ and $\mathrm{Cr}_m(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$.

3.1. Distortion and monomial transformations

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Here, we show that all elements of $\mathsf{PGL}_{m+1}(\mathbf{k})$ are distorted in $\mathrm{Cr}_m(\mathbf{k})$, and we compute their distortion rate.

3.1.1. *Monomial transformations and distortion of semisimple automorphisms*

The group $\mathsf{GL}_m(\mathbf{Z})$ acts by automorphisms on the m-dimensional multiplicative group $\mathbb{G}_{\mathsf{m}}^m$: if $A = [a_{i,j}]$ is in $\mathsf{GL}_m(\mathbf{Z})$, then $A(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ with

$$y_j = \prod_i x_i^{a_{i,j}}. \tag{3.1}$$

The group $\mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$ acts also on itself by translations. Altogether, we get an embedding of $\mathsf{GL}_m(\mathbf{Z}) \ltimes \mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$ in $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$.

If s is a fixed element of $\mathbf{k}^\times$, we denote by $\varphi_s : \mathbf{Z}^m \to \mathbb{G}_{\mathsf{m}}^m$ the homomorphism defined by $\varphi_s(n_1, \ldots, n_d) = (s^{n_1}, \ldots, s^{n_d})$. This homomorphism is injective if and only if s is not a root of unity. Its image $\varphi_s(\mathbf{Z}^m)$ is normalized by the monomial group $\mathsf{GL}_d(\mathbf{Z})$; in this way, every element $s \in \mathbf{k}^\times$ of infinite order determines an embedding of $\mathsf{GL}_m(\mathbf{Z}) \ltimes \mathbf{Z}^m$ into $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$, the image of which is $\mathsf{GL}_m(\mathbf{Z}) \ltimes \varphi_s(\mathbf{Z}^m)$. The following lemma is classical (see [24,25] for instance).

Lemma 3.1. *For every $m \geq 2$, the Abelian subgroup $\mathbf{Z}^m$ is exponentially distorted in $\mathsf{GL}_m(\mathbf{Z}) \ltimes \mathbf{Z}^m$. More precisely, $|g^n| \simeq \log(n)$ for every non-trivial element g in the* (*multiplicative*) *Abelian group $\mathbf{Z}^m$.*

For $u \in \mathbf{k}^\times$, the subgroup $\varphi_u(\mathbf{Z}^m)$ of $\mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$ acts by translations on $\mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$. This determines a subgroup V_u of $\mathrm{Cr}_m(\mathbf{k})$ acting by diagonal transformations $(x_1, \ldots, x_m) \mapsto (u^{n_1}x_1, \ldots, u^{n_m}x_m)$. By the previous lemma, the distortion of every element in V_u is at least exponential in $\mathrm{Cr}_m(\mathbf{k})$ (when u is a root of unity, the distortion is infinite).

Now let u be an arbitrary diagonal transformation: $u(x) = (u_1x_1, \ldots, u_mx_m)$, where $(u_i) \in \mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$. Consider the transformations $g_i = (x_1, \ldots, x_{i-1}, u_ix_i, x_{i+1}, \ldots, x_m)$. Then the g_i pairwise commute and $u = g_1 \ldots g_m$. Since $g_i \in V_{u_i}$, it is at least exponentially distorted in $\mathsf{GL}_m(\mathbf{Z}) \ltimes \mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$. Thus, u is at least exponentially distorted in $\mathsf{GL}_m(\mathbf{Z}) \ltimes \mathbb{G}_{\mathsf{m}}^m(\mathbf{k})$. We have proved:

Lemma 3.2. *Let* $\mathbf{k}$ *be a field and* $m \geq 2$ *be an integer. In* $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$*, every linear, diagonal transformation is at least exponentially distorted.*

3.1.2. *Distortion of unipotent automorphisms*

Lemma 3.3. *If* U *is a unipotent element of* $\mathsf{SL}_{m+1}(\mathbf{k})$*, then* U *is at least exponentially distorted in* $\mathsf{SL}_{m+1}(\mathbf{k})$*, and it is at least doubly exponentially distorted in* $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$ *for* $m \geq 2$.

Consequently, the image of U has finite order in every linear representation of (large enough subgroups of) the Cremona group. Note that (in characteristic zero) this already indicates that $\mathrm{Cr}_1(\mathbf{k}) \subset \mathrm{Cr}_2(\mathbf{k})$ is distorted in the sense that the translation $x \mapsto x+1$, which has exponential distortion in $\mathrm{Cr}_1(\mathbf{k}) \simeq \mathsf{PGL}_2(\mathbf{k})$, has double exponential distortion in $\mathrm{Cr}_2(\mathbf{k})$.

Proof. Unipotent elements of $\mathsf{SL}_{m+1}(\mathbf{k})$ have finite order if the characteristic of the field is positive; hence, we assume that $\mathrm{char}(\mathbf{k}) = 0$. Consider the element

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

of $\mathsf{SL}_2(\mathbf{k})$. Let $A \in \mathsf{SL}_2(\mathbf{k})$ be the diagonal matrix with coefficients 2 and $1/2$ on the diagonal: $A^n U A^{-n} = U^{4^n}$ and U is exponentially distorted in the subgroup of $\mathsf{SL}_2(\mathbf{k})$ generated by U and A. Similarly, consider a unipotent matrix $U_{i,j} = \mathrm{Id} + E_{i,j}$, where $E_{i,j}$ is the $(m+1)\times(m+1)$ matrix with only one non-zero coefficient, namely $e_{i,j} = 1$; then $U_{i,j}$ is exponentially distorted in $\mathsf{SL}_{m+1}(\mathbf{k})$: there is a diagonal matrix A such that $|U_{i,j}^n| \simeq \log(n)$ in the group $\langle U_{i,j}, A\rangle$, for all $n \geq 1$. This implies that unipotent matrices are exponentially distorted in $\mathsf{SL}_{m+1}(\mathbf{k})$.

As a second step, consider a 3×3 Jordan block and its iterates:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^n = \begin{pmatrix} 1 & n & n(n-1)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

We want to prove that U is doubly exponentially distorted in $\mathrm{Cr}_2(\mathbf{k})$. Take iterates U^{K^n} for some integer $K > 1$. Then, conjugating by A^n, and multiplying by B, with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4}$$

we get a new matrix $BA^{-n}U^{K^n}A^n = [v_{i,j}(n)]$ which is upper triangular; its coefficients are equal to 1 on the diagonal, $v_{1,2} = K^n$, $v_{1,3} = K^n(K^n-1)/2$ and $v_{2,3} = 0$. Conjugating with C^n changes $v_{1,2}$ into $v'_{1,2} = 1$ and $v_{1,3}$ into $v'_{1,3} = (K^n-1)/2$. Multiplying by the unipotent matrix $D = \mathrm{Id} - E_{1,2} + 1/2E_{1,3}$ changes $v'_{1,2}$ into 0 and $v'_{1,3}$ into K^n. One more conjugacy by C^n gives a matrix E with constant coefficients. Thus U^{K^n} is a word of finite length (independent of n) in A^n, C^n, and a fixed, finite number of unipotent matrices (B, D, E). Since A and C are diagonal matrices, they satisfy $|A^n| \sim \log(n)$ and $|C^n| \sim \log(n)$ in some finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{k})$. Thus, U is doubly exponentially distorted.

This argument and a recursion starting at $m = 2$ proves the general result. □

3.1.3. *Distortion of linear projective transformations*

Every $A \in \mathsf{PGL}_{m+1}(\mathbf{k})$ is the product of a semisimple element S_A with a unipotent element U_A such that S_A and U_A commute. When $\mathbf{k}$ is algebraically closed, S_A is diagonalizable. By Lemmas 3.2 and 3.3, A is at least exponentially distorted (respectively doubly exponentially distorted if S_A has finite order).

3.2. Heights and upper bounds

Theorem 3.4. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero. Let* A *be an element of* $\mathsf{Aut}(\mathbb{P}^m_{\mathbf{k}})$ *given by a matrix in* $\mathsf{SL}_{m+1}(\mathbf{k})$ *of infinite order. Then, its distortion in the Cremona group* $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$ *is doubly exponential if the matrix is virtually unipotent, and simply exponential otherwise.*

To prove this result, we use basic properties of heights of polynomial functions. We start with a proof of this theorem when $\mathbf{k} = \overline{\mathbf{Q}}$ is an algebraic closure of the field of rational number; the general case is obtained by a specialization argument.

3.2.1. *Heights of polynomial functions*

Let $\mathbf{K}$ be a finite extension of $\mathbf{Q}$, and let $M_{\mathbf{K}}$ be the set of places of $\mathbf{K}$; to each place, we associate a unique absolute value $|\cdot|_v$ on K, normalized as follows (see [6, Section 1.4]). First, for each prime number p, the p-adic absolute value on $\mathbf{Q}$ satisfies $|p|_p = 1/p$, and $|\cdot|_\infty$ is the standard absolute value. Then, if $v \in M_{\mathbf{K}}$ is a place that divides p, with p prime or ∞, then

$$|x|_v = \left|\mathrm{Norm}_{\mathbf{K}/\mathbf{Q}}(x)\right|_p^{1/[\mathbf{K}:\mathbf{Q}]} \tag{3.5}$$

for every $x \in \mathbf{K}$. With such a choice, the product formula reads

$$\sum_{v \in M_{\mathbf{K}}} \log|x|_v = 0 \tag{3.6}$$

for every $x \in \mathbf{K} \setminus \{0\}$.

Let m be a natural integer. If $f(\mathbf{x}) = \sum_I a_I \mathbf{x}^I$ is a polynomial function in the variables $\mathbf{x} = (x_0, \ldots, x_m)$, with $a_I \in \mathbf{K}$ for each multi-indice $I = (i_0, \ldots, i_m)$, we set

$$|f|_v = \max_I |a_I|_v \tag{3.7}$$

for every place $v \in M_{\mathbf{K}}$. If $f \neq 0$, we define its *height* $h(f)$ by

$$h(f) = \sum_{v \in M_{\mathbf{K}}} \log|f|_v. \tag{3.8}$$

If $\hat{f} = (f_0, \dots, f_m)$ is an endomorphism of $\mathbb{A}^{m+1}_{\mathbf{K}}$, the height $h(\hat{f})$ is the maximum of the heights $h(f_i)$, and $|\hat{f}|_v$ is the maximum of the $|f_i|_v$. (Note that the affine coordinates system $\mathbf{x}$ is implicitely fixed.)

Remark 3.5. Let f and g be non-zero elements of $\mathbf{K}[x_0, \dots, x_m]$.

(1) The product formula implies that $h(af) = h(f), \forall a \in \mathbf{K} \setminus \{0\}$;
(2) From this, we see that $h(f) \geq 0$ for all $f \in \mathbf{K}[x_0, \dots, x_m] \setminus \{0\}$. Indeed, one can multiply f by the inverse of a coefficient $a_I \neq 0$ without changing the value of its height; then, one of the coefficients is equal to 1 and $|f|_v \geq 1$ for all $v \in M_{\mathbf{K}}$;
(3) The Gauss Lemma says that $|fg|_v = |f|_v |g|_v$ when v is not Archimedean. This multiplicativity property fails for places at infinity;
(4) If $\mathbf{L}$ is an extension of $\mathbf{K}$, then the height of $f \in \mathbf{K}[x_0, \dots, x_m]$ is the same as its height as an element of $\mathbf{L}[x_0, \dots, x_m]$ (see [6, Lemma 1.3.7)]. Thus, the height is well defined on $\overline{\mathbf{Q}}[x_0, \dots, x_m]$.

Theorem 3.6 (see [6, 1.6.13]). *Let $f_1, \dots, f_s$ be non-zero elements of $\overline{\mathbf{Q}}[x_0, \dots, x_m]$, and let f be their product $f_1 \cdots f_s$. Let $\Delta(f)$ be the sum of the partial degrees of f with respect to each of the variables x_i. Then*

$$-\Delta(f)\log(2) + \sum_{i=1}^{s} h(f_i) \;\leq\; h(f) \;\leq\; \Delta(f)\log(2) + \sum_{i=1}^{s} h(f_i).$$

If $\deg(f)$ denotes the degree of f, then $\Delta(f) \leq (m+1)\deg(f)$. For $s = 2$ we get

$$h(f_1) \leq h(f) - h(f_2) + (m+1)\log(2)\deg(f). \tag{3.9}$$

3.2.2. *Heights of birational transformations*

Consider a birational transformation $f \colon \mathbb{P}^m_{\overline{\mathbf{Q}}} \dashrightarrow \mathbb{P}^m_{\overline{\mathbf{Q}}}$, and write it in homogeneous coordinates

$$f\big[x_0 : \dots : x_m\big] = \big[f_0 : \dots : f_m\big], \tag{3.10}$$

where the $f_i \in \overline{\mathbf{Q}}[x_0, \dots, x_m]$ are homogeneous polynomial functions of the same degree d with no common factor of positive degree. Then, d is the degree of f (see Section 1.2.1), and the f_i are uniquely determined modulo multiplication by a common constant $a \in \overline{\mathbf{Q}} \setminus \{0\}$. Thus, Remark 3.5(1) shows that the real number

$$h(f) = \max_i h(f_i) \tag{3.11}$$

is well defined. This number $h(f)$ is, by definition, the *height* of the birational transformation f. It coincides with the height of the lift of f as the endomorphism $\hat{f} = (f_0, \dots, f_{m+1})$ of $\mathbb{A}^{m+1}_{\mathbf{k}}$ (see Section 3.2.1).

3.2.3. *Growth of heights under composition*

Let $S = \{f^1, \dots, f^s\}$ be a finite symmetric set of birational transformations of $\mathbb{P}^m_{\overline{\mathbf{Q}}}$; the symmetry means that $f \in S$ if and only if $f^{-1} \in S$. Consider the homomorphism from the free group $\mathbb{F}_s = \langle a_1, \dots, a_s | \emptyset \rangle$ to $\mathsf{Bir}(\mathbb{P}^m_{\overline{\mathbf{Q}}})$ defined by mapping each generator a_j to f^j. Then, to every reduced word $w_\ell(a_1, \dots, a_s)$ of length ℓ in the generators a_j corresponds an element

$$w_l(S) = w_l\big(f^1, \dots, f^s\big) \tag{3.12}$$

of the Cremona group $\mathsf{Bir}(\mathbb{P}^m_{\overline{\mathbf{Q}}})$.

For each $f^i \in S$, we fix a system of homogeneous polynomials $f^i_j \in \overline{\mathbf{Q}}[x_0 : \dots : x_m]$ defining f, as in Section 3.2.2: $f^i = [f^i_0 : \dots : f^i_m]$ and the f^i_j have degree $d_i = \deg(f^i)$. Moreover, we choose the f^i_j so that for every i at least one of the coefficients of the f^i_j is equal to 1. Once the f^i_j have been fixed, we have a canonical lift of each f^i to a homogeneous endomorphism $\hat{f}^i$ of $\mathbb{A}^{m+1}_{\overline{\mathbf{Q}}}$, given by

$$\hat{f}^i(x_0, \dots, x_m) = \left(f^i_0, \dots, f^i_m\right). \tag{3.13}$$

Thus, every reduced word w_ℓ of length ℓ in $\mathbb{F}_s$ determines also an endomorphism $\hat{w}_\ell(S) = w_\ell(\hat{f}^1, \dots, \hat{f}^s)$ of the affine space.

Let d_S be the maximum of $\{2, d_1, \dots, d_s\}$, so that $d_S \geq 2$. Then, the degree of the endomorphism $\hat{w}_\ell(S)$ is at most d_S^ℓ.

Let $\mathbf{K}$ be the finite extension of $\mathbf{Q}$ which is generated by all the coefficients $a^i_{j,I}$ of the polynomial functions $f^i_j = \sum a^i_{j,I} \mathbf{x}^I$. We shall say that a place $v \in M_{\mathbf{K}}$ is *active* if $|a^i_{j,I}|_v > 1$ for at least one of these coefficients; the set of active places is finite, because there are only finitely many coefficients. For each place $v \in M_{\mathbf{K}}$, we set

$$M(v) = \max \left| a^i_{j,I} \right|_v = \max \left| \hat{f}^i \right|_v, \tag{3.14}$$

the maximum of the absolute values of the coefficients; our normalization implies that $M(v) \geq 1$ and $M(v) = 1$ if and only if v is not active.

Lemma 3.7. *Let v be a non-Archimedean place. If $w_\ell \in \mathbb{F}_s$ is a reduced word of length ℓ, then*

$$\log |\hat{w}_\ell(S)|_v \leq \log(M(v)) d_S^\ell.$$

Thus, if v is not active, then $\log |\hat{w}_\ell(S)|_v = 0$.

Proof. Set $d = d_S$. Write $\hat{w}_\ell(S)$ as a composition $\hat{g}^\ell \circ \cdots \circ \hat{g}^1$, where each $\hat{g}^k$ is one of the $\hat{f}^i$ (here we use that S is symmetric). By definition, $|\hat{g}^1|_v \leq M(v)$. Then, assume that $|\hat{g}^{k-1} \circ \cdots \circ \hat{g}^1|_v \leq M(v)^{1+d+\cdots+d^{k-2}}$ for some integer $2 \leq k \leq \ell$.

Write $\hat{g}^{k-1} \circ \cdots \circ \hat{g}^1 = (u_0, \ldots, u_m)$ for some homogeneous polynomials u_j. The Gauss lemma (see Remark 3.5) says that

$$\left| u_0^{i_0} \cdots u_m^{i_m} \right|_v = |u_0|_v^{i_m} \cdots |u_m|_v^{i_0} \leq \left(M(v)^{1+d+\cdots+d^{k-2}} \right)^d \tag{3.15}$$

for every multi-index $I = (i_0, \ldots, i_m)$ of length $\sum i_j \leq d$. The endomorphism $\hat{g}^k$ has degree $\leq d$, and the absolute values of its coefficients are bounded by $M(v)$, hence

$$\left| \hat{g}^k \circ \cdots \circ \hat{g}^1 \right|_v \leq M(v)^{1+d+\cdots+d^{k-1}}. \tag{3.16}$$

By recursion, this upper bound holds up to $k = \ell$. For $k = \ell$ we obtain the estimate $\log |\hat{w}_\ell(S)|_v \leq \log(M(v)) d^\ell$ because $1 + d + \cdots + d^{\ell-1} \leq d^\ell$. □

Lemma 3.8. *Let v be an Archimedean place. If $w_\ell \in \mathbb{F}_s$ is a reduced word of length ℓ, then*

$$\log |\hat{w}_\ell(S)|_v \leq 2 \log(M(v)) d_S^\ell + \log \left(m d_S^m \right) d_S^{2\ell}.$$

Proof. Consider a monomial $\mathbf{x}^I = x_0^{i_0} \cdots x_m^{i_m}$ of degree $\leq d$. Let $u_0, \ldots, u_m$ be homogeneous polynomials of degree $\leq D$ with $D \geq 2$ and with all coefficients satisfying $|c|_v \leq C$. Note that the space of homogeneous polynomials of degree D in m variables has dimension $\binom{D+m}{m}$.Then

$$\left| u_0^{i_0} \cdots u_m^{i_m} \right|_v \leq \binom{D+m}{m}^d C^d \leq (D+m)^{md} C^d \leq \left(m D^m \right)^d C^d. \tag{3.17}$$

Indeed, every coefficient in the product $u_0^{i_0} \cdots u_m^{i_m}$ is obtained as a sum of at most $\binom{D+m}{m}^d$ terms, each of which is a product of at most d coefficients of the u_j.

Then, to estimate the absolute values of the coefficients of $\hat{w}_\ell(S)$, we proceed by recursion as in the proof of Lemma 3.7. Set $B = m d_S^m$. For a composition $\hat{g}^k \circ \cdots \circ \hat{g}_1$ of length k we obtain

$$\left| \hat{g}^k \circ \cdots \circ \hat{g}^1 \right|_v \leq B^{(k-1) d_S^{k-1}} M(v)^{2 d_S^{k-1}}. \tag{3.18}$$

The conclusion follows from $\ell d_S^\ell \leq d_S^{2\ell}$. □

Putting these lemmas together, we get

$$h(\hat{w}_\ell(S)) \leq \sum_{v \text{ active}} 2 \log(M(v)) d_S^\ell + \sum_{v | \infty} \log \left(m d_S^m \right) d_S^{2\ell}. \tag{3.19}$$

This inequality concerns the height of the endomorphism $\hat{w}_\ell(S)$; to obtain the birational transformation $w_\ell(S)$, we might need to divide by a common factor

$q(x_0, \dots, x_m)$. Since the degree of $\hat{w}_\ell(S)$ is no more than d_S^ℓ, Theorem 3.6 provides the upper bound

$$h(w_\ell(S)) \leq \left((m+1)\log(2) + \sum_{v \text{ active}} \log(M(v)) + \sum_{v|\infty} \log\left(md_S^m\right)\right) d_S^{2\ell}. \quad (3.20)$$

This proves the following proposition.

Proposition 3.9. *Let $\mathbb{F}_s = \langle a_1, \dots, a_s | \emptyset \rangle$ be a free group of rank $s \geq 1$. For every homomorphism $\rho \colon \mathbb{F}_s \to \mathsf{Bir}(\mathbb{P}^m_{\overline{\mathbf{Q}}})$, there exist two constants $C_m(\rho)$ and $d(\rho) \geq 1$ such that $h(\rho(w)) \leq C_m(\rho) d(\rho)^{|w|}$ for every $w \in \mathbb{F}_s$, where $|w|$ is the length of w as a reduced word in the generators a_i.*

3.3. Proof of Theorem 3.4

We may now prove Theorem 3.4. When $\mathbf{k} = \overline{\mathbf{Q}}$, this result is a direct corollary of Proposition 3.9 and Section 3.1.3; we start with this case and then treat the general case via a specialization argument.

3.3.1. *Number fields*

Let A be an element of $\mathsf{SL}_{m+1}(\overline{\mathbf{Q}})$ of infinite order. After conjugation, we may assume A to be upper triangular. First, suppose that A is virtually unipotent (all its eigenvalues are roots of unity). Then $h(A^n)$ grows like $\tau \log(n)$ as n goes to $+\infty$. Thus, if A^n is a word of length $\ell(n)$ in some fixed, finitely generated subgroup of $\mathsf{Bir}(\mathbb{P}^m_{\overline{\mathbf{Q}}})$, Proposition 3.9 shows that

$$\tau \log(n) \leq C d^{\ell(n)} \quad (3.21)$$

for some positive constants C and $d > 1$. Thus, A is at most doubly exponentially distorted; from Lemma 3.3, it is exactly doubly exponentially distorted. Now, suppose that an eigenvalue α of A is not a root of unity. Kronecker's lemma provides a place $v \in M_{\mathbf{Q}(\alpha)}$ for which $|\alpha|_v > 1$ (see [6, Theorem 1.5.9]). Thus, $h(A^n)$ grows like τn for some positive constant τ as n goes to $+\infty$ (see Remark 3.5(2)), and A is at most exponentially distorted in $\mathsf{Bir}(\mathbb{P}^m_{\overline{\mathbf{Q}}})$. From Section 3.1.3, we obtain Theorem 3.4 when $\mathbf{k} = \overline{\mathbf{Q}}$.

3.3.2. *Fields of characteristic zero*

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let A be an element of $\mathsf{SL}_{m+1}(\mathbf{k})$. Let $S = \{f^1, \dots, f^m\}$ be a finite symmetric subset of $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$ such that the group generated by S contains A. For each n, denote by $\ell(n)$ the length of A^n as a reduced word in the f^i.

Write each f^i in homogeneous coordinates $f^i = [f^i_0 : \dots : f^i_m]$, as in Section 3.2.3; and denote by $\mathcal{C}$ the set of coefficients of the matrix A and of the polynomial functions $f^i_j = \sum a^i_{j,I} \mathbf{x}^I$. This is a finite subset of $\mathbf{k}$, generating a finite extension $\mathbf{K}$ of $\mathbf{Q}$. This finite extension is an algebraic extension of a purely trancendental extension $\mathbf{Q}(t_1, \ldots, t_r)$, where r is the transcendental degree of $\mathbf{K}$ over $\mathbf{Q}$. Then, the elements of $\mathcal{C}$ are algebraic functions with coefficients in $\overline{\mathbf{Q}}$ (such as $(2t_1t_3^2 - 1)^{1/3} + t_2^5$); the ring of functions generated by $\mathcal{C}$ (over $\overline{\mathbf{Q}}$) may be viewed as the ring of functions of some algebraic variety $V_{\mathcal{C}}$ (defined over $\overline{\mathbf{Q}}$).

If u is a point of $V_{\mathcal{C}}(\overline{\mathbf{Q}})$ and $c \in \mathcal{C}$ is one of the coefficients, we may evaluate c at u to obtain an algebraic number $c(u)$. Similarly, we may evaluate, or specialize, A and the f^i at u. This gives an element A_u in $\mathsf{SL}_{m+1}(\overline{\mathbf{Q}})$ (the determinant is 1), and rational transformations f^i_u of $\mathbb{P}^m_{\overline{\mathbf{Q}}}$. For some values of u, f^i_u may be degenerate, identically equal to $[0 : \ldots : 0]$; but for u in a dense, Zariski open subset of $V_{\mathcal{C}}$, the f^i_u are birational transformations of degree $\deg(f^i_u) = \deg(f^i)$. Pick such a point $u \in V_{\mathcal{C}}(\overline{\mathbf{Q}})$. If A^n is a word of length $\ell(n)$ in the f^i, then A^n_u is a word of the same length in the f^i_u. From the previous section we deduce that A is at most doubly exponentially distorted. Moreover, if one of the eigenvalues $\alpha \in \mathbf{k}$ of A is not a root of unity, we may add α to the set $\mathcal{C}$ and then choose the point u such that $\alpha(u)$ is not a root of unity either. Then, A_u and thus A is at most exponentially distorted. This concludes the proof of Theorem 3.4.

4. Non-distortion

In this section we prove Theorem 4.1, which provides an upper bound for the distortion of parabolic isometries in certain groups of isometries of hyperbolic spaces.

4.1. Hyperbolic spaces and parabolic isometries

4.1.1. *Hyperbolic spaces*

Let $\mathcal{H}$ be a real Hilbert space of dimension $m + 1$ (m can be infinite). Fix a unit vector $\mathbf{e}_0$ of $\mathcal{H}$ and a Hilbert basis $(\mathbf{e}_i)_{i \in I}$ of the orthogonal complement of $\mathbf{e}_0$. Define a new scalar product on $\mathcal{H}$ by

$$\langle u | u' \rangle = a_0 a_0' - \sum_{i \in I} a_i a_i' \tag{4.1}$$

for every pair $u = a_0\mathbf{e}_0 + \sum_i a_i \mathbf{e}_i$, $u' = a_0'\mathbf{e}_0 + \sum_i a_i'\mathbf{e}_i$ of vectors. Define $\mathbb{H}_m$ to be the connected component of the hyperboloid $\{u \in \mathcal{H} |\ \langle u|u \rangle = 1\}$ that contains $\mathbf{e}_0$, and let dist be the distance on $\mathbb{H}_m$ defined by (see [3])

$$\cosh(\mathsf{dist}(u, u')) = \langle u | u' \rangle. \tag{4.2}$$

The metric space $(\mathbb{H}_m, \mathsf{dist})$ is a model of the hyperbolic space of dimension m (see [3]). The projection of $\mathbb{H}_m$ into the projective space $\mathbb{P}(\mathcal{H})$ is one-to-one onto its image. In what follows, $\mathbb{H}_m$ is identified with its image in $\mathbb{P}(\mathcal{H})$ and its boundary is denoted by $\partial \mathbb{H}_m$; hence, boundary points correspond to isotropic lines in the space $\mathcal{H}$ for the scalar product $\langle \cdot | \cdot \rangle$.

4.1.2. *Hyperbolic plane*

A useful model for $\mathbb{H}_2$ is the Poincaré model: $\mathbb{H}_2$ is identified to the upper half-plane $\{z \in \mathbf{C}; \mathrm{Im}(z) > 0\}$, with its Riemanniann metric given by $ds^2 = (x^2 + y^2)/y^2$. Its group of orientation preserving isometries coincides with $\mathsf{PSL}_2(\mathbf{R})$, acting by linear fractional transformations. The distance between two points z_1 and z_2 satisfies

$$\sinh\left(\frac{1}{2}\mathsf{dist}_{\mathbb{H}_2}(z_1, z_2)\right) = \frac{|z_1 - z_2|}{2(\mathrm{Im}(z_1)\mathrm{Im}(z_2))^{1/2}}. \tag{4.3}$$

4.1.3. *Isometries*

Denote by $\mathsf{O}_{1,m}(\mathbf{R})$ the group of linear transformations of $\mathcal{H}$ preserving the scalar product $\langle\cdot|\cdot\rangle$. The group of isometries $\mathsf{Iso}(\mathbb{H}_m)$ coincides with the index 2 subgroup $\mathsf{O}^+_{1,m}(\mathbf{R})$ of $\mathsf{O}(\mathcal{H})$ that preserves the chosen sheet $\mathbb{H}_m$ of the hyperboloid $\{u \in \mathcal{H} \mid \langle u|u\rangle = 1\}$. This group acts transitively on $\mathbb{H}_m$, and on its unit tangent bundle.

If $h \in \mathsf{O}^+_{1,m}(\mathbf{R})$ is an isometry of $\mathbb{H}_m$ and $v \in \mathcal{H}$ is an eigenvector of h with eigenvalue λ, then either $|\lambda| = 1$ or v is isotropic. Moreover, since $\mathbb{H}_m$ is homeomorphic to a ball, h has at least one eigenvector v in $\mathbb{H}_m \cup \partial\mathbb{H}_m$. Thus, there are three types of isometries [8]:

(1) An isometry h is *elliptic* if and only if it fixes a point u in $\mathbb{H}_m$. Since $\langle\cdot|\cdot\rangle$ is negative definite on the orthogonal complement $u^\perp$, the linear transformation h fixes pointwise the line $\mathbf{R}u$ and acts by rotation on $u^\perp$ with respect to $\langle\cdot|\cdot\rangle$;

(2) An isometry h is *parabolic* if it is not elliptic and fixes a vector v in the isotropic cone. The line $\mathbf{R}v$ is uniquely determined by the parabolic isometry h. If z is a point of $\mathbb{H}_m$, there is an increasing sequence of integers m_i such that $h^{m_i}(z)$ converges towards the boundary point ξ determined by v;

(3) An isometry h is *loxodromic* if and only if h has an eigenvector v_h^+ with eigenvalue $\lambda > 1$. Such an eigenvector is unique up to scalar multiplication, and there is another, unique, isotropic eigenline $\mathbf{R}v_h^-$ corresponding to an eigenvalue < 1; this eigenvalue is equal to $1/\lambda$. On the orthogonal complement of $\mathbf{R}v_h^+ \oplus \mathbf{R}v_h^-$, h acts as a rotation with respect to $\langle\cdot|\cdot\rangle$. The boundary points determined by v_h^+ and v_h^- are the two fixed points of h in $\mathbb{H}_\infty \cup \partial\mathbb{H}_\infty$: the first one is an attracting fixed point, the second is repelling. Moreover, $h \in \mathsf{Iso}(\mathbb{H}_\infty)$ is loxodromic if and only if its *translation length*

$$L(h) = \inf\{\mathsf{dist}(x, h(x)) \mid x \in \mathbb{H}_\infty\} \tag{4.4}$$

is positive. In that case, $\lambda = \exp(L(h))$ is the largest eigenvalue of h and $\mathsf{dist}(x, h^n(x))$ grows like $nL(h)$ as n goes to $+\infty$ for every point x in $\mathbb{H}_m$.

When h is elliptic or parabolic, the translation length vanishes (there is a point u in $\mathbb{H}_m$ with $L(h) = \mathsf{dist}(u, h(u))$ if h is elliptic, but no such point exists if h is parabolic).

4.1.4. *Horoballs*

Let ξ be a boundary point of $\mathbb{H}_m$, and let ϵ be a positive real number. The *horoball* $H_\xi(\epsilon)$ in $\mathbb{H}_\infty$ is the subset

$$H_\xi(\epsilon) = \{v \in \mathbb{H}_m\,;\ 0 < \langle v|\xi\rangle < \epsilon\}.$$

It is a limit of balls with centers converging to the boundary point ξ. An isometry h fixing the boundary point ξ maps $H_\xi(\epsilon)$ to $H_\xi(e^{L(h)}\epsilon)$.

4.2. Distortion estimate

Our goal is to prove the following theorem.

Theorem 4.1. *Let m be any (possibly infinite) cardinal. Let G be a subgroup of $\mathsf{Iso}(\mathbb{H}_m)$. Let f be a parabolic element of G, and let $\xi \in \partial\mathbb{H}_m$ be the fixed point of f. Suppose that the following two properties are satisfied.*

(i) *There are positive constants C and $C' > 0$ and a point $x_0 \in \mathbb{H}_m$ such that*

$$\mathsf{dist}\big(f^n(x_0), x_0\big) \geq C \log n - C'$$

for all large enough values of n;

(ii) *There exists a horoball B centered at ξ such that for every $g \in G$ either $gB = B$ or $gB \cap B = \emptyset$.*

Then f is at most $n^{2/C}$-distorted in G. In particular $C \leq 2$ and if $C = 2$ then f is undistorted in G.

When looking at the Cremona group $\mathrm{Cr}_2(\mathbf{k})$, we shall see examples of isometries in $\mathbb{H}_\infty$ with $\mathsf{dist}(f^n(x_0), x_0) \sim C \log n$ for $C = 1$ or $C = 2$.

4.2.1. *Complements of horoballs*

Let X be a metric space. Let W be a subset of X. Let dist_{W^c} (or d_X when $W = \emptyset$) be the induced intrinsic distance on the complement of W; namely, for x and y in W^c, we have $\mathsf{dist}_{W^c}(x, y) = \sup_{\varepsilon>0} d_{W^c,\varepsilon}(x, y)$ with

$$d_{W^c,\varepsilon}(x, y) = \inf\left\{\sum_{i=0}^{n-1} d(x_i, x_{i+1}) : n \geq 0, x_0 = x, x_n = y, \sup_i d(x_i, x_{i+1}) \leq \varepsilon\right\}$$

for points x_i which are all in $X \setminus W$. It is a distance as soon as it does not take the ∞ value. In the cases we shall consider, X will be the hyperbolic space $\mathbb{H}_m$, W will be a union of horoballs, and W^c will be path connected. In that case, $\mathsf{dist}_{W^c}(x, y)$ is the infimum of the length of paths connecting x to y within W^c.

Similarly, if Y is a subset of X, we denote by dist_Y the induced intrinsic distance on Y (hence, $\mathsf{dist}_Y = \mathsf{dist}_{(X\setminus Y)^c}$).

Lemma 4.2. *Let $B \subset \mathbb{H}_m$ be an open horoball, with boundary ∂B. Let x and y be points on the horosphere ∂B. Then*

$$\mathsf{dist}_{B^c}(x, y) = \mathsf{dist}_{\partial B}(x, y) = 2\sinh(\mathsf{dist}(x, y)/2).$$

Proof. The statement being trivial when $x = y$, we assume $x \neq y$ in what follows. Let ξ be the center at infinity of B. Then x, y, and the boundary point ξ are contained in a unique geodesic plane P. Since the projection of $\mathbb{H}_m$ onto P is a 1-Lipschitz map, we have $\mathsf{dist}_{\partial B}(x, y) = \mathsf{dist}_{\partial B \cap P}(x, y)$ and $\mathsf{dist}_{B^c}(x, y) = \mathsf{dist}_{B^c \cap P}(x, y)$. Hence, we can replace $\mathbb{H}_m$ by the 2-dimensional hyperbolic space $P \simeq \mathbb{H}_2$. To conclude, we use the Poincaré half-plane model of $\mathbb{H}_2$. There is an isometry $P \to \mathbb{H}_2$ mapping the horosphere $\partial B \cap P$ to the line $i + \mathbf{R}$, the points x and y to $i + t$ and $i + t'$, and ξ to ∞. If $\gamma(s) = x(s) + iy(s)$, $s \in [a, b]$, is a path in $P \cap B^c$ that connects x to y, its length satisfies

$$\text{length}(\gamma) = \int_a^b \frac{(x'(s)^2 + y'(s)^2)^{1/2}}{y(s)} ds \leq \int_a^b |x'(s)| ds$$

because $y \leq 1$ in $B^c \cap P$. Thus, the geodesic segment from x to y for dist_{B^c} (respectively $\mathsf{dist}_{\partial B}$) is the euclidean segment $\gamma(s) = i + s$, with $s \in [t, t']$, and

$$\mathsf{dist}_{B^c}(x, y) = \mathsf{dist}_{\partial B}(x, y) = |t - t'|.$$

We conclude with Formula (4.3), that gives $\sinh(\mathsf{dist}_{\mathbb{H}_2}(x, y)/2) = |t - t'|/2$. □

Lemma 4.3. *Let (B_i) be a family of open horoballs in $\mathbb{H}_m$ with pairwise disjoint closures and let $Q = \bigcup B_i$. Then*

1. dist_{Q^c} *is a distance on* $\mathbb{H}_m \setminus Q$*;*
2. *For every index i and every pair of points (x, y) on the boundary of ∂B_i, we have* $\mathsf{dist}_{Q^c}(x, y) = \mathsf{dist}_{\partial B_i}(x, y)$.

Proof. Let (x, y) be a pair of points in Q^c. Consider the unique geodesic segment of $\mathbb{H}_m$ that joins x to y. Denote by $[u_j, u'_j]$ the intersection of this segment with B_j. Let C be a positive constant such that $2\sinh(s/2) \leq Cs$ for all $s \in [0, \mathsf{dist}(x, y)]$ (such a constant depends on $\mathsf{dist}(x, y)$). From Lemma 4.2 we obtain

$$\mathsf{dist}_{Q^c}(x, y) \leq \mathsf{dist}(x, y) + \sum_j \mathsf{dist}_{\partial B_j}\big(u_j, u'_j\big) \tag{4.5}$$

$$\leq \mathsf{dist}(x, y) + \sum_j C\mathsf{dist}\big(u_j, u'_j\big) \tag{4.6}$$

$$\leq (1 + C)\mathsf{dist}(x, y). \tag{4.7}$$

Thus, $\mathsf{dist}_{Q^c}(x, y)$ is finite: this proves (1).

For (2), note that $Q^c \subset B_i^c$ and Lemma 4.2 imply $\mathsf{dist}_{Q^c}(x, y) \geq \mathsf{dist}_{B_i^c}(x, y) = \mathsf{dist}_{\partial B_i}(x, y)$, and that $\mathsf{dist}_{\partial B_i}(x, y) \geq \mathsf{dist}_{Q^c}(x, y)$ because $\partial B_i \subset \overline{Q^c}$. □

4.2.2. *Proof of Theorem* 4.1

Changing B in a smaller horoball, we can suppose that B is open and that $g\bar{B} \cap \bar{B}$ is empty for all $g \in G$ with $gB \neq B$. Let Q be the union of the horoballs gB for $g \in G$. Let x_1 be a point on the horosphere ∂B. Let $D > 1$ satisfy $2\log(D) = C' + 2\mathsf{dist}(x_1, x_0)$. From the first hypothesis, we know that

$$\mathsf{dist}\big(f^n(x_1), x_1\big) \geq C \log n - 2\log(D)$$

for all sufficiently large values of n. By Lemmas 4.2 and 4.3, we get

$$\begin{aligned} \mathsf{dist}_{Q^c}(x_1, f^n(x_1)) &= 2\sinh\big(\mathsf{dist}(x_1, f^n(x_1))/2\big) \\ &\geq 2\sinh(C\log(n)/2 - \log(D)) \\ &\geq D^{-1}n^{C/2} - Dn^{-C/2} \end{aligned}$$

for large enough n. Let us now estimate the distortion of f in G. Let S be a finite symmetric subset of G and let D_S be the maximum of the distances $\mathsf{dist}_{Q^c}(g(x_1),x_1)$ for g in S. Suppose that $f^n = g_1 \circ g_2 \circ \cdots \circ g_\ell$ is a composition of ℓ elements $g_i \in S$. The group generated by the g_i acts by isometries on Q^c for the distance dist_{Q^c}. Thus,

$$\begin{aligned} \mathsf{dist}_{Q^c}(f^n(x_1), x_1) &= \mathsf{dist}_{Q^c}(g_1 \circ \cdots \circ g_\ell(x_1), x_1) \\ &\leq \mathsf{dist}_{Q^c}(g_1 \circ \cdots \circ g_\ell(x_1), g_1 \circ \cdots \circ g_{\ell-1}(x_1)) \\ &\quad + \mathsf{dist}_{Q^c}(g_1 \circ \cdots \circ g_{\ell-1}(x_1), x_1) \\ &\leq \mathsf{dist}_{Q^c}(g_\ell(x_1), x_1) + \mathsf{dist}_{Q^c}(g_1 \circ \cdots \circ g_{\ell-1}(x_1), x_1) \\ &\leq \sum_{j=1}^{\ell} \mathsf{dist}_{Q^c}(g_j(x_1), x_1) \\ &\leq \ell D_S. \end{aligned}$$

This shows that $D^{-1}n^{C/2} - Dn^{-C/2} \leq D_S \times \ell$ for large values of n (and ℓ), and the conclusion follows.

5. The Picard-Manin space and hyperbolic geometry

In this section we recall the construction of the Picard-Manin space of a projective surface X (see [13,26] for details).

5.1. Picard-Manin spaces

Let X be a smooth, irreducible, projective surface. We denote its Néron-Severi group by $\mathrm{Num}(X)$; when $\mathbf{k} = \mathbf{C}$, $\mathrm{Num}(X)$ can be identified to $H^{1,1}(X;\mathbf{R}) \cap H^2(X;\mathbf{Z})$. The *intersection form*

$$(C, D) \mapsto C \cdot D \tag{5.1}$$

is a non-degenerate quadratic form on $\mathrm{Num}(X)$ of signature $(1, \rho(X) - 1)$. The Picard-Manin space $\mathcal{Z}(X)$ is the limit $\lim_{\pi\colon X' \to X} \mathrm{Num}(X')$ obtained by looking at all birational morphisms $\pi\colon X' \to X$, where X' is smooth and projective. By construction, $\mathrm{Num}(X)$ embeds naturally as a proper subspace of $\mathcal{Z}(X)$, and the intersection form is negative definite on the infinite dimensional space $\mathrm{Num}(X)^{\perp}$.

Example 5.1. The group $\mathrm{Pic}(\mathbb{P}^2_{\mathbf{k}})$ is generated by the class $\mathbf{e}_0$ of a line. Blow-up one point q_1 of the plane, to get a morphism $\pi_1\colon X_1 \to \mathbb{P}^2_{\mathbf{k}}$. Then, $\mathrm{Pic}(X_1)$ is a free Abelian group of rank 2, generated by the class $\mathbf{e}_1$ of the exceptional divisor E_{q_1}, and by the pull-back of $\mathbf{e}_0$ under π_1 (still denoted $\mathbf{e}_0$ in what follows). After n blow-ups one obtains

$$\mathrm{Pic}(X_n) = \mathrm{Num}(X_n) = \mathbf{Z}\mathbf{e}_0 \oplus \mathbf{Z}\mathbf{e}_1 \oplus \ldots \oplus \mathbf{Z}\mathbf{e}_n\,, \tag{5.2}$$

where $\mathbf{e}_0$ (respectively $\mathbf{e}_i$) is the class of the total transform of a line (respectively of the exceptional divisor E_{q_i}) by the composite morphism $X_n \to \mathbb{P}^2_{\mathbf{k}}$ (respectively $X_n \to X_i$). The direct sum decomposition (5.2) is orthogonal with respect to the intersection form:

$$\mathbf{e}_0\cdot\mathbf{e}_0 = 1, \quad \mathbf{e}_i\cdot\mathbf{e}_i = -1\ \forall\, 1 \le i \le n, \quad \text{and} \quad \mathbf{e}_i\cdot\mathbf{e}_j = 0\ \forall\, 0 \le i \ne j \le n. \tag{5.3}$$

Taking limits, $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ splits as a direct sum $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}}) = \mathbf{Z}\mathbf{e}_0 \oplus \bigoplus_q \mathbf{Z}\mathbf{e}_q$ where q runs over all possible points of the so-called bubble space $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ of $\mathbb{P}^2_{\mathbf{k}}$ (see [4,19,26]).

5.2. The hyperbolic space $\mathbb{H}_\infty(X)$

Denote by $\mathcal{Z}(X,\mathbf{R})$ and $\mathrm{Num}(X,\mathbf{R})$ the tensor products $\mathcal{Z}(X)\otimes_{\mathbf{Z}}\mathbf{R}$ and $\mathrm{Num}(X)\otimes_{\mathbf{Z}} \mathbf{R}$. Elements of $\mathcal{Z}(X,\mathbf{R})$ are finite sums $u_X + \sum_i a_i \mathbf{e}_i$ where u_X is an element of $\mathrm{Num}(X,\mathbf{R})$, each $\mathbf{e}_i$ is the class of an exceptional divisor, and the coefficients a_i are real numbers. Allowing infinite sums with $\sum_i a_i^2 < +\infty$, one gets a new space $\mathsf{Z}(X)$, on which the intersection form extends continuously [7,12]. Fix an ample class $\mathbf{e}_0$ in $\mathrm{Num}(X) \subset \mathcal{Z}(X)$. The subset of elements u in $\mathsf{Z}(X)$ such that $u \cdot u = 1$ is a hyperboloïd, and

$$\mathbb{H}_\infty(X) = \{u \in \mathsf{Z}(X) \mid \quad u\cdot u = 1 \quad \text{and} \quad u \cdot \mathbf{e}_0 > 0\} \tag{5.4}$$

is the sheet of that hyperboloid containing ample classes of $\mathrm{Num}(X,\mathbf{R})$. With the distance $\mathsf{dist}(\cdot,\cdot)$ defined by

$$\cosh \mathsf{dist}(u, u') = u \cdot u', \tag{5.5}$$

$\mathbb{H}_\infty(X)$ is isometric to the hyperbolic space $\mathbb{H}_\infty$ described in Section 4.1.

We denote by $\mathsf{Iso}(\mathsf{Z}(X))$ the group of isometries of $\mathsf{Z}(X)$ with respect to the intersection form, and by $\mathsf{Iso}(\mathbb{H}_\infty(X))$ the subgroup that preserves $\mathbb{H}_\infty(X)$. As explained in [12,13,26], the group $\mathsf{Bir}(X)$ acts by isometries on $\mathbb{H}_\infty$. The homomorphism

$$f \in \mathsf{Bir}(X) \mapsto f_\bullet \in \mathsf{Iso}(\mathbb{H}_\infty(X)) \tag{5.6}$$

is injective.

5.3. Types and degree growth

Since $\mathsf{Bir}(X)$ acts faithfully on $\mathbb{H}_\infty(X)$, there are three types of birational transformations: *Elliptic*, *parabolic*, and *loxodromic*, according to the type of the associated isometry of $\mathbb{H}_\infty(X)$. We now describe how each type can be characterized in algebro-geometric terms.

5.3.1. *Degrees, distances, translation lengths and loxodromic elements*

Let $\mathbf{h} \in \mathrm{Num}(X, \mathbf{R})$ be an ample class with self-intersection 1. The degree of f with respect to the polarization $\mathbf{h}$ is $\deg_{\mathbf{h}}(f) = f_\bullet(\mathbf{h}) \cdot \mathbf{h} = \cosh(\mathsf{dist}(\mathbf{h}, f_\bullet \mathbf{h}))$. Consider for instance an element f of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, with the polarization $\mathbf{h} = \mathbf{e}_0$ given from the class of a line; then the image of a general line by f is a curve of degree $\deg_{\mathbf{h}}(f)$ which goes through the base points q_i of f^{-1} with certain multiplicities a_i, and

$$f_\bullet \mathbf{e}_0 = \deg_{\mathbf{h}}(f)\mathbf{e}_0 - \sum_i a_i \mathbf{e}_i\ , \tag{5.7}$$

where $\mathbf{e}_i$ is the class corresponding to the exceptional divisor that one gets when blowing up the point q_i.

If the translation length $L(f_\bullet)$ is positive, we know that the distance $\mathsf{dist}(f_\bullet^n(x), x)$ grows like $nL(f_\bullet)$ for every $x \in \mathbb{H}_\infty(X)$ (see Section 4.1). We get: the logarithm $\log(\lambda_1(f))$ of the dynamical degree of f is the translation length $L(f_\bullet)$ of the isometry $f_\bullet$. In particular, f is loxodromic if and only if $\lambda_1(f) > 1$.

5.3.2. *Classification*

Elliptic and parabolic transformations are also classified in terms of degree growth. Say that a sequence of real numbers $(d_n)_{n\geq 0}$ grows linearly (respectively quadratically) if $n/c \leq d_n \leq cn$ (respectively $n^2/c \leq d_n \leq cn^2$) for some $c > 0$.

Theorem 5.2 (Gizatullin, Cantat, Diller and Favre, see [10,11,16,20]). *Let X be a projective surface, defined over an algebraically closed field* $\mathbf{k}$, *and* $\mathbf{h}$ *be a polarization of X. Let f be a birational transformation of X.*

(1) *f is elliptic if and only if the sequence* $\deg_{\mathbf{h}}(f^n)$ *is bounded. In this case, there exists a birational map $\phi\colon Y \dashrightarrow X$ and an integer $k \geq 1$ such that $\phi^{-1} \circ f \circ \phi$ is an automorphism of Y and $\phi^{-1} \circ f^k \circ \phi$ is in the connected component of the identity of the group* $\mathsf{Aut}(Y)$;

(2) *f is parabolic if and only if the sequence* $\deg_{\mathbf{h}}(f^n)$ *grows linearly or quadratically with n. If f is parabolic, there exists a birational map $\psi\colon Y \dashrightarrow X$ and a fibration $\pi\colon Y \to B$ onto a curve B such that $\psi^{-1} \circ f \circ \psi$ permutes the fibers of π. The fibration is rational if the growth is linear, and elliptic (or quasi-elliptic if char*$(\mathbf{k}) \in \{2, 3\}$*) if the growth is quadratic;*

(3) *f is loxodromic if and only if* $\deg_{\mathbf{h}}(f^n)$ *grows exponentially fast with n: There is a constant* $b_{\mathbf{h}}(f) > 0$ *such that* $\deg_{\mathbf{h}}(f^n) = b_{\mathbf{h}}(f)\lambda(f)^n + O(1)$.

5.4. Elliptic elements of $\mathrm{Cr}_2(\mathbf{k})$

Every elliptic, infinite order element of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is conjugate to an automorphism $f \in \mathsf{PGL}_3(\mathbf{k})$ when $\mathbf{k}$ is algebraically closed (see [5]). Thus, Theorem 3.4 stipulates that elliptic elements of infinite order are

- Exactly doubly exponentially distorted if they are conjugate to a virtually unipotent element of $\mathsf{PGL}_3(\mathbf{k})$;
- Exactly exponentially distorted otherwise.

5.5. Loxodromic elements of $\mathrm{Cr}_2(\mathbf{k})$

Loxodromic elements have an exponential degree growth; by Proposition 2.1, they are not distorted. This result applies to all loxodromic elements $f \in \mathsf{Bir}(X)$, for all projective surfaces.

5.6. Parabolic elements of $\mathrm{Cr}_2(\mathbf{k})$

According to Theorem 5.2, there are two types of parabolic elements, depending on the growth of the sequence $\deg(f^n)$: Jonquières and Halphen twists. Here, we collect extra informations on these transformations, and study their distortion properties in Sections 6 and 7.

5.6.1. *Jonquières twists*

Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$ for which the sequence $\deg(f^n)$ grows linearly with n. Then, f is called a *Jonquières twist*. Examples are given by the transformations $f(X, Y) = (X, Q(X)Y)$ with $Q \in \mathbf{k}(X)$ of degree ≥ 1. The following properties follow from [4,5,16].

Normal form. There is a birational map $\varphi\colon \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ that conjugates f to an element g of $\mathsf{Bir}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$ which preserves the projection $\pi\colon \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$ onto the first factor. More precisely, there is an automorphism A of $\mathbb{P}^1_{\mathbf{k}}$ such that $\pi \circ g = A \circ \pi$. If x and y are affine coordinates on each of the factors, then

$$g(x, y) = (A(x), B(x)(y)), \tag{5.8}$$

where (A, B) is an element of the semi-direct product $\mathsf{PGL}_2(\mathbf{k}) \ltimes \mathsf{PGL}_2(\mathbf{k}(x))$. Alternatively, f is conjugate to an element g' of $\mathrm{Cr}_2(\mathbf{k})$ that preserves the pencil of lines through the point $[0 : 0 : 1]$.

Action on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$. Assume now that g' preserves the pencil of lines through the point $q_1 := [0 : 0 : 1]$. Let $\mathbf{e}_1 \in \mathcal{Z}(\mathbb{P}^2_{\mathbf{k}}; \mathbf{R})$ be the class of the exceptional divisor E_1 that one gets by blowing-up q_1. Then $g'_\bullet$ preserves the isotropic vector $\mathbf{e}_0 - \mathbf{e}_1$ (corresponding to the class of the linear system of lines through q_1), and the unique fixed point of $g'_\bullet$ on $\partial\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ is determined by $\mathbf{e}_0 - \mathbf{e}_1$. Let d denote the degree $\deg_{\mathbf{e}_0}(g')$. Let q_i denote the base points of $(g')^{-1}$ (including infinitely near base points) and $\mathbf{e}(q_i)$ be the corresponding classes of exceptional divisors. From [1,4], one knows that there are $2d-1$ base points (including q_1), and that

$$g'_{\bullet}\mathbf{e}_0 = d\mathbf{e}_0 - (d-1)\mathbf{e}(q_1) - \sum_{i=2}^{2d-1} \mathbf{e}(q_i) \tag{5.9}$$

$$g'_{\bullet}\mathbf{e}(q_1) = (d-1)\mathbf{e}_0 - (d-2)\mathbf{e}(q_1) - \sum_{i=2}^{2d-1} \mathbf{e}(q_i). \tag{5.10}$$

Degree growth. The sequence $\frac{1}{n}\deg_{\mathbf{e}_0}(f^n)$ converges toward a number $\alpha(f)$. The set $\{\alpha(hfh^{-1}); h \in \mathrm{Cr}_2(\mathbf{k})\}$ admits a minimum; this minimum is of the form $\frac{1}{2}\mu(f)$ for some integer $\mu(f) > 0$, and there is an integer $a \geq 1$ such that $\alpha(f) = \frac{1}{2}\mu(f)a^2$. Blanc and Déserti prove also that $a = 1$ precisely when f preserves a pencil of lines in $\mathbb{P}^2_{\mathbf{k}}$ (thus, the conjugate g' of f satisfies $\alpha(g') = \frac{1}{2}\mu(f)$). Moreover, when f preserves such a pencil, one knows from [4, Lemma 5.7], that $\deg_{\mathbf{e}_0}(f^n)$ is a subadditive sequence. Thus, $\frac{1}{n}\deg_{\mathbf{e}_0}(f^n) \geq \mu/2$, and $\mu/2$ is the infimum of $\frac{1}{n}\deg_{\mathbf{e}_0}(f^n)$. In Section 7, we shall describe how Blanc and Déserti interpret $\mu(f)$ as an asymptotic number of base points.

5.6.2. *Halphen twists*

Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$ for which the sequence $\deg(f^n)$ grows quadratically with n. Then, f is called a *Halphen twist*. The following properties follow from [4,14,15].

Normal form. There is a rational surface X, together with a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ and a genus 1 fibration $\pi : X \to \mathbb{P}^1_{\mathbf{k}}$ such that $g = \varphi^{-1} \circ f \circ \varphi$ is a regular automorphism of X that preserves the fibration π. More precisely, there is an element A in $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}})$ of finite order such that $\pi \circ g = A \circ \pi$. Changing g into g^k where k is the order of g, we may assume that the action on the base of π is trivial; then, g acts by translations along the fibers of π.

There is a classification of genus 1 pencils of the plane up to birational conjugacy, which dates back to Halphen (see [18,21]): a Halphen pencil of index l is a pencil of curves of degree $2l$ with 9 base-points of multiplicity l. Every Halphen twist f preserves such a pencil; on X, the pencil corresponds to the genus 1 fibration which is g-invariant.

Action on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ and degree growth. Let $\mathbf{c}$ be the class of the fibers of π in $\mathrm{Num}(X)$ (respectively in $\mathcal{Z}(X) = \mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$). This class is g-invariant (respectively $f_\bullet$-invariant) and isotropic. Thus $\mathbf{c} \in \mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ determines the unique fixed point of the parabolic isometry $f_\bullet$ on $\partial\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$.

After conjugacy, we may assume that the genus 1 fibration π comes from a Halphen pencil of the plane of index l with nine base points $q_1, \ldots, q_9$. This linear system corresponds to the class $\mathbf{c}$ such that

$$\frac{1}{l}\mathbf{c} = 3\mathbf{e}_0 - \sum_{j=1}^{9} \mathbf{e}(q_j). \tag{5.11}$$

Thus, after conjugacy, we may assume that the Halphen twist g fixes such a class. Under this hypothesis, of [4, Lemma 5.10] provides the following inequality

$$\sqrt{\deg_{\mathbf{e}_0}(g^{n+m})} \leq \sqrt{\deg_{\mathbf{e}_0}(g^n)} + \sqrt{\deg_{\mathbf{e}_0}(g^m)} \tag{5.12}$$

for all integers $n, m \geq 0$. In particular, the number

$$\tau(g) = \inf_{n>0} \frac{1}{n}\sqrt{\deg_{\mathbf{e}_0}(g^n)} = \lim_{n\to+\infty} \frac{1}{n}\sqrt{\deg_{\mathbf{e}_0}(g^n)} \tag{5.13}$$

is a well defined positive real number, and $\deg_{\mathbf{e}_0}(g^n) \geq \tau(g)n^2$ for all $n \geq 1$. Blanc and Déserti prove that the minimum $\kappa(g) = \min \tau(hgh^{-1})^2$ for $h \in \mathrm{Cr}_2(\mathbf{k})$ is a positive rational number and that $\lim_{n\to+\infty} \frac{1}{n^2} \deg_{\mathbf{e}_0}(g^n) = \frac{\kappa(g)}{9} a^2$ for some integer $a \geq 3$.

6. Parabolic elements of $\mathrm{Cr}_2(\mathbf{k})$ and their invariant horoballs

For simplicity, the hyperbolic space $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ will be denoted by $\mathbb{H}_\infty$. In this section we prove Theorem 6.1, which states that sufficiently small horoballs invariant by Jonquières or Halphen twists are pairwise disjoint. Combined with Theorem 4.1, this result implies that Halphen twists are not distorted.

6.1. Small horoballs associated to Halphen and Jonquières twists

6.1.1. *Fixed points of Jonquières and Halphen twists*

Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$ acting as a parabolic isometry on the hyperbolic space $\mathbb{H}_\infty$. Then, f fixes a unique point ξ on the boundary $\partial\mathbb{H}_\infty$. Up to conjugacy, there are two possibilities:

- f is a Jonquières twist, and f preserves the pencil of lines through a point q_1 of $\mathbb{P}^2_{\mathbf{k}}$. Then, setting $\mathbf{e}_1 = \mathbf{e}(q_1)$, the boundary point ξ is represented by the ray $\mathbf{R}^+ w$, where
$$w_J = \mathbf{e}_0 - \mathbf{e}_1; \tag{6.1}$$
- f is a Halphen twist. Then, up to conjugacy, ξ is $\mathbf{R}^+ w$ with
$$w_H = 3\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8 - \mathbf{e}_9, \tag{6.2}$$
where the $\mathbf{e}_i$ are the classes given by the blow-up of the base-points of a Halphen pencil.

6.1.2. *Disjonction of horoballs*

If w is an element of the Picard-Manin space with $w^2 = 0$ and $w \cdot \mathbf{e}_0 > 0$, the ray $\mathbf{R}^+ w$ determines a boundary point of $\mathbb{H}_\infty$. Let ϵ be a positive real number. The horoball $H_w(\epsilon)$ is defined in Section 4.1.4; its elements are characterized by the following three constraints:

$$v^2 = 1, \quad v \cdot \mathbf{e}_0 > 0, \quad 0 < v \cdot w < \epsilon. \tag{6.3}$$

When f is a Jonquières or Halphen twist then, after conjugacy, $f_\bullet$ preserves the horoballs centered $H_{w_J}(\epsilon)$ or $H_{w_H}(\epsilon)$. Define

$$\epsilon_J = \frac{\sqrt{3}-1}{\sqrt{2}} \simeq 0.5176 \quad \text{and} \quad \epsilon_H := \frac{1}{3\sqrt{2}} \simeq 0.2357. \tag{6.4}$$

Theorem 6.1. *Let w_J be the class $\mathbf{e}_0 - \mathbf{e}_1 \in \mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ determined by the pencil of lines through a point q_1. If $0 < \epsilon < \epsilon_J$, the horoballs $h(H_{w_J}(\epsilon))$, for $h \in \mathrm{Cr}_2(\mathbf{k})$, are pairwise disjoint; more precisely, given h in $\mathrm{Cr}_2(\mathbf{k})$,*

$$\text{either} \quad h(H_{w_J}(\epsilon)) = H_{w_J}(\epsilon) \quad \text{or} \quad h(H_{w_J}(\epsilon)) \cap H_{w_J}(\epsilon) = \emptyset.$$

Let w_H be the class $3\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8 - \mathbf{e}_9$ determined by a Halphen pencil. If $0 < \epsilon \leq \epsilon_H$, the horoballs $h(H_{w_H}(\epsilon))$, $h \in \mathrm{Cr}_2(\mathbf{k})$, are pairwise disjoint; more precisely, given h in $\mathrm{Cr}_2(\mathbf{k})$,

$$\text{either} \quad h(H_{w_H}(\epsilon)) = H_{w_H}(\epsilon) \quad \text{or} \quad h(H_{w_H}(\epsilon)) \cap H_{w_H}(\epsilon) = \emptyset.$$

6.2. Proof of the first assertion

6.2.1. For simplicity, we write w instead of w_J. Let h be an element of $\mathrm{Cr}_2(\mathbf{k})$. If $h_\bullet$ fixes the line $\mathbf{R}_+ w$, then it fixes w and its dynamical degree is equal to 1; thus, h fixes the horoballs $H_w(\epsilon)$. We may therefore assume that $h_\bullet$ does not fix w. Write

$$h_\bullet(w) = h_\bullet(\mathbf{e}_0 - \mathbf{e}_1) = m\mathbf{e}_0 - \sum_i r_i \mathbf{e}_i \tag{6.5}$$

for some multiplicities r_i in $\mathbf{Z}^+$. Since $w^2 = 0$, we get

$$m^2 = \sum_i r_i^2. \tag{6.6}$$

For later purpose, we shall write $r_1 = m - s_1$ for some integer $s_1 \geq 0$. Then,

$$s_1^2 + \sum_{j \geq 2} r_j^2 = 2ms_1. \tag{6.7}$$

Remark 6.2. We have $h_\bullet(w) = w$ if and only if $m = 1$ and $r_1 = 1$, if and only if $s_1 = 0$. Indeed, if $s_1 = 0$, then the last equation implies that all r_j vanish for $j \geq 2$. Hence, $h_\bullet(w) = mw$ for some $m \geq 1$, h is parabolic, and m must be equal to the dynamical degree of h, so that $m = 1$.

6.2.2. Assume that $h_\bullet(H_w(\epsilon))$ intersects $H_w(\epsilon)$. Then, there exists a point u in the intersection. Write

$$u = \alpha_0 e_0 - \sum_i \alpha_i e_i. \tag{6.8}$$

By definition of $H_w(\epsilon)$, we have $0 < w \cdot u < \epsilon$ and $0 < h_\bullet(w) \cdot u < \epsilon$, *i.e.*,

$$0 < \alpha_0 - \alpha_1 < \epsilon \quad \text{and} \quad 0 < m\alpha_0 - \sum_i r_i \alpha_i < \epsilon. \tag{6.9}$$

We shall write $\alpha_1 = \alpha_0 - \tau$ with $0 < \tau < \epsilon$. Since $u \cdot e_0 > 0$ we know that $\alpha_0 > 0$, and since $u^2 = 1$ we have

$$\sum_i \alpha_i^2 = \alpha_0^2 - 1, \tag{6.10}$$

and therefore

$$\tau^2 + \sum_{j \geq 2} \alpha_j^2 = 2\alpha_0 \tau - 1. \tag{6.11}$$

6.2.3. In a first step, we prove a lower estimate for α_0. By Equation (6.9),

$$m\alpha_0 < \epsilon + \sum_i \alpha_i r_i. \tag{6.12}$$

Apply Cauchy-Schwartz inequality and use Equations (6.5) and (6.10) to obtain

$$m\alpha_0 < \epsilon + \left(\sum_i \alpha_i^2\right)^{1/2} \left(\sum_i r_i^2\right)^{1/2} = \epsilon + \left(\alpha_0^2 - 1\right)^{1/2} \left(m^2\right)^{1/2}. \tag{6.13}$$

This gives

$$m\alpha_0 \left(1 - \left(1 - 1/\alpha_0^2\right)^{1/2}\right) < \epsilon. \tag{6.14}$$

Then, remark that $(1-t)^{1/2} \leq 1 - t/2$, to deduce $1 - (1 - 1/\alpha_0^2)^{1/2} \geq \frac{1}{2\alpha_0^2}$, and inject this relation in the previous inequality to get

$$\frac{m}{2\epsilon} < \alpha_0. \tag{6.15}$$

6.2.4. Isolate $r_1\alpha_1$ in Equation (6.9), *i.e.* write $m\alpha_0 - r_1\alpha_1 - \sum_{j\geq 2} \alpha_j r_j < \epsilon$, to obtain

$$s_1 \alpha_0 + m\tau < \epsilon + s_1 \tau + \sum_{j \geq 2} \alpha_j r_j. \tag{6.16}$$

Then, remark that $m\tau \geq 0$, and apply Cauchy-Schwartz estimate to the vectors $(s_1, (r_j)_{j\geq 2})$ and $(\tau, (\alpha_j)_{j \geq 2})$; from Equations (6.11) and (6.7) we get

$$s_1 \alpha_0 < \epsilon + (2\alpha_0\tau - 1)^{1/2} (2ms_1)^{1/2} \tag{6.17}$$

$$< \epsilon + 2(\alpha_0 \epsilon)^{1/2} (ms_1)^{1/2} \tag{6.18}$$

because $0 < \tau < \epsilon$. This gives

$$\left(\frac{s_1}{m}\alpha_0\right)^{1/2} < \frac{\epsilon}{(ms_1\alpha_0)^{1/2}} + 2(\epsilon)^{1/2}$$

and the inequality $\alpha_0 > m/(2\epsilon)$ gives

$$\left(\frac{s_1}{2\epsilon}\right)^{1/2} < \frac{\epsilon^{3/2}}{(m^2 s_1/2)^{1/2}} + 2(\epsilon)^{1/2}.$$

Since $s_1 \geq 1$ and $m \geq 1$ we get $(\sqrt{2})^{-1} < \sqrt{2}\epsilon^2 + 2\epsilon$, in contradiction with $\epsilon < \epsilon_J$.

6.3. Proof of the second assertion

The proof follows the same lines.

6.3.1. For simplicity, we write w instead of w_H. Let h be an element of $\mathrm{Cr}_2(\mathbf{k})$. If $h_\bullet$ the line $\mathbf{R}w$, it fixes also the class w, and its dynamical degree is equal to 1; thus, $h_\bullet$ fixes the horoballs $H_w(\epsilon)$. Thus, we may assume that $h_\bullet$ does not fix w. Write

$$h_\bullet(w) = me_0 - \sum_i r_i e_i \tag{6.19}$$

for some r_i in $\mathbf{Z}^+$. Since $w^2 = 0$, we get

$$m^2 = \sum_i r_i^2. \tag{6.20}$$

For later purpose, we shall write $r_i = (m/3) - s_i$ for each index $1 \leq i \leq 9$. Then

$$\sum_{i=1}^{9} s_i^2 + \sum_{j\geq 10} r_j^2 = (2/3)mS, \tag{6.21}$$

with

$$S := \sum_{i=1}^{9} s_i. \tag{6.22}$$

Remark 6.3. We have $h_\bullet(w) = w$ if and only if $m = 3$ and $r_i = 1$ for $1 \leq i \leq 9$. This is equivalent to $S = 0$. Indeed, if $S = 0$, then the last inequality implies that all multiplicities r_j vanish for $j \geq 10$, and all s_i vanish for $1 \leq i \leq 9$. Thus, $h_\bullet(w) = mw$, m must be equal to the dynamical degree of h, and $m = 1$.

6.3.2. Assume that $h_\bullet(H_w(\epsilon))$ intersects $H_w(\epsilon)$. Then, there exists a point u in the intersection. Write $u = \alpha_0 e_0 - \sum_i \alpha_i e_i$. By definition, we have $0 < w \cdot u < \epsilon$ and $0 < h_\bullet(w) \cdot u < \epsilon$, *i.e.*,

$$0 < 3\alpha_0 - \sum_{i=1}^{9} \alpha_i < \epsilon \quad \text{and} \quad 0 < m\alpha_0 - \sum_i r_i \alpha_i < \epsilon. \tag{6.23}$$

We shall write $\alpha_i = (1/3)\alpha_0 - \tau_i$ for $1 \le i \le 9$, and $T = \sum_{i=1}^{9} \tau_i$. Then,

$$0 < T < \epsilon. \tag{6.24}$$

Since $u \cdot e_0 > 0$ we know that $\alpha_0 > 0$, and since $u^2 = 1$ we have

$$\sum_i \alpha_i^2 = \alpha_0^2 - 1. \tag{6.25}$$

Thus,

$$\sum_{i=1}^{9} \tau_i^2 + \sum_{j \ge 10} \alpha_j^2 = (2/3)\alpha_0 T - 1. \tag{6.26}$$

6.3.3. The following lower estimate is obtained as in the case $w = w_J$:

$$\frac{m}{2\epsilon} < \alpha_0. \tag{6.27}$$

6.3.4. Now, isolate the terms $r_i\alpha_i$, for i between 1 and 9, in Equation (6.23):

$$m\alpha_0 - \sum_{i=1}^{9} r_i\alpha_i - \sum_{j \ge 10} \alpha_j r_j < \epsilon. \tag{6.28}$$

We obtain

$$(m - (1/3)\sum_{i=1}^{9} r_i)\alpha_0 + \sum_i r_i \tau_i < \epsilon + \sum_{j \ge 10} \alpha_j r_j \tag{6.29}$$

i.e.,

$$(1/3)S\alpha_0 + (1/3)mT < \epsilon + \sum_{i=1}^{9} s_i \tau_i + \sum_{j \ge 10} \alpha_j r_j. \tag{6.30}$$

Apply again, the fact that $mT \ge 0$ and Cauchy-Schwartz estimate:

$$(1/3)S\alpha_0 - \epsilon < ((2/3)\alpha_0 T - 1)^{1/2}((2/3)mS)^{1/2} < (2/3)(\alpha_0\epsilon)^{1/2}(mS)^{1/2} \tag{6.31}$$

because $0 < T < \epsilon$. This gives

$$\frac{1}{3}\left(\frac{S}{m}\alpha_0\right)^{1/2} < \frac{\epsilon}{(mS\alpha_0)^{1/2}} + \frac{2}{3}(\epsilon)^{1/2} \tag{6.32}$$

and the inequality $\alpha_0 > m/(2\epsilon)$ implies

$$\frac{1}{3}\left(\frac{S}{2\epsilon}\right)^{1/2} < \frac{\epsilon^{3/2}}{(m^2 S/2)^{1/2}} + \frac{2}{3}(\epsilon)^{1/2}. \tag{6.33}$$

Since $S \geq 1$ and $m \geq 1$, we get $(3\sqrt{2})^{-1} < \sqrt{2}\epsilon^2 + (2/3)\epsilon$, in contradiction with $\epsilon < \epsilon_H$.

6.4. Consequence: Halphen twists are not distorted

Let $h \in \mathrm{Cr}_2(\mathbf{k})$ be a Halphen twist. After conjugacy, we may assume that $h_\bullet$ preserves the class w_H associated to some Halphen pencil. We know from Section 5.6.2 that the degree growth of h is quadratic, with

$$\deg_{\mathbf{e}_0}(h^n) \geq (\tau(h)n)^2. \tag{6.34}$$

Since $\deg_{\mathbf{e}_0}(h^n)$ is equal to $\cosh(\mathsf{dist}(h_\bullet \mathbf{e}_0, \mathbf{e}_0))$, we obtain the lower bound

$$\log \mathsf{dist}(h_\bullet \mathbf{e}_0, \mathbf{e}_0)) \geq 2\log(n) - 2\log(\tau(h)). \tag{6.35}$$

Set $\mathbb{H}_m = \mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$, $f = g_\bullet$, $G = \mathrm{Cr}_2(\mathbf{k})$, $B = H_{w_H}(\epsilon_H/2)$, and $C = 2$. By Theorem 6.1 if g is an element of G then $g(B) = B$ or $g(B) \cap B = \emptyset$. Thus, we may apply Theorem 4.1 to $f = h_\bullet$ and we get the desired result: h is undistorted in $\mathrm{Cr}_2(\mathbf{k})$.

6.5. Non-rational surfaces

The previous paragraph makes use of the explicit description of Halphen pencils in $\mathbb{P}^2_{\mathbf{k}}$. Here, we consider a smooth projective surface X, over the algebraically closed field $\mathbf{k}$, and assume that

- X is not rational;
- f is a birational transformation of X with $\deg(f^n) \simeq n^2$ (we shall say that f is a Halphen twist of X).

Then, from Theorem 5.2, we know that f preserves a unique pencil of genus 1.

Lemma 6.4. *The Kodaira dimension of X is equal to* 0 *or* 1. *The surface X has a unique minimal model X_0, and* $\mathsf{Bir}(X_0) = \mathsf{Aut}(X_0)$.

Proof. A Halphen twist has infinite order, thus $\mathsf{Bir}(X)$ is infinite, and the Kodaira dimension of X is < 2. If it is equal to $-\infty$, then X is a ruled surface, and since X is not rational, the ruling is unique and $\mathsf{Bir}(X)$-invariant. Thus, f must preserve two pencils. These two rational fibrations determine two $f_\bullet$-invariant isotropic classes in $\mathcal{Z}(X)$, in contradiction with the fact that $f_\bullet$ is parabolic. This proves the first assertion. The second one is a well-known consequence of the first. □

We can therefore conjugate f to an automorphism f_0 of X_0, and assume that $\mathsf{Bir}(X_0) = \mathsf{Aut}(X_0)$. Thus, the distortion of f in $\mathsf{Bir}(X)$ is now equivalent to the distortion of f_0 in $\mathsf{Aut}(X_0)$. Instead of looking at the infinite dimensional vector space $\mathcal{Z}(X)$, we can look at the action of $\mathsf{Aut}(X_0)$ on the Néron-Severi group $\mathrm{Num}(X_0)$.

Identify $\mathrm{Num}(X_0)$ to $\mathbf{Z}^r$, where r is the Picard number of X, and denote by q_0 the intersection form on $\mathrm{Num}(X_0)$. Then, the image of $\mathsf{Aut}(X_0)$ in $\mathsf{GL}\,(\mathrm{Num}(X))$ is a subgroup of the orthogonal group $O^+(q_0; \mathbf{Z})$ preserving the hyperbolic space $\mathbb{H}_r \subset \mathrm{Num}(X_0; \mathbf{R})$ defined by q_0. The quotient $V = \mathbb{H}_r / O^+(q_0; \mathbf{Z})$ is a hyperbolic orbifold, and the fixed point ξ of f_0 in $\partial\mathbb{H}_r$ gives a cusp of V. A sufficiently small horoball B centered at ξ determines a neighborhoods of this cusp (see [30]). Thus, if g is an element of $O^+(q_0; \mathbf{Z})$, then $g(B) = B$ or $g(B) \cap B = \emptyset$, as in Theorem 6.1. From Theorem 4.1, we deduce that f is undistorted. We have proved:

Theorem 6.5. *Let* $\mathbf{k}$ *be an algebraically closed field. Let X be a smooth projective surface, defined over* $\mathbf{k}$. *If $f \in \mathsf{Bir}(X)$ is a Halphen twist* (i.e. $\deg(f^n) \simeq n^2$)*, then f is not distorted in* $\mathsf{Bir}(X)$.

7. Jonquières twists are undistorted

The argument presented in Section 6.4 to show that Halphen twists are undistorted is not sufficient for Jonquières twists; it only gives a quadratic upper bound on the distortion function. As we shall see, the following result follows from [5].

Theorem 7.1. *Let* $\mathbf{k}$ *be an algebraically closed field, and let $X_{\mathbf{k}}$ be a projective surface. Let f be an element of* $\mathsf{Bir}(X)$. *If f is a Jonquières twist (*i.e. *if the sequence* $\deg(f^n)$ *grows linearly) then f is not distorted in* $\mathsf{Bir}(X)$.

7.1. In the Cremona group

We first describe the proof when X is the projective plane. Denote by $\mathrm{bp}\colon \mathrm{Cr}_2(\mathbf{k}) \to \mathbf{Z}_+$ the function *number of base-points*: $\mathrm{bp}(f)$ is the number of base-points of the homaloidal net of f, *i.e.* of the linear system of curves obtained by pulling-back the system of lines in $\mathbb{P}^2$. Indeterminacy points are examples of base-points, but the base-point set may also include infinitely near points. The number of base-points is also the number of blow-ups needed to construct a minimal resolution of the indeterminacies of f. If $f_\bullet$ denotes the action of f on the Picard-Manin space, and e_0 is the class of a line, then

$$\left(f^{-1}\right)_\bullet e_0 = d e_0 - \sum_i m_i e(p_i), \tag{7.1}$$

where d is the degree of f and m_i is the multiplicity of the homaloidal system $f^*\mathcal{O}(1)$ at the base-point p_i; thus, $\mathrm{bp}(f)$ is just the number of classes for which the multiplicity m_i is positive. The number of base-points is non-negative, is subadditive, and is symmetric (see [5]): $\mathrm{bp}(f \circ g) \leq \mathrm{bp}(f) + \mathrm{bp}(g)$ and $\mathrm{bp}(f) = \mathrm{bp}(f^{-1})$. As a consequence, the limit

$$\alpha(f) = \lim_{n \to +\infty} \frac{1}{n} \mathrm{bp}(f^n) \tag{7.2}$$

exists and is non-negative. It is symmetric, *i.e.* $\alpha(f^{-1}) = \alpha(f)$, invariant under conjugacy, and it vanishes if f is distorted, because if f is distorted its stable length vanishes (Lemma 1.4) and this implies $\alpha(f) = 0$ by the subadditivity of bp.

Blanc and Déserti prove that $\alpha(f)$ is a non-negative integer, and that it vanishes if and only if f is conjugate to an automorphism by a birational map $\pi : X \dashrightarrow \mathbb{P}^2$. In particular, $\mathrm{bp}(f) > 0$ for Jonquières twists because they are not conjugate to automorphisms; but the result of Blanc and Déserti is even more precise: if f is a Jonquières twist, $\alpha(f)$ coïncides with the integer $\mu(f)$ which was defined in Section 5.6.1. Theorem 7.1 follows from those results.

7.2. In $\mathsf{Bir}(X)$

The definition of $\mathrm{bp}(f)$ extends to birational transformations of arbitrary smooth surfaces; again, its stable version $\alpha(f)$ is invariant under conjugacy, vanishes when f is distorted, and may be interpreted as the number of terms in the decomposition $f^n \mathbf{e}_0 = \mathbf{u}_X + \sum_i a_i \mathbf{e}_i$ in the Picard-Manin space $\mathcal{Z}(X) = \mathrm{Num}(X) \oplus_i \mathbf{Z}\mathbf{e}_i$ (see Section 5), where $\mathbf{e}_0$ is any ample class in $\mathrm{Num}(X)$. (The proofs of Blanc and Déserti extend directly to this general situation.)

If $f \in \mathsf{Bir}(X)$ is a parabolic element with $\deg(f^n) \simeq n$, and if X is not a rational surface, one can do a birational conjugacy to assume that X is the product $C \times \mathbb{P}^1_{\mathbf{k}}$ of a curve of genus $g(C) \geq 1$ with the projective line. Then, $\mathsf{Bir}(X)$ preserves the projection $\pi : X \to C$, acting by automorphisms on the base.

The Néron-Severi group of X has rank 2, and is generated by the class $\mathbf{v}$ of a vertical line $\{x_0\} \times \mathbb{P}^1_{\mathbf{k}}$ and by the class $\mathbf{h}$ of a horizontal section $C \times \{y_0\}$. The canonical class $\mathbf{k}_X$ is $2(g-1)\mathbf{v} - 2\mathbf{h}$, where g is the genus of the curve C. Blowing-up X, the canonical class of the surfaces $X' \to X$ determines a limit

$$\tilde{\mathbf{k}} = 2(g-1)\mathbf{v} - 2\mathbf{h} + \sum_i \mathbf{e}_i\,, \tag{7.3}$$

where the $\mathbf{e}_i$ are the classes of all exceptional divisors, as in Section 5. This limit is not an element of the Picard-Manin space $\mathsf{Z}(X)$, but it determines a linear form on the $\mathbf{Z}$-module $\mathcal{Z}(X)$; this form is invariant under the action of $\mathsf{Bir}(X)$ on $\mathcal{Z}(X)$.

As an ample class, take $\mathbf{e}_0 = \sqrt{2}^{-1}(\mathbf{v} + \mathbf{h})$. This is an element of $\mathbb{H}_\infty(X)$. If f is an element of $\mathsf{Bir}(X)$, it preserves the class $\mathbf{v}$ of the fibers of $\pi : X \to C$; hence $\sqrt{2} f_\bullet(\mathbf{e}_0) = \mathbf{h} + d\mathbf{v} - \sum a_i \mathbf{e}_i$ for some multiplicities $a_i \in \mathbf{Z}_+$. Applied to $\sqrt{2} f_\bullet(\mathbf{e}_0)$, the invariance of the canonical class leads to the following constraint:

$$2(d-1) = \sum_i a_i. \tag{7.4}$$

And the invariance of the intersection form gives

$$2(d-1) = \sum_i a_i^2. \tag{7.5}$$

Thus, $a_i = 1$ or 0, and there are exactly $2(d-1)$ non-zero terms in the sum $\sum_i a_i \mathbf{e}_i$. We get

$$\sqrt{2} f_\bullet(\mathbf{e}_0) = \mathbf{h} + d\mathbf{v} - \sum_{i=1}^{2(d-1)} \mathbf{e}_i. \tag{7.6}$$

When f is a Jonquières twist, then $\deg(f^n) \simeq n$, and the number $\mathrm{bp}(f^n)$ of terms in the sum also grows linearly, like $2\deg(f^n)$. Thus, $\alpha(f) > 0$, extending the result of Blanc and Déserti to all surfaces. This concludes the proof of Theorem 7.1.

8. Appendix: two examples

8.1. Baumslag-Solitar groups

Fix a pair of integers $k, \ell \geq 2$. In the Baumslag-Solitar group $B_k = \langle t, x \mid txt^{-1} = x^k \rangle$, we have $\delta_x(n) \simeq \exp(n)$ (see [22], Section 3.K1). In the "double" Baumslag-Solitar group

$$B_{k,\ell} = \left\langle t, x, y \mid txt^{-1} = x^k, xyx^{-1} = y^\ell \right\rangle,$$

we have $t^n x t^{-n} = x^{k^n} \in S^{2n+1}$ and $x^{k^n} y x^{-k^n} = y^{\ell^{k^n}} \in S^{4n+3}$; hence, $\delta_{y,S}(4n+3) \geq \ell^{k^n}$ and the distortion of y in $B_{k,\ell}$ is at least doubly exponential. In fact, we can check $\delta_y(n) \simeq \exp\exp(n)$ in $B_{k,\ell}$ as follows. Consider the homeomorphisms of the real lines $\mathbf{R}$ which are defined by $Y(s) = s+1$, $X(s) = \ell s$, and $T(s) = \mathrm{sign}(s)|s|^k$; the relations satisfied by t, x and y in $B_{k,\ell}$ are also satisfied by T, X and Y in $\mathsf{Homeo}(\mathbf{R})$: this gives a homomorphism from $B_{k,\ell}$ to $\mathsf{Homeo}(\mathbf{R})$. If f is any of the three homeomorphisms T, X and Y or their inverses, it satisfies $|f(s)| \leq \max(2\ell, |s|^k)$. Thus, a recursion shows that every word w of length n in the generators is a homeomorphism satisfying $|w(0)| \leq (2\ell)^{k^n}$. Since $Y^m(0) = m$, this shows that the distortion of y is at most doubly exponential.

8.2. Locally nilpotent groups

Consider the group M of upper triangular (infinite) matrices whose entries are indexed by the ordered set $\mathbf{Q}$ of rational numbers, the coefficients are rational numbers, and the diagonal coefficients are all equal to 1:

(1) M is perfect (it coincides with it derived subgroup), and torsion free;
(2) M is locally nilpotent (every finitely generated subgroup is nilpotent);
(3) For every integer $d \geq 1$, the elementary matrix $U = \mathrm{Id} + E_{0,1}$ is in the d-th derived subgroup of a finitely generated, nilpotent subgroup N_d of M.

The first two assertions are described in [31, Section 6.2]; the last one follows from the following two simple remarks: the elementary matrix $\mathrm{Id}+E_{d,d+1}$ is in the center of the group of upper triangular matrices of $\mathsf{SL}_{d+1}(\mathbf{Q})$; the translation $\alpha \mapsto \alpha - d$ is an order preserving permutation of $\mathbf{Q}$, and this action determines an automorphism of the group M that maps $\mathrm{Id}+E_{d,d+1}$ to U. Property (3) implies that the distortion of U in N_d is n^d. This implies that the distortion of U in M is at least n^d for all d; but its distortion is polynomial in every finitely generated subgroup of M.

References

[1] M. ALBERICH-CARRAMIÑANA, "Geometry of the Plane Cremona Maps", Lecture Notes in Mathematics, Vol. 1769, Springer-Verlag, Berlin, 2002.

[2] A. AVILA, *Distortion in* $\mathrm{Diff}^\infty(\mathbb{R}/\mathbb{Z})$, unpublished manuscript, arXiv:0808.2334v1.

[3] R. BENEDETTI and C. PETRONIO, "Lectures on Hyperbolic Geometry", Universitext, Springer-Verlag, Berlin, 1992.

[4] J. BLANC and S. CANTAT, *Dynamical degrees of birational transformations of projective surfaces*, J. Amer. Math. Soc. **29** (2016), 415–471.

[5] J. BLANC and J. DÉSERTI, *Degree growth of birational maps of the plane*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), 507–533.

[6] E. BOMBIERI and W. GUBLER, "Heights in Diophantine Geometry", New Mathematical Monographs, Vol. 4, Cambridge University Press, Cambridge, 2006.

[7] S. BOUCKSOM, C. FAVRE and M. JONSSON, *Degree growth of meromorphic surface maps*, Duke Math. J. **141** (2008), 519–538.

[8] M. BURGER, A. IOZZI and N. MONOD, *Equivariant embeddings of trees into hyperbolic spaces*, Int. Math. Res. Not. **22** (2005), 1331–1369.

[9] D. CALEGARI and M. H. FREEDMAN, *Distortion in transformation groups*, Geom. Topol. **10** (2006), 267–293.

[10] S. CANTAT, *Dynamique des automorphismes des surfaces K*3, Acta Math. **187** (2001), 1–57.

[11] S. CANTAT, *Sur la dynamique du groupe d'automorphismes des surfaces K*3, Transform. Groups **6** (2001), 201–214.

[12] S. CANTAT, *Sur les groupes de transformations birationnelles des surfaces*, Ann. of Math. **174** (2011), 299–340.

[13] S. CANTAT, *The Cremona groups*, In: "Proceedings of 2015 Summer Institute on Algebraic Geometry", AMS Proceedings of Symposia in Pure Mathematics, 2016, 1–48.

[14] S. CANTAT and I. DOLGACHEV, *Rational surfaces with a large group of automorphisms*, J. Amer. Math. Soc. **25** (2012), 863–905.

[15] S. CANTAT and C. FAVRE, *Symétries birationnelles des surfaces feuilletées*, J. Reine Angew. Math. **561** (2003), 199–235.

[16] J. DILLER and C. FAVRE, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), 1135–1169.

[17] T.-C. DINH and N. SIBONY, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, Ann. of Math. (2) **161** (2005), 1637–1644.

[18] I. V. DOLGAČHEV, *Rational surfaces with a pencil of elliptic curves*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 1073–1100.

[19] I. V. DOLGAČHEV, "Classical Algebraic Geometry. A modern view", Cambridge University Press, Cambridge, 2012.

[20] M. H. GIZATULLIN, *Rational G-surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 110–144.

[21] J. GRIVAUX, *Parabolic automorphisms of projective surfaces (after M. H. Gizatullin)*, Mosc. Math. J. **16** (2016), 275–298.

[22] M. GROMOV, *Asymptotic invariants of infinite groups*, In: "Geometric Group Theory", Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., Vol. 182, Cambridge Univ. Press, Cambridge, 1993, 1–295.

[23] F. LE ROUX and K. MANN, *Strong distortion in transformation groups*, Bull. Lond. Math. Soc. **50** (2018), 46–62.

[24] A. LUBOTZKY, S. MOZES and M. S. RAGHUNATHAN, *The word and Riemannian metrics on lattices of semisimple groups*, Inst. Hautes Études Sci. Publ. Math. **91** (2001), 5–53.

[25] A. LUBOTZKY, S. MOZES and M.S. RAGHUNATHAN, *Cyclic subgroups of exponential growth and metrics on discrete groups*, C.R. Acad. Sci. Paris Sér. I Math. **317** (1993), 735–735.

[26] Y. I. MANIN, "Cubic Forms", North-Holland Mathematical Library, Vol. 4, North-Holland Publishing Co., Amsterdam, second edition, 1986, Algebra, geometry, arithmetic, translated from the Russian by M. Hazewinkel.

[27] E. MILITON, *Éléments de distorsion de* $\mathrm{Diff}_0^\infty(M)$, Bull. Soc. Math. France **141** (2013), 35–46.

[28] E. MILITON, *Distortion elements for surface homeomorphisms*, Geom. Topol. **18** (2014), 521–614.

[29] B.-D. NGUYEN, *Degrees of iterates of rational transformations of projective varieties*, arXiv:1701.07760

[30] J. G. RATCLIFFE, "Foundations of Hyperbolic Manifolds", Graduate Texts in Mathematics, Vol. 149, Springer, New York, second edition, 2006.

[31] D. J. S. ROBINSON, "Finiteness Conditions and Generalized Soluble Groups. Part 2", Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 63, Springer-Verlag, New York-Berlin, 1972.

[32] T. T. TRUONG, *Relative dynamical degrees of correspondances over fields of arbitrary characteristic*, J. Reine Angew. Math. **758** (2020), 139–182.

[33] C. URECH, *Remarks on the degree growth of birational transformations*, Math. Res. Lett. **25** (2018), 291–308.

[34] A. WEIL, *On algebraic groups of transformations*, Amer. J. Math. **77** (1955), 355–391.

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