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Counting lines on surfaces, especially quintics

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Abstract. We introduce certain rational functions on a smooth projective surface $X \subset \mathbb{P}^3$ which facilitate counting the lines on X. We apply this to smooth quintics in characteristic zero to prove that they contain no more than 127 lines, and that any given line meets at most 28 others. We construct examples which demonstrate that the latter bound is sharp.

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1. Introduction

Recently, there has been considerable progress in the understanding of configurations of lines on smooth quartic surfaces in $\mathbb{P}^3$. If the ground field K has $\mathrm{char}(K) \neq 2, 3$, the maximal number of lines on a smooth X_4 is 64, and a line can be met by at most 20 other lines (see [14,18]). For $\mathrm{char}(K) = 3$ the maximal number is 112 and a line meets at most 30 other lines [15]. The cases $\mathrm{char}(K) = 2$ as well as $K = \mathbb{R}$, or even $K = \mathbb{Q}$ were studied in [5,6,16]. In fact, the former two papers deal more generally, and to great extent, with smooth quartic surfaces with many lines, not just the maximum. Most of this progress was made possible by the fact that smooth quartics are K3 surfaces, and lines endow them with elliptic fibrations (or genus one fibrations). Moreover, examples can be constructed with the help of lattice-theoretical techniques based on the Torelli theorem.

In contrast, for smooth surfaces $X_d \subset \mathbb{P}^3$ of degree $d \geq 5$ very little seems to be known, even if $K = \mathbb{C}$. Although the question what is the maximal number of lines appears in various places in the modern literature (see, *e.g.* [2,9,10]), the best known bound essentially stems from work of Salmon (see [4,17]) and has since then been only slightly improved by Segre ([18]):

$$M(d) \leq d(11d - 28) + 12 \tag{1.1}$$

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where $M(d)$ is the maximum number of lines on a smooth degree-d surface (at least over $\mathbb{C}$). In comparison, the Fermat surface of degree d contains exactly $3d^2$ lines (over any algebraically closed field of characteristic zero or exceeding the degree d) and there are examples with more lines known only in degrees $6, 8, 12, 20$ (see [3]).

Even less seems to be known about the maximum *valency* $v(\ell)$ of a line ℓ where

$$v(\ell) = \#\{\text{lines } \ell' \subset X_d; \quad \ell'.\ell = 1\}$$

and X_d is again assumed to be smooth. By [18] if ℓ is a line of the first kind (to be defined below, see Definition 2.1), $v(\ell)$ cannot exceed $(8d - 14)$ (see Proposition 2.2), but this bound does not generalize to the so-called lines of the second kind (*cf.* [11,14]).

In the case of quartics, the proof of the sharp bound $v(\ell) \leq 20$ for $\ell \subset X_4$ in [14] (outside characteristics 2, 3) is based on the study of the elliptic fibration given by the pencil of cubics residual to ℓ in X_4. Already for $\ell \subset X_5$, *i.e.*, on quintics, this approach leads to a genus 3-fibration, that is much more difficult to control. Instead, for a line $\ell \subset X_d$ we will define certain rational functions $\mathfrak{b}_0, \ldots, \mathfrak{b}_{d-3}$ on the line ℓ (Definition 3.3) which form the main novelty of this paper. The key feature of these functions, especially on the explicit and computational side, is that their common zeroes encode the points where ℓ is met by other lines on X_d (Proposition 3.7).

A line of the second kind on a quartic surface is always met by at least twelve other lines (see [13,18]), whereas one can construct examples of quintic surfaces that contain only one line and the latter is of the second kind. In this note we give another geometric interpretation of the notion of a line of the second kind (see Proposition 3.9) that explains why special properties of lines of the second kind on quartics do not carry over to lines on surfaces of degree $d \geq 5$. As an application, we sketch an elementary proof of the valency bound for lines on quartics from [14] in Example 3.11.

After those general preparations we shall focus on the case $d = 5$, *i.e.*, of quintics $X_5 \subset \mathbb{P}^3$. Our first main result concerns the valency on smooth quintics:

Theorem 1.1. *Let K be a field of characteristic zero. If $X_5 \subset \mathbb{P}^3_K$ is a smooth quintic containing a line ℓ, then*

$$v(\ell) \leq 28\,. \tag{1.2}$$

More precisely, the maximal value can possibly be attained only for lines of three certain ramification types (Corollary 6.2). For one of these configurations, all surfaces containing it can be exhibited explicitly using our methods (Example 6.3), so the bound from Theorem 1.1 is indeed sharp. In particular, this implies that a quintic surface with a fivetuplet of coplanar lines can contain at most 125 lines (Corollary 6.1). Combined with a technique using the flecnodal divisor and some basic topological and intersection-theoretical arguments, this leads to the following bound for the maximum number of lines on X_5 which is our second main result:

Theorem 1.2. *Let K be a field of characteristic zero. Any smooth quintic $X_5 \subset \mathbb{P}^3_K$ may contain at most* 127 *lines. In other words,*

$$M(5) \leq 127.$$

We have to admit that we do not expect Theorem 1.2 to be sharp, or even close to it. In fact, the current record for the number of lines on a smooth quintic surface (outside characteristic $2, 5$) stands at 75, attained both by the Fermat quintic and Barth's quintic [12]. Yet the given bound provides a substantial improvement compared to (1.1).

The organization of the paper is as follows. At first we recall Segre's argument for lines of the first kind (see Section 2). In Section 3 we define the rational functions $\mathfrak{b}_0, \ldots, \mathfrak{b}_{d-3}$ for a line on a smooth degree-d surface and describe their relevant properties. In the subsequent three sections we study lines of the second kind on quintics to complete the proof of Theorem 1.1. Finally, in Sections 8, 9 we investigate quintics without fivetuplets of coplanar lines in order to derive Theorem 1.2.

Convention 1.3. Since the statements of this paper remain valid under base extension (or restriction), we work over an algebraically closed field K of characteristic zero.

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2. Segre's bound for the lines of the first kind

Let $d \geq 4$ and let $X_d \subset \mathbb{P}^3$ be a smooth degree-d surface that contains a line ℓ (*i.e.*, a degree-one curve). The linear system $|\mathcal{O}_{X_d}(1) - \ell|$ endows the surface in question with a fibration, *i.e.*, a morphism

$$\pi \,:\, X_d \to \mathbb{P}^1 \tag{2.1}$$

whose fibers are plane curves of degree $d-1$. We let F_P denote the fiber of (2.1) contained in the tangent space $\mathrm{T}_P X_d$ for a point $P \in \ell$.

The restriction of the fibration (2.1) to the line ℓ defines the degree-$(d-1)$ map

$$\pi|_\ell \,:\, \ell \to \mathbb{P}^1. \tag{2.2}$$

Let R_ℓ be the ramification divisor of (2.2). By the Hurwitz formula, one has

$$\deg(R_\ell) = 2d - 4\,.$$

The line ℓ is said to be *of ramification type* (1^{2d-4}) if and only if all ramification points are simple, *i.e.*, no ramification point occurs in R_ℓ with multiplicity greater

than one. The lines of other ramification types are defined in an analogous way; *e.g.*, type $(2, 1^{2d-6})$ means that only one double point appears in R_ℓ. We highlight that in the case of $d \geq 5$ a phenomenon occurs that was impossible for quartics: two ramification points of the map (2.2) may belong to the same fiber (see Subsection 4.3 for an explicit example). To simplify our notation we say that the line ℓ is of ramification type

$$(1^{2d-6}, [1, 1])$$

if it is of the type (1^{2d-4}) and exactly two ramification points of index 1 lie in the same fiber of (2.2). Moreover, a line $\ell' \neq \ell$, $\ell' \subset X_d$ is called *ℓ-unramified* if and only if it meets ℓ away from the set of ramification points of the map (2.2).

The general fiber of (2.1) is a smooth planar curve with $3(d-1)(d-3)$ inflection points. In his work on lines on surfaces Segre introduces the following notion (*cf.* [18, page 87]):

Definition 2.1 (Lines of Second Kind). We call the line ℓ a line of the second kind if and only if it is contained in the closure of the flex locus of the smooth fibers of the fibration (2.1). Otherwise, ℓ is called a line of the first kind.

Obviously each line $\ell' \neq \ell$ on X_d that meets ℓ is a component of a fiber of (2.1). In particular, it meets ℓ in a point where both the equation of the degree-$(d-1)$ curve (= the fiber of the fibration (2.1)) and its Hessian vanish. The resultant of the restrictions of both polynomials to the line ℓ is of degree $(8d-14)$ in the homogeneous coordinates of ℓ (see [18, page 88], [14, Lemma 5.2]). After verifying that multiple lines meeting ℓ in the same point result in a multiple zero of the resultant, this yields the following bound for the valency:

Proposition 2.2 (Segre). *If $\ell \subset X_d$ is a line of the first kind, then it is met by at most $(8d-14)$ other lines on X_d:*

$$v(\ell) \leq 8d - 14. \tag{2.3}$$

For quintics, for instance, the bound reads $v \leq 26$ which is a little better than what we stated in Theorem 1.1. Indeed, examples of smooth degree-d surfaces that contain a line met by $(d(d-2)+2)$ other lines (thus eventually violating (2.3)) can be found in [11], whereas quartic surfaces that violate the above bound are studied in [14]. It is due to those examples that in this note we will mostly deal with lines of the second kind; in particular, this will be necessary and sufficient to complete the proof of Theorem 1.1.

Remark 2.3. Assume that the ideal of a degree-d surface $X_d \subset \mathbb{P}^3$ is generated by

$$f = \sum_{i,j=0}^{d} \alpha_{i,j} \cdot x_3^i x_4^j = \sum_{i,j,k,l=0}^{d} \alpha_{i,j,k,l} \cdot x_3^i x_4^j x_1^k x_2^l \in K[x_1, x_2, x_3, x_4], \tag{2.4}$$

where $\ell = V(x_3, x_4)$ is a line of the second kind, $\alpha_{i,j} \in K[x_1, x_2]$ is homogeneous of degree $(d-i-j)$, and $\alpha_{i,j,k,l} \in K$. The degree-$(d-1)$ curve residual to ℓ in

the intersection $X_d \cap V(x_4 - \lambda \cdot x_3)$ is given by the polynomial

$$f_\lambda(x_1, x_2, x_3) := f(x_1, x_2, x_3, \lambda x_3)/x_3 \,. \tag{2.5}$$

We put $h_\lambda(x_1, x_2, x_3)$ to denote the determinant of the Hessian of f_λ. Let $r \in K[x_1, \lambda]$ be the remainder of division of $h_\lambda(x_1, 1, 0)$ by $f_\lambda(x_1, 1, 0)$. We consider the expansion

$$r = \sum_{i,j} r_{i,j} x_1^i \lambda^j \,. \tag{2.6}$$

By definition, we have that

$$\ell \text{ is of the second kind if and only if all } r_{i,j} \in K[\alpha_{i,j,k,l}] \text{ vanish identically.} \tag{2.7}$$

3. Counting lines with rational functions

In this section we introduce the main new tool of this note – the rational functions $\mathfrak{b}_k$ (see Definition 3.3). The definition is preceded by some elementary lemmata which we need in order to show that the functions we introduce are in fact well-defined.

Let $X_d \subset \mathbb{P}^3$ be a smooth surface of degree d that contains a line ℓ. To simplify our notation we assume as in Remark 2.3 that $\ell = V(x_3, x_4)$ and let f denote a generator of the ideal $\mathrm{I}(X_d)$ with expansion (2.4).

We define

$$\mathfrak{d}_j(f) := \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_j \partial x_3} & \dfrac{\partial^2 f}{\partial x_j \partial x_4} \\ \dfrac{\partial f}{\partial x_3} & \dfrac{\partial f}{\partial x_4} \end{vmatrix} .$$

Let $P \in \ell$ be a point such that $\frac{\partial f}{\partial x_4}(P) \neq 0$. Since the tangent space $T_P X_d$ can be parametrized by the map

$$\begin{array}{ccc} \mathbb{P}^2 & \to & T_P X_d \\ (x_1 : x_2 : x_3) & \mapsto & (x_1 : x_2 : x_3 : C(P) \cdot x_3) \end{array} \quad \text{where } C(P) := -\frac{\partial f/\partial x_3(P)}{\partial f/\partial x_4(P)}, \tag{3.1}$$

the fiber F_P is given by the vanishing of $f_{C(P)}$ (*i.e.*, (2.5) with $\lambda = C(P)$). One can easily check that for $j = 1, 2$ the following equality holds

$$\frac{\partial f}{\partial x_4}(P) \cdot \frac{\partial f_{C(P)}}{\partial x_j}|_\ell = \frac{\partial^2 f}{\partial x_j \partial x_3}|_\ell \cdot \frac{\partial f}{\partial x_4}(P) - \frac{\partial^2 f}{\partial x_j \partial x_4}|_\ell \cdot \frac{\partial f}{\partial x_3}(P). \tag{3.2}$$

Note that the right-hand side gives $\mathfrak{d}_j(f)(P)$ upon substituting P. As an immediate consequence we obtain the following lemma (which has an analogous statement for $\mathfrak{d}_1(f)$ as the proof reflects).

Lemma 3.1. *Suppose that* $P := (1 : p_2 : 0 : 0) \in \ell$ *is a point such that* $\frac{\partial f}{\partial x_4}(P) \neq 0$. *Then the following conditions are equivalent:*

- $\mathfrak{d}_2(f)(P) = 0$;
- P *is a ramification point of the map* (2.2).

Proof. The point P is a ramification point of the map (2.2) if and only if the line ℓ is tangent to the fiber F_P in the point P, *i.e.*,

$$\frac{\partial f_{C(P)}}{\partial x_1}(P) = \frac{\partial f_{C(P)}}{\partial x_2}(P) = 0\,. \tag{3.3}$$

Suppose $\mathfrak{d}_2(f)(P) = 0$. From (3.2) we obtain $\frac{\partial f_{C(P)}}{\partial x_2}(P) = 0$. The Euler identity yields $\frac{\partial f_{C(P)}}{\partial x_1}(P) = 0$.

Assume P is a ramification point of the map (2.2). From (3.2) and (3.3) we obtain $\mathfrak{d}_2(f)(P) = 0$. □

Lemma 3.2. *Let* $X_d \subset \mathbb{P}^3$ *be a smooth surface that contains the line* $\ell = V(x_3, x_4)$ *and let* $\mathrm{I}(X_d) = \langle f \rangle$. *Then* f *satisfies the following conditions*

$$\frac{\partial f}{\partial x_4} \notin \mathrm{I}(\ell) \quad \text{and} \quad \mathfrak{d}_2(f) \notin \mathrm{I}(\ell)\,. \tag{3.4}$$

Proof. Suppose to the contrary that the formal partial derivative $\frac{\partial f}{\partial x_4}$ belongs to the ideal $\mathrm{I}(\ell)$. Since $f \in \mathrm{I}(\ell)$, we can write

$$f = x_3 \cdot g(x_1, x_2, x_3, x_4) + x_4 \cdot h(x_1, x_2, x_4)\,.$$

Thus, we have $\frac{\partial f}{\partial x_4} \in \mathrm{I}(\ell)$ if and only if $h(x_1, x_2, x_4) \in \mathrm{I}(\ell)$. The latter implies that the partials $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_4}$ vanish along ℓ. Hence, the zeroes of g on ℓ are singularities of X_d. That contradiction shows that $\frac{\partial f}{\partial x_4}$ cannot belong to the ideal of the line ℓ.

Suppose that $\mathfrak{d}_2(f) \in \mathrm{I}(\ell)$. vanishes along the line ℓ. By Lemma 3.1 every point of the line ℓ where $\frac{\partial f}{\partial x_4}$ does not vanish is a ramification point of the map (2.2). Contradiction. □

After those preparations we can define the main tool of this note (recall that the monomial $x_3^i x_4^j x_1^k x_2^l$ appears in (2.4) with the coefficient $\alpha_{i,j,k,l}$).

Definition 3.3. Let $\ell = V(x_3, x_4)$ be a line on a smooth surface $X_d \subset \mathbb{P}^3$ and let f be a generator of the ideal $\mathrm{I}(X_d)$ given as (2.4). For $k = 0, \ldots, d-3$ we define the k-th function $\mathfrak{b}_k := \mathfrak{b}_k(f) \in K(x_1, x_2)$ as the rational function given by the formula

$$\sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 \leq k+3 \\ j_3 \leq d-(k+3)}} \binom{d-(j_1+j_2+j_3)}{k+3-(j_1+j_2)} \alpha_{j_1, j_2, j_3, d-(j_1+j_2+j_3)} A^{d-(k+3)-j_3} B^{(k+3)-(j_1+j_2)} C^{j_2}$$

where

$$A := -\frac{\mathfrak{d}_1(f)}{\mathfrak{d}_2(f)}|_\ell,\ B := \frac{(\partial f/\partial x_3)\cdot \mathfrak{d}_4(f) - (\partial f/\partial x_4)\cdot \mathfrak{d}_3(f)}{2\,\mathfrak{d}_2(f)\cdot(\partial f/\partial x_4)}|_\ell,\ C := -\frac{\partial f/\partial x_3}{\partial f/\partial x_4}|_\ell.$$

Originally, we introduced the functions $\mathfrak{b}_0, \ldots, \mathfrak{b}_{d-3}$ to find quartic surfaces with interesting configurations of lines. Indeed, for an explicitly given surface, the $\mathfrak{b}$-functions will provide a simple tool to check whether a line is met by many lines (see Proposition 3.7 and Example 3.11). For quintic surfaces, the number of possible ramification types will still be small enough to successfully apply a similar approach. This will be crucial for the proof of Theorem 1.1.

The main feature of the rational functions $\mathfrak{b}_k$ will be presented in Proposition 3.7. Before we get there, we discuss some basic properties.

Remark 3.4. Given a pair (X_d, ℓ) where $\ell \subset X_d$ is a line, the functions $\mathfrak{b}_k$ do depend on the choice of four ordered linear forms h_1, h_2, h_3, h_4 that constitute a system of homogenous coordinates on $\mathbb{P}^3$ such that $\langle h_3, h_4\rangle = \mathrm{I}(\ell)$. Indeed, one can easily see that some poles of $\mathfrak{b}_k(f)$ move when we change the homogenous coordinates. Still, as we are about to show, both the set of common zeroes of $\mathfrak{b}_0(f), \ldots, \mathfrak{b}_{d-3}(f)$ and the vanishing of $\mathfrak{b}_0(f), \ldots, \mathfrak{b}_j(f)$ along the line ℓ have purely geometric meanings for the pair (X_d, ℓ). We prefer to define $\mathfrak{b}_k$ for fixed homogenous coordinates (x_1, x_2, x_3, x_4) and a polynomial $f \in \langle x_3, x_4\rangle$ to keep our exposition concise.

Remark 3.5. In the definition of the functions $\mathfrak{b}_k$ we assumed the surface X_d to be smooth. Obviously, one can define the functions $\mathfrak{b}_k$ as soon as there exists a system of homogenous coordinates (x_1, x_2, x_3, x_4) on $\mathbb{P}^3$ such that the pair (X_d, ℓ) satisfies the conditions (3.4). This can be used in study of configurations of lines on singular surfaces (*cf.* [19]).

For later use, we continue with another interpretation of the polynomials $\mathfrak{d}_1(f), \ldots, \mathfrak{d}_4(f)$. For a point $P \in X_d := V(f)$, we follow [15] to define the *Hessian quadric* $V_P := V_P X_d$ as the quadric in $\mathbb{P}^3$ given by the quadratic form

$$q_P = \frac{1}{2}(x_1, x_2, x_3, x_4)\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(P)\right)_{1\le i,j\le 4}{}^t(x_1, x_2, x_3, x_4). \tag{3.5}$$

Suppose that $P \in \ell$ is a point such that $\frac{\partial f}{\partial x_4}(P) \neq 0$. As one can easily check, the pre-image of the intersection of the quadric V_P with the tangent space $T_P X_d$ under the parametrization (3.1) is the quadric given by the (3×3)-matrix

$$\begin{pmatrix} 0 & 0 & \mathfrak{d}_1(f)(P) \\ 0 & 0 & \mathfrak{d}_2(f)(P) \\ \mathfrak{d}_1(f)(P) & \mathfrak{d}_2(f)(P) & (-B(P))\mathfrak{d}_2(f)(P) \end{pmatrix}. \tag{3.6}$$

In particular, we obtain the following useful observation:

Observation 3.6. If $P \in \ell$ is a point such that $\frac{\partial f}{\partial x_4}(P) \neq 0$ and $\mathfrak{d}_2(f)(P) \neq 0$ then the Hessian quadric $V_P X_d$ does not contain the tangent space $T_P X_d$.

Finally we can state the proposition that justifies our interest in the rational functions $\mathfrak{b}_0(f), \ldots, \mathfrak{b}_{d-3}(f)$ (and the way we defined them).

Proposition 3.7. *Let $X_d \subset \mathbb{P}^3$ be a smooth surface that contains the line $\ell = V(x_3, x_4)$ and let $I(X_d) = \langle f \rangle$. Suppose that $P := (1 : p_2 : 0 : 0)$ is a point such that*

$$\frac{\partial f}{\partial x_4}(P) \neq 0 \quad \textit{and} \quad \mathfrak{d}_2(f)(P) \neq 0.$$

Then there is at most one line $\ell' \neq \ell$ such that $P \in \ell' \subset X_d$, and the following conditions are equivalent:

(a) *there exists a line $\ell' \neq \ell$ such that $P \in \ell' \subset X_d$;*
(b) *all functions $\mathfrak{b}_j$ vanish at P,* i.e.

$$\mathfrak{b}_0(P) = \ldots = \mathfrak{b}_{d-3}(P) = 0\,.$$

Proof. We assumed the surface X_d to be smooth, so if we have two lines $\ell' \neq \ell''$ on X_d such that $\ell', \ell'' \neq \ell$ and $P \in \ell', \ell''$, then the lines in question are coplanar, so P is a ramification point of the map (2.2). The latter is impossible by Lemma 3.1.

By Observation 3.6 the Hessian quadric V_P does not contain the tangent space $T_P X_d$, so the latter and V_P meet along two lines, one of which is the line ℓ. An elementary computation (see (3.1) and (3.6)) shows that the other line can be parametrized by the map $\Phi := \Phi_P$ given as

$$\mathbb{P}^1 \ni (t_1 : t_3) \mapsto (t_1 : A(P) \cdot t_1 + B(P) \cdot t_3 : t_3 : C(P) \cdot t_3)\,.$$

Observe that we have in particular $\Phi(1 : 0) = P$ (*e.g.*, by (3.6)).

The line $\Phi(\mathbb{P}^1)$ meets X_d with multiplicity at least 3 in the point P, so the composition $(f \circ \Phi)$ has a triple root in the point $(1 : 0)$. By direct check, one has

$$f \circ \Phi = t_3^3 \cdot \sum_{j=0}^{d-2} \mathfrak{b}_j(P) \cdot t_3^j \cdot t_1^{d-3-j} \tag{3.7}$$

so the line $\Phi(\mathbb{P}^1)$ lies on X_d if and only if (b) holds. □

In particular, since the surface X_d contains only finitely many lines for any $d \geq 3$, Proposition 3.7 shows that the $\mathfrak{b}_k$-functions cannot all vanish identically:

Observation 3.8. If X_d is smooth and $d \geq 3$, then there is an integer $k \in \{0, \ldots, d-3\}$ such that $\mathfrak{b}_k \not\equiv 0$.

The proof of Proposition 3.7 enables us to reinterpret Segre's notion of the line of the second kind (Definition 2.1).

Proposition 3.9. *Let $X_d \subset \mathbb{P}^3$ be a smooth surface that contains a line ℓ. The following conditions are equivalent:*

(a) *ℓ is a line of the second kind;*
(b) *the function $\mathfrak{b}_0(f)$ vanishes along the line ℓ.*

Remark 3.10. Observe that the claim of Proposition 3.9 does not depend on the choice of homogenous coordinates on $\mathbb{P}^3$ (*cf.* Remark 3.4).

Proof. We maintain the notation of the proof of Proposition 3.7.

Let us assume that the map (2.2) is not ramified at P. By definition, we have $\Phi_P(\mathbb{P}^1) \neq \ell$ and the line $\Phi_P(\mathbb{P}^1)$ is tangent to F_p in the point P. The point P is an inflection point of the fiber in question if and only if $(f \circ \Phi_P)$ has a zero of multiplicity ≥ 4 in the point $(1:0)$. By (3.7) the latter means that $\mathfrak{b}_0(P) = 0$. This completes the proof of the proposition. □

Obviously, for a fixed line $\ell \subset X_d$ we can choose the homogenous coordinates (x_1, x_2, x_3, x_4) in such a way that the hyperplane section $V(x_3) \cap X_d$, say, contains no lines and no lines on X_d run through the intersection point $V(x_1) \cap \ell$ (which is missing from the description in Proposition 3.7). Once we make such a choice, by Proposition 3.7, the number of ℓ-unramified lines on X_d cannot exceed the minimum of the degrees of the numerators of the non-zero functions $\mathfrak{b}_k(f)$, where $k = 0, \ldots, d-3$. Thus the bound on the number of ℓ-unramified lines on X_d depends on the presentations

$$\mathfrak{b}_j(1, x_2) = \frac{\tilde{\mathfrak{b}}_j}{\mathfrak{c}_j}, \tag{3.8}$$

where $\tilde{\mathfrak{b}}_j, \mathfrak{c}_j \in K[x_2]$ are relatively prime.

As an illustration, we compute the functions $\mathfrak{b}_0$, $\mathfrak{b}_1$ for the quartic surfaces from the family $\mathcal{Z}$ which was central for [14, Section 4].

Example 3.11. Let X_4 be a quartic that contains a line ℓ of the second kind of ramification type (2^2). For now, let K denote an algebraically closed field of characteristic $\neq 2, 3$. By an elementary computation (see [14, Lemma 4.5]) we can assume that X_4 is given by the equation

$$x_3 x_1^3 + x_4 x_2^3 + x_1 x_2 q(x_3, x_4) + p(x_3, x_4,$$

where $q = \sum_{j=0}^{2} q_j x_3^j x_4^{2-j}$, $p = \sum_{j=0}^{4} p_j x_3^j x_4^{4-j}$ and $\ell := V(x_3, x_4)$. The ramification points are $P_1 := (1:0:0:0)$, $P_2 := (0:1:0:0)$ and we have $T_{P_j} X_4 = V(x_{j+2})$. We obtain $\mathfrak{b}_0 = 0$ and

$$\begin{aligned}\mathfrak{b}_1 = \frac{1}{27 x_1^3 x_2^{15}} \cdot \Big(& - q_2^3 x_2^{18} + \left(27\, p_4 + 3\, q_1\, q_2^2\right) x_2^{15} x_1^3 \\ & - \left(27\, p_3 + 3\, q_0\, q_2^2 + 3\, q_1^2 q_2\right) x_2^{12} x_1^6 + \left(q_1^3 + 6\, q_0\, q_1\, q_2 + 27\, p_2\right) x_2^9 x_1^9 \\ & - \left(3\, q_0\, q_1^2 + 27\, p_1 + 3\, q_0^2 q_2\right) x_2^6 x_1^{12} + \left(3\, q_0^2 q_1 + 27\, p_0\right) x_2^3 x_1^{15} - q_0^3 x_1^{18} \Big).\end{aligned}$$

By Proposition 3.7 the line ℓ is met by exactly 18 ℓ-unramified lines provided $q_2q_0 \neq 0$.

We want to check what happens when the ramified fibers of (2.1) contain lines. One can easily see that the fiber F_{P_1} contains exactly one (respectively three) lines if and only if $p_0 = 0$ (respectively $q_0 = 0$). A similar argument shows that F_{P_2} contains exactly one (respectively three) lines if and only if $p_4 = 0$ (respectively $q_2 = 0$).

Whenever a ramified fiber (respectively both ramified fibers) acquires a triplet of lines, Proposition 3.7 implies that the number of ℓ-unramified lines drops by three (respectively six) and we arrive at exactly 18 lines that meet ℓ if $p_0p_4 \neq 0$, or at least 19 if $p_0p_4 = 0$. The maximal number 20 is attained precisely when both p_0, p_4 vanish, but no q_0, q_2. In this case, the line ℓ is met by 20 other lines on the quartic X_4: exactly one line through each of ramification points and 18 ℓ-unramified lines.

In Example 3.11 we gave an elementary proof of the most important part of [14, Proposition 1.1]. The computations we just carried out depict a useful property. Apparently the presentation (3.8) can be used only to count the ℓ-unramified lines. Still, the functions $\mathfrak{b}_0, \ldots, \mathfrak{b}_{d-3}$ detect the existence of many lines through the ramified points of (2.2) as well (*i.e.*, the degree of the numerator of $\tilde{\mathfrak{b}}_j$ decreases when many lines on X_d run through a ramification point of the map (2.2)). On the other hand, the functions $\mathfrak{b}_0, \ldots, \mathfrak{b}_{d-3}$ can also fail to detect the existence of a line $\neq \ell$ through a ramification point (see 6.3.1).

4. Lines of the second kind on quintic surfaces

Let $X_5 \subset \mathbb{P}^3$ be a smooth quintic surface that contains a line ℓ. In this case we obtain the degree-4 map (2.2) and the ramification divisor R_ℓ is of degree six, so there are seven possible ramification types of a line (plus the subcases when two ramification points lie in the same fiber). In this section we discuss important properties of quintic surfaces with lines of the second kind.

If $P \in \ell$ is an n-fold ramification point, then generally there can be at most $n + 1$ other lines on X_5 containing P. For simple ramification points on a line of the second kind, there is a useful strengthening (which is a consequence of (2.7)).

Lemma 4.1. *Let ℓ be a line of the second kind on a smooth quintic X_5 and let P occur in the divisor R_ℓ with multiplicity one. Then at most two lines on X_5 run through the point P (one of them, by assumption, being ℓ).*

Proof. We maintain the notation of Remark 2.3, put $P = (1 : 0 : 0 : 0)$,

$$\alpha_{1,0} = x_1^2(x_1 - c_1x_2)(x_1 - c_2x_2), \quad \alpha_{0,1} = x_2^2(x_2 - d_1x_1)(x_2 - d_2x_1)\,, \quad (4.1)$$

and assume, after some linear transformation, that the line $\ell' := V(x_2, x_3)$ lies on the quintic surface X_5. By assumption, we have $d_1d_2 \neq 0$, since otherwise P would

not be a simple ramification point. Hence (2.7) yields $\alpha_{0221} = \alpha_{0311} = 0$. One can easily check that the line ℓ' is the (reduced) tangent cone $C_P F_P$. □

The condition (2.7) reduces the problem of finding surfaces with lines of the second kind to solving (relatively large) systems of polynomial equations. In the remainder of this section we explain how one can find all quintics with a line ℓ of the second kind such that

$$R_\ell \in \{(2^2, 1^2),\ (2^3),\ (3, 1^3),\ (3, 2, 1),\ (3^2)\}\,. \tag{4.2}$$

The remaining two ramification types, namely (1^6) and $(2, 1^4)$, are harder to parametrize, so we will use different methods to deal with them (in the next section) in order to prove Theorem 1.1.

Let us assume that X_5 is given by (2.4) with $d = 5$, $\ell = V(x_3, x_4)$ and r is the remainder defined in Remark 2.3. One can easily check that each $r_{i,j}$

- is of degree at most 2 with respect to $\alpha_{i,j,k,l}$, when $i + j = 2$;
- is linear with respect to $\alpha_{i,j,k,l}$, for $i + j = 3$;
- is constant with respect to $\alpha_{i,j,k,l}$ with $i + j \geq 4$.

By a linear coordinate change that does not alter the ramification type of P (of the form $\tilde{x}_i = x_i + \sum_{j=3,4} b_{i,j} x_j$ where $i = 1, 2$, $j = 3, 4$), we can assume that four of the coefficients $\alpha_{i,2-i,j,k}$ vanish (the most convenient choice of the four coefficients will depend on the ramification type we study – see, *e.g.* (4.4)). Moreover, a direct computation shows that $\alpha_{2003}, \alpha_{0230}$ are zero (a geometric interpretation of this fact for some ramification types can be found in [18, Section 5]).

Throughout the remainder of this section, we denote two of the ramification points (after some linear transformation) by $P_1 := P = (1 : 0 : 0 : 0)$, $P_2 := (0 : 1 : 0 : 0)$. Then we fix the polynomials $\alpha_{1,0}, \alpha_{0,1}$ appropriate for the given ramification type. A determinant computation shows that for all quintic surfaces we consider one can choose twelve $r_{i,j}$ such that the linear system of equations they define (with the indeterminates $\alpha_{i,3-i,k,l}$) has a unique solution. In this way we obtain a map

$$K^6 \times K^{16} \to \mathcal{O}_{\mathbb{P}^3}(5) \tag{4.3}$$

that associates to $\alpha_{i,j,k,l}$, where $(i + j) \in \{2, 4, 5\}$, the quintic given by (2.4).

The substitution of (4.3) into all $r_{i,j}$'s yields several affine quadrics in the affine parameter space K^6 which corresponds to $\alpha_{i,2-i,k,l}$. Let us denote their (set-theoretic) intersection by $\mathcal{W}$. By definition, the quintics with a line of the second kind of the considered ramification type can be parametrized by the image of the restriction of (4.3) to

$$\mathcal{W} \times K^{16}\,.$$

It should be pointed out that in all cases we study $\mathcal{W} \subset K^6$ happens to be an affine subspace.

Below we discuss the ramification types from (4.2) one-by-one in a series of subsections. We omit certain formulae to keep our exposition compact, but all

formulae can be obtained with the help of any computer algebra system without difficulty.

Remark 4.2. A careful analysis, for instance based on a parameter count, reveals that, just like for quartics [13], quintics with a line of the second kind do not form a single irreducible family, as different ramification types can yield distinct families. This can also be inferred from the degree of a certain surface in $\mathbb{P}^3$ (S_{11}^r to be investigated in Section 5, see Remark 5.8).

4.1. Two triple ramification points

Let $\alpha_{1,0} = x_1^4$ and $\alpha_{0,1} = x_2^4$. In particular we have $R_\ell = 3(P_1 + P_2)$ and $T_{P_i} X_5 = V(x_{i+2})$ for both ramification points.

After a linear change of coordinates we can assume that

$$\alpha_{1130} = \alpha_{1103} = \alpha_{2030} = \alpha_{0203} = 0. \tag{4.4}$$

We consider the system of equations given by $r_{0,2}, \ldots, r_{0,5}$ and $r_{i,j}$ where $i = 2, 3$, $j = 1, 2, 3, 4$, to arrive at (4.3). One can easily check that $\mathcal{W} = V(\alpha_{2012}, \alpha_{0221})$ in this case.

Consequently, every smooth quintic X_5 with a line of the second kind of ramification type (3^2) can be given by (2.4) where

$$\begin{aligned} \alpha_{3002} &= \frac{1}{8}{\alpha_{2021}}^2,\ \alpha_{0320} = \tfrac{1}{8}{\alpha_{0212}}^2,\ \alpha_{2102} = \frac{1}{4}\alpha_{1121}\alpha_{2021}, \\ \alpha_{2120} &= \frac{1}{8}{\alpha_{1112}}^2,\ \alpha_{1202} = \tfrac{1}{8}{\alpha_{1121}}^2,\ \alpha_{1220} = \frac{1}{4}\alpha_{0212}\alpha_{1112} \end{aligned} \tag{4.5}$$

and the other coefficients $\alpha_{i,3-i,k,l}$ vanish.

Remark 4.3. Suppose that ℓ is of ramification type (3^2). Computing the (formal) Taylor expansion of the equation of F_{P_1} (respectively F_{P_2}) around P_1 (respectively P_2) one checks that the line ℓ is the tangent cone of F_{P_1} (respectively F_{P_2}) provided either $\alpha_{0212} \neq 0$ or $\alpha_{0410} \neq 0$ (respectively either $\alpha_{2021} \neq 0$ or $\alpha_{4001} \neq 0$).

4.2. One triple ramification point

In order to study ramification types with a single triple point we put

$$\alpha_{1,0} = x_1^4, \quad \alpha_{0,1} = x_2^2(x_2 - x_1)(x_2 - cx_1)$$

in (2.4). As in Subsection 4.1, we can assume that (4.4) holds and check that the same system of twelve linear equations has a unique solution for every $c \in K$.

A discriminant computation reveals that for three values of c (namely $c = 0$ and $9c^2 - 14c + 9 = 0$), we obtain the ramification type $(3, 2, 1)$, whereas $c = 1$ uniquely leads to $(3, 1, [1, 1])$. Otherwise the line ℓ is of the (non-degenerate) type $(3, 1^3)$ (*i.e.*, no pair of ramification points lies in one and the same fiber).

One can check that for $c = 0$ (*i.e.*, ramification type $(3, 2, 1)$) the set $\mathcal{W}$ is given by vanishing of α_{2012} and

$$\alpha_{0212} = -\frac{9}{64}\alpha_{1121} - \frac{4}{3}\alpha_{0221} - \frac{243}{16384}\alpha_{2021} - \frac{27}{256}\alpha_{1112}\,.$$

For the values of c such that ℓ is not of ramification type $(3, 2, 1)$, $\mathcal{W}$ is a 3-dimensional affine subspace of K^6 given, among others, by the vanishing of α_{2012}.

In the remark below we maintain the notation of Subsection 4.1. Recall that the fiber F_{P_1} (respectively F_{P_2}) is the quartic curve residual to ℓ in the intersection of X_5 with the plane $V(x_3)$ (respectively $V(x_4)$).

Remark 4.4. a. Suppose that ℓ is of the type $(3, 2, 1)$. Without loss of generality we can assume that P_1 (respectively P_2) is a double, (respectively a triple) ramification point and the fiber F_{P_1} meets the line ℓ in the point $(1 : 1 : 0 : 0)$. This amounts to the choice $c = 0$ in Subsection 4.2 (in particular, we do not have to consider the other values of c for the ramification type $(3, 2, 1)$). Then

$$R_\ell = 2P_1 + 3P_2 + P_3, \text{ with } P_3 = (4 : 3 : 0 : 0).$$

As in Remark 4.3, one can check that if the fiber F_{P_2} contains a line through the point P_2, then $\alpha_{2021}, \alpha_{4001}$ vanish. By Lemma 4.1 the quintic X_5 contains at most one line $\neq \ell$ through the point P_3.

b. For the ramification $(3, 1^3)$ (including the case $(3, 1, [1, 1])$) we have the same condition for F_{P_2} to have a degree-1 component that contains the triple ramification point P_2. Lemma 4.1 shows that X_5 contains at most one line $\neq \ell$ through each of the other ramification points.

4.3. Double ramification points

Let us assume that

$$\alpha_{1,0} = x_1^3(x_1 - x_2), \quad \alpha_{0,1} = x_2^3(x_2 - cx_1) \quad \text{and} \quad c \neq 0, 1.$$

A discriminant computation shows that if $c \neq 0, 1, 4, -8$, the line ℓ is of the non-degenerate type $(2^2, 1^2)$. Obviously, $c = 1$ yields a singular quintic. For $c = 4$ we have $R_\ell = 2(P_1 + P_2 + P_3)$, where $P_3 = (1 : 2 : 0 : 0)$, so the line ℓ is of ramification type (2^3). Finally, if $c = -8$, then the fiber of the parametrization (2.2) defined by the plane $V(x_3 - 64x_4)$ consists of exactly two (different) ramification points. Hence, the line ℓ is of the ramification type $(2^2, [1, 1])$.

After some linear transformation, we can assume that $\alpha_{2021}, \alpha_{2030}, \alpha_{1130}, \alpha_{0212}$ vanish. In order to determine the map (4.3) for a fixed $c \neq 0$, we solve the system of equations given by the vanishing of $r_{3,0}, \ldots, r_{3,4}$ and $r_{i,j}$ where $i = 1, 2$ and $j = 1, 2, 4, 5$.

For $c = 4$ the intersection of quadrics $\mathcal{W}$ is given by the single equation

$$\alpha_{0203} = -128\alpha_{2012} + 16\alpha_{1103} + 8\alpha_{1112} + 4\alpha_{1121} - \alpha_{0221}/4. \tag{4.6}$$

If $c = -8$ the set $\mathcal{W}$ is a codimension-2 vector subspace of K^6 cut out by:

$$\alpha_{0203} = -64\alpha_{1103} + 16\alpha_{1112} + 1024\alpha_{2012},$$
$$\alpha_{0221} = -128\alpha_{1112} - 64\alpha_{1121} - 8192\alpha_{2012}.$$

For $c \neq 0, 1, 4, -8$, we obtain a codimension-2 affine subspace again. Indeed, one can easily check that α_{0203}, α_{1103} can be uniquely expressed as elements of $K[c, 1/c][\alpha_{0221}, \alpha_{1112}, \alpha_{1121}, \alpha_{2012}]$.

Remark 4.5. a. By a tangent cone argument if $a_{0221} \neq 0$ (respectively $a_{2012} \neq 0$) then at most one line $\neq \ell$ on X_5 runs through the point P_1 (respectively P_2).

b. A similar argument shows that if a_{0221}, a_{0410}, a_{0401} (respectively a_{2012}, a_{4010}, a_{4001}) vanish simultaneously, then the fiber F_{P_1} (respectively F_{P_2}) contains no lines at all through the point P_1 (respectively P_2).

5. Segre's surface of principal lines along a line

Having discussed the five ramification types from (4.2), it remains to analyse the ramification types (1^6) and $(2, 1^4)$ (which, as we pointed out before, are relatively hard to parametrize explicitly). Their analysis is the content of the present section using an idea that implicitly appears in [18] to count the lines that meet a line of the second kind of the given ramification type.

To this end, we consider the following subset of the Grassmannian $\mathrm{Gr}(1, 3)$,

$$\mathfrak{S}_\ell := \{\tilde{\ell} \,:\, \tilde{\ell} \text{ is a line}, \exists\, P \in \ell \text{ such that } \tilde{\ell} \subset \mathrm{T}_P X_5 \text{ and } \mathrm{i}(\tilde{\ell}, F_P; P) \geq 2\},$$

where F_P denotes the residual fiber through P and $\mathrm{i}(\cdot)$ stands for the intersection multiplicity of the planar curves $\tilde{\ell}$ and F_P in the tangent plane $\mathrm{T}_P X_5$ computed in the point P. We have the following lemma (maintaining the notation of Remark 2.3 and [13, Lemma 2.3]).

Lemma 5.1. *The total space* $\cup\{\tilde{\ell} \,:\, \tilde{\ell} \in \mathfrak{S}_\ell\} =: S_{11}$ *is an algebraic surface of degree at most eleven.*

Proof. Let $P = (p_1 : p_2 : 0 : 0) \in \ell$ be a point such that $\alpha_{0,1}(p_1, p_2) \neq 0$. The residual quartic curve $F_P \subset T_P X_5$ is given by the polynomial

$$g(x_1, x_2, x_3) := f(\alpha_{0,1}(P)x_1, \alpha_{0,1}(P)x_2, \alpha_{0,1}(P)x_3, -\alpha_{1,0}(P)x_3)/x_3\,.$$

Indeed, this is just $(\alpha_{0,1}(P)^5 \cdot f_{C(P)})$, where $f_{C(P)}$ is defined by (2.5).

By direct computation, there exist $h_1, h_2 \in K[x_1, x_2]$ (respectively h_3) of degree 7 (respectively 11) such that

$$\frac{\partial g}{\partial x_3}(p_1, p_2, 0) = \alpha_{0,1}^3(p_1, p_2) \cdot h_3(p_1, p_2),$$
$$\frac{\partial g}{\partial x_j}(p_1, p_2, 0) = \alpha_{0,1}^4(p_1, p_2) \cdot h_j(p_1, p_2) \quad \text{for } j = 1, 2.$$

We define bihomogenous polynomials $H_{11} \in K[z_1, z_2][x_1, \ldots, x_4]$ (respectively H_4) of bidegree (11, 1) (respectively (4, 1))

$$H_{11} := \alpha_{0,1}(z_1, z_2) \cdot h_1(z_1, z_2) \cdot x_1 + \alpha_{0,1}(z_1, z_2) \cdot h_2(z_1, z_2) \cdot x_2 + h_3(z_1, z_2) \cdot x_3,$$
$$H_4 := \alpha_{1,0}(z_1, z_2) \cdot x_3 + \alpha_{0,1}(z_1, z_2) \cdot x_4,$$

and repeat almost verbatim the proof of [13, Lemma 2.3], dehomogenizing by putting $z_2 = 1$ and computing the resultant with respect to z_1. (In [13] one deals with a quartic surface X_4 and considers bihomogeneous polynomials H_8, H_3 instead of H_{11}, H_4, but that is about the only difference.) □

Definition 5.2. The surface S_{11} will be called Segre's *surface of principal lines* along ℓ.

Remark 5.3. Suppose $P \in \ell$ and the tangent space $T_P X_5$ meets the Hessian quadric V_P along two lines (obviously one of them is ℓ). Then, by definition both lines are contained in S_{11}.

Segre's surface of principal lines has the following useful properties (the proofs of claims (a) and (b) use the standard topology of $\mathbb{P}^3(\mathbb{C})$, so here the initial characteristic-0 assumption of the paper is relevant).

Lemma 5.4. *Let $\ell \subset X_5$ be a line of the second kind.*

(a) *If a point P appears in R_ℓ with multiplicity one or two, then the tangent plane $\mathrm{T}_P X_5$ is a component of the ruled surface S_{11};*
(b) *If P appears in R_ℓ with multiplicity one, then the surface residual to $\mathrm{T}_P X_5$ in S_{11} contains all lines on X_5 that run through P;*
(c) *If the support $\mathrm{supp}(R_\ell)$ consists of at least five points and a fiber F_{P_2} contains two ramification points, then $\mathrm{T}_{P_2} X_5$ appears with multiplicity at least two in S_{11}.*

Proof. (a) Let F_P be a fiber of (2.1). According to [18, Section 6], if $\mathrm{i}(\tilde{\ell}, F_P; P) \geq 2$, then either $P \in \mathrm{sing}(F_P)$ or $\mathrm{i}(\tilde{\ell}, F_P; P) \geq 4$. The latter is ruled out by the assumption on multiplicity, so the claim of the lemma follows directly from the definition of the family $\mathfrak{S}_\ell$.

(b) We maintain the notation of the proof of Lemma 4.1 and assume that the line ℓ' runs through P on X_5. One can easily check that for each point $\tilde{P}$ from a (punctured) neighbourhood of P on ℓ (in the complex analytic topology) the Hessian quadric $V_{\tilde{P}}$ meets the tangent plane $T_{\tilde{P}} X_5$ properly (*i.e.*, along two lines: ℓ and $\ell_{\tilde{P}}$). Moreover, the line ℓ' lies in the closure of the union of the lines $\ell_{\tilde{P}}$.

(c) The claim results from an elementary but tedious computation. We maintain the notation of the proof of Lemma 4.1, assume that the points $P := (1 : 0 : 0 : 0)$, $P_2 := (0 : 1 : 0 : 0)$ belong to $\mathrm{supp}(R_\ell)$ and that the fiber F_P (respectively F_{P_2}) is the quartic curve residual to ℓ in the intersection of X_5 with the plane $V(x_3)$ (respectively $V(x_4)$).

At first we assume that P, P_2 are simple ramification points and $P_3 := (1 : 1 : 0 : 0) \in \mathrm{supp}(R_\ell) \cap F_{P_2}$ (*i.e.*, $c_1 = c_2 = 1$, $d_1 \neq d_2$ and $d_1 d_2 \neq 0$ in (4.1)). Then

we apply (2.7) to compute the map (4.3), and we find the defining polynomial h_{11} of S_{11} using the explicit resultant approach from the proof of Lemma 5.1. Finally, we check that h_{11} is divisible by x_4^2, so the claim follows for the ramification types $(1^4, [1, 1])$, $(1^2, [1, 1]^2)$, $(2, 1^2, [1, 1])$.

To deal with the ramification type $([1, 1]^3)$ we assume $c_1 = c_2 = 1$ and $0 \neq d_1 = d_2$ in (4.1). Using the discriminant one can check that $([1, 1]^3)$ occurs only for $d_1 = 2$, whereas the ramification type $(2, [1, 1]^2)$ is impossible. Then for $d_1 = 2$ we repeat the reasoning we used to deal with the other cases, to complete the proof of the claim. □

Lemma 5.4 has an immediate consequence for the valency of a line ℓ on a smooth quintic X_5 granted that all ramification points (relative to ℓ) are simple:

Corollary 5.5. *If $\ell \subset X_5$ is a line of the second kind of ramification type (1^6), then ℓ is met by at most 24 other lines on X_5:*

$$v(\ell) \leq 24.$$

Proof. By Lemma 5.4 all lines that meet ℓ on X_5 lie on a degree-5 surface contained in S_{11} (residual to the six distinct tangent planes at the ramification points). The bound then follows straight from Bezout's theorem. □

In the remaining case of ramification type $(2, 1^4)$, we shall see that Segre's surface of principal lines still yields a bound that is sufficient for our purposes. However, one has to carry out a more detailed analysis of its behaviour. For this purpose, we put S_{11}^r to denote the surface residual in S_{11} to the union of tangent planes of X_5 in ramification points.

Lemma 5.6. *Let $\ell \subset X_5$ be a line of the second kind of ramification type $(2, 1^4)$ and let P appear in R_ℓ with multiplicity two.*

(a) *If there are more than two lines through P on X_5, then*

$$\deg(S_{11}^r) \leq 5;$$

(b) *If $\deg(S_{11}^r) = 6$, then S_{11}^r and X_5 meet along the line ℓ with multiplicity ≥ 3.*

Proof. We maintain the notation of the proof of Lemma 4.1, put $d_1 = 0$ (*i.e.*, $P = (1 : 0 : 0 : 0)$ is the double ramification point), and apply (2.7) to find the parametrization (4.3). We use the resultant to compute the polynomial h_{11} that defines Segre's surface S_{11} (as indicated in the proofs of Lemmata 5.1, 5.4). It is divisible by x_3x_4, so set

$$h_9 = h_{11}/(x_3x_4) \quad \text{and define the auxiliary surface} \quad S_9 = V(h_9) \supset S_{11}^r.$$

(a) The assumption on the number of lines through P implies that α_{0221} vanishes (*cf.* Remark 4.5.(a). By direct check, the polynomial h_9 is divisible by x_3 (respectively

by x_3x_4 when $V(x_4)$ contains two ramification points). The claim thus follows from Lemma 5.4.a.

(b) In general, one checks that the polynomial h_9 belongs to the fifth power of the ideal $\langle x_3, x_4\rangle$. Thus, if S_{11}^r is a sextic, then it is singular along the line ℓ (this gives intersection multiplicity at least two). The claim will be proved by studying the intersection number of appropriate hyperplane sections of X_5 and S_9. In detail, we cut X_5 and S_9 with the hyperplane $H_t := V(x_1 - tx_2)$ that meets ℓ transversally in the point $P' = (t : 1 : 0 : 0)$). By direct check, the line $T_{P'}X_5 \cap H_t$ is always a component of the tangent cone of the curve $H_t \cap S_9$, so the intersection multiplicity of X_5 and S_9 along ℓ does exceed five. Since S_{11}^r is a sextic by assumption, the hypersurface S_9 contains exactly three planes, each of which meets X_5 along ℓ with multiplicity one. Thus the claim of Lemma 5.4.b follows. □

It should be pointed out that for a double ramification point the statement of Lemma 5.4.b does not carry over (*i.e.*, sometimes the surface S_{11}^r does not contain all lines through the point in question). Yet Bezout's theorem (using Lemma 5.6) immediately gives the following for ramification type $(2, 1^4)$.

Corollary 5.7. *If $\ell \subset X_5$ is a line of the second kind of ramification type $(2, 1^4)$, then ℓ is met by at most 28 other lines on X_5:*

$$v(\ell) \leq 28.$$

For completeness, we emphasize that together with Corollary 5.5 this covers the two cases missing from Section 4 for the proof of Theorem 1.1.

Remark 5.8. As explored for quartics in [13], one can construct examples of quintics with lines of other ramification types and S_{11}^r of degree up to 9. For such surfaces the approach we just followed yields weaker bounds for the valency than those needed for the proof of Theorem 1.1. In the next section we will overcome this subtlety using the results of Section 3.

6. Counting lines along a line on a quintic surface

In this section we can finally come to the problem that was our original motivation. We complete the proof of Theorem 1.1 and show that the given bound for the valency is in fact sharp (Example 6.3). Before the proof, let us formulate an immediate consequence of Theorem 1.1 which already points towards Theorem 1.2:

Corollary 6.1. *If a smooth quintic surface contains five coplanar lines, then it contains at most 125 lines.*

Proof of Corollary 6.1. Any line in $\mathbb{P}^3$ meets the hyperplane which decomposes into the 5 given lines when intersected with the quintic X_5. Thanks to Theorem 1.1, the total number of lines on X_5 is thus bounded by $5 + 5 \cdot (28 - 4) = 125$ as claimed. □

6.1. Proof of Theorem 1.1

Let $\ell \subset X_5$ be a line. By Proposition 2.2 and Corollaries 5.5, 5.7, we can assume that ℓ is of the second kind and $\mathrm{supp}(R_\ell)$ consists of at most four points (as given in (4.2)). The proof is based on the case-by-case study of the $\mathfrak{b}_k$-functions from Definition 3.3 for each possible ramification type.

Without loss of generality we assume that the quintic surface X_5 is given by (2.4) and $\ell = V(x_3, x_4)$.

Recall that Lemma 3.9 implies that the rational function $\mathfrak{b}_0$ vanishes identically. For $j = 1, 2$ we define the polynomials $\tilde{\mathfrak{b}}_j, \mathfrak{c}_j \in K[x_2]$ by the formula (3.8). In order to apply Proposition 3.7, we have to check the condition $\partial f/\partial x_4 \neq 0$ on ℓ. Presently, one finds that

$$V\left(\frac{\partial f}{\partial x_4}\right) \cap \ell = F_{P_1} \cap \ell \subset V(x_3).$$

Hence all the lines on X_5 intersecting ℓ are either

- contained in the plane $V(x_3)$ or
- ℓ-ramified outside the plane $V(x_3)$ (so that restrictions imposed by the index of ramification apply, including Lemma 4.1) or
- ℓ-unramified outside the plane $V(x_3)$ (so that the total number is bounded by both $\deg(\tilde{\mathfrak{b}}_1)$ and $\deg(\tilde{\mathfrak{b}}_2)$).

In what follows, we will balance out these three cases against each other in order to prove the valency bound from Theorem 1.1.

6.2. Triple ramification points

At first we study the ramification types with a triple point and assume that X_5 lies in the image of $\mathcal{W} \times K^{16}$ under the parametrization (4.3) where $\mathcal{W} \subset K^6$ is one of the affine subspaces we discussed in Subsection 4.1 and Subsection 4.2.

6.2.1. *Ramification type* (3^2)

We maintain the notation of Subsection 4.1. Observe that the fiber F_{P_1} meets ℓ only in P_1, so F_{P_1} contains no ℓ-unramified lines. On the other hand, all ℓ-ramified lines (if any) are contained in the fibres F_{P_1}, F_{P_2}.

Suppose that $\mathfrak{b}_1 \not\equiv 0$. Since, by direct computation, we have

$$\deg(\tilde{\mathfrak{b}}_1) \leq 17,$$

and there are at most four lines in the fibers F_{P_1} (respectively F_{P_2}), Proposition 3.7 yields that the line ℓ is met by at most 25 other lines on X_5.

Assume that $\mathfrak{b}_1 \equiv 0$. We have $\deg(\tilde{\mathfrak{b}}_2) \leq 28$, so the valency bound stated in Theorem 1.1 follows directly from Proposition 3.7 provided the fibers F_{P_1}, F_{P_2}

contain no lines. To complete the proof in this case, it remains to study $\tilde{\mathfrak{b}}_2$ more precisely. We have (with short-hand notation t.o.d. = terms of degree)

$$\tilde{\mathfrak{b}}_2 = \alpha_{2021} \cdot (\text{t.o.d. } 28, 26) + (\text{t.o.d. } 24, \ldots, 4) + \alpha_{0212} \cdot (\text{t.o.d. } 3, 0) .$$

Therefore, if one of the fibers F_{P_1}, F_{P_2} contains a line $\tilde{\ell} \neq \ell$, then by Remark 4.3 and Proposition 3.7, we have $\alpha_{0212} = 0$ or $\alpha_{2021} = 0$, and the line ℓ is met by at most 24 ℓ-unramified lines (respectively at most 20 if both fibers contain a line). This completes the proof of Theorem 1.1 for ramification type (3^2).

6.2.2. *Ramification type* $(3, 2, 1)$

We consider the case $c = 0$ in Subsection 4.2 and check that the denominator of $\mathfrak{b}_j$ divides the product $(x_1^4(x_1 - x_2)^5 x_2^{19})$ for $j = 1, 2$ (here the factor $(x_1 - x_2)^5$ comes from the vanishing of the partial $\partial f/\partial x_4$, whereas the factor given by the simple ramification point P_3 (*cf.* Remark 4.4.a cancels out).

Assume $\mathfrak{b}_1 \not\equiv 0$. One can easily check that $\deg(\tilde{\mathfrak{b}}_1) \leq 20$. Moreover, if F_{P_1} contains an ℓ-unramified line (*i.e.*, a line through $(1 : 1 : 0 : 0)$) then $\deg(\tilde{\mathfrak{b}}_1)$ cannot exceed 19. Since the fiber F_{P_1} (respectively F_{P_2}) contains at most three (respectively four) lines through the point P_1 (respectively P_2), Proposition 3.7 combined with Lemma 4.1 and Remark 4.4.a implies that ℓ is met by at most 28 other lines on X_5.

If $\mathfrak{b}_1 \equiv 0$, then $\alpha_{0221} = 0$ and the degree of $\tilde{\mathfrak{b}}_2$ is at most 24. Moreover, vanishing of $\alpha_{2021}, \alpha_{4001}$ results in $\deg(\tilde{\mathfrak{b}}_2) \leq 20$, whereas the existence of an ℓ-unramified line in F_{P_1} diminishes $\deg(\tilde{\mathfrak{b}}_2)$ by one. The required bound thus results from Proposition 3.7 combined with Remark 4.4.a.

6.2.3. *Ramification type* $(3, 1^3)$

Again, we maintain the notation of Subsection 4.2 and fix any $c \in K$ that does not result in the ramification type $(3, 2, 1)$. We obtain

$$\deg(\tilde{\mathfrak{b}}_1) \leq 17 \quad \text{and} \quad \deg(\tilde{\mathfrak{b}}_2) \leq 24.$$

If $\tilde{\mathfrak{b}}_1 \not\equiv 0$, then the claim follows directly from Lemma 4.1 (indeed, we have at most eight lines in the fibers F_{P_1}, F_{P_2} and at most one line through each of the other two simple ramification points). Otherwise, we use Remark 4.4.b to check that $\deg(\tilde{\mathfrak{b}}_2) \leq 20$ whenever F_{P_2} contains a line through P_2.

6.3. Double ramification points

It remains to deal with the ramification types with two double points. As in Subsection 6.2, we assume that X_5 lies in the image of $\mathcal{W} \times K^{16}$ under the parametrization (4.3), where $\mathcal{W} \subset K^6$ is one of the affine subspaces we discussed in Subsection 4.3.

We maintain the notation of Subsection 4.3. Observe that the partial derivative $\partial f/\partial x_4$ vanishes in P_1 and in the point $(1 : c : 0 : 0) \in F_{P_1}$ (thus the latter is the only unramified point where the first assumption of Proposition 3.7 is not fulfilled (but the second is)) .

6.3.1. *Ramification type* (2^3)

We assume $c = 4$ in the formulas from Subsection 4.3.

Suppose that $\mathfrak{b}_1 \not\equiv 0$. By direct computation we have $\deg(\tilde{\mathfrak{b}}_1) \leq 26$ (respectively $\deg(\tilde{\mathfrak{b}}_1) \leq 25$ when there is an extra line through $(1 : 4 : 0 : 0)$). Moreover, if there are at least two extra lines through P_2, then Remark 4.5.a yields $\alpha_{2012} = 0$ and one gets $\deg(\tilde{\mathfrak{b}}_1) \leq 23$. More generally, one can check that if there are at least two lines $\neq \ell$ through each of k ramification points, than $\deg(\tilde{\mathfrak{b}}_1) \leq 26 - 3k$. Consequently, Proposition 3.7 yields the required bound for the valency from Theorem 1.1 provided $\mathfrak{b}_1 \not\equiv 0$ and either ℓ is the unique line through the ramification point P_1 or there are at least three lines on X_5 which contain the point P_1.

It remains to discuss the case when there are exactly 26 ℓ-unramified lines and exactly one line $\neq \ell$ runs through each point P_j for $j = 1, \ldots, 3$. By Proposition 3.7 every root of $\tilde{\mathfrak{b}}_1$ must be a root of $\tilde{\mathfrak{b}}_2$, so the remainder of the division of $\tilde{\mathfrak{b}}_2$ by $\tilde{\mathfrak{b}}_1$ vanishes. An elementary (but tedious) computation yields $\alpha_{2012} = 0$ and at most 23 ℓ-unramified lines. Contradiction.

Finally suppose that $\mathfrak{b}_1 \equiv 0$. Then one can easily see that $\alpha_{2012}, \alpha_{0221}, \alpha_{4001}, \alpha_{4010}, \alpha_{0410}, \alpha_{0401}$ vanish. By Remark 4.5.b none of the fibers F_{P_1}, F_{P_2} contains a line. Moreover, we have $\deg(\tilde{\mathfrak{b}}_2) \leq 24$, so this completes the proof of Theorem 1.1 for ramification type (2^3).

6.3.2. *Ramification type* $(2^2, 1^2)$

We assume that $c \neq 0, 1, 4, -8$ in Subsection 4.3 and put P_3, P_4 to denote the ramification points of index 1 (they can be found with the help of $\mathfrak{d}_2(f)$). Recall that, by Lemma 4.1, there is at most one line $\neq \ell$ through P_3 (respectively P_4). Moreover, one can easily check that once there is a line (different from ℓ) on X_5 through $(1 : c : 0 : 0)$, the degree of $\tilde{\mathfrak{b}}_1$ drops by one.

Suppose that $\mathfrak{b}_1 \not\equiv 0$. We have $\deg(\tilde{\mathfrak{b}}_1) \leq 23$. If neither α_{2012} nor α_{0221} vanishes, the bound $v \leq 28$ follows from Remark 4.5.a. In the remaining cases, a direct computation shows that if $\alpha_{2012} = 0$ (respectively $\alpha_{0221} = 0$), then we have $\deg(\tilde{\mathfrak{b}}_1) \leq 20$ (respectively $\tilde{\mathfrak{b}}_1$ is divisible by x_2^3). Proposition 3.7 thus yields the claim again.

Eventually we are led to assume $\mathfrak{b}_1 \equiv 0$. Then $\alpha_{2012}, \alpha_{0221}, \alpha_{4001}, \alpha_{4010}, \alpha_{0410}, \alpha_{0401}$ vanish and we have $\deg(\tilde{\mathfrak{b}}_2) \leq 20$. Remark 4.5.b yields that there are no lines $\neq \ell$ through P_1, P_2, which completes the proof of Theorem 1.1 for non-degenerate ramification type $(2^2, 1^2)$. The proof for the degenerate type follows the same lines, so we omit it for the sake of brevity. □

6.4. Record valency

In the above part of the proof of Theorem 1.1 we did not analyze (very precisely) the behaviour of $\mathfrak{b}_2$ under the assumption that $\mathfrak{b}_1$ vanishes along ℓ. Such an analysis (the details of which we omit) yields the following corollary.

Corollary 6.2. *Let $X_5 \subset \mathbb{P}^3$ be a smooth quintic surface and let $\ell \subset X_5$ be a line. If ℓ is met by* 28 *other lines on X_5, then one of the following holds:*

(28_a) *the line ℓ is of the type (3^2) and $\mathfrak{b}_1$ vanishes along ℓ;*
(28_b) *the line ℓ is of the type (2^3), and $\mathfrak{b}_1$ does not vanish along ℓ;*
(28_c) *the line ℓ is of the type $(2, 1^4)$.*

Finite field experiments suggest that the configurations (28_b), (28_c) might not exist, but proving this conjecture exceeds the scope of this paper. Below we describe the family of all quintics that carry the configuration (28_a). In particular this shows that Theorem 1.1 is sharp.

Example 6.3 (Quintics of type (28_a)). Let us maintain the notation of Subsection 4.1 and assume that X_5, endowed with a line with two triple ramification points, satisfies the conditions (4.5). We solve the system of equations given by the vanishing of $\mathfrak{b}_1$ along the line ℓ. An elementary computation yields that $\alpha_{4001}, \alpha_{0410}$ vanish, we have

$$\alpha_{4010} = -\frac{1}{16}\,\alpha_{2021}{}^2\alpha_{1112}, \quad \alpha_{0401} = -\frac{1}{16}\,\alpha_{1121}\alpha_{0212}^2,$$
$$\alpha_{3101} = -\frac{1}{16}\,\alpha_{2021}\alpha_{1112}{}^2, \quad \alpha_{1310} = -\frac{1}{16}\,\alpha_{1121}{}^2\alpha_{0212},$$

and the following equalities hold

$$\begin{aligned}
\alpha_{3110} &= -\frac{1}{8}\,\alpha_{2021}\alpha_{1112}\alpha_{1121} - \frac{1}{16}\,\alpha_{2021}{}^2\alpha_{0212},\\
\alpha_{2201} &= -\frac{1}{8}\,\alpha_{2021}\alpha_{1112}\alpha_{0212} - \frac{1}{16}\,\alpha_{1112}{}^2\alpha_{1121},\\
\alpha_{2210} &= -\frac{1}{8}\,\alpha_{2021}\alpha_{1121}\alpha_{0212} - \frac{1}{16}\,\alpha_{1112}\alpha_{1121}{}^2,\\
\alpha_{1301} &= -\frac{1}{8}\,\alpha_{1112}\alpha_{1121}\alpha_{0212} - \frac{1}{16}\,\alpha_{2021}\alpha_{0212}{}^2.
\end{aligned}$$

In this way we may define a map

$$\mathcal{Z}_5 \,:\, K^4 \times K^6 \to \mathcal{O}_{\mathbb{P}^3}(5). \tag{6.1}$$

Consider the image of $\mathcal{Z}_5$, a family of quintic surfaces. A standard Groebner basis computation shows that the quintic surface $Z = \mathcal{Z}_5(1, \ldots, 1)$ is smooth. By Remark 4.3 none of the (ramified) fibers F_{P_1}, F_{P_2} contain a line. Moreover, for the line $\ell := V(x_3, x_4)$ on Z we have $\mathfrak{b}_1 = 0$, $\deg(\tilde{\mathfrak{b}}_2) = 28$ and the discriminant of $\tilde{\mathfrak{b}}_2$ does not vanish. By Proposition 3.7, the line ℓ meets exactly 28 other lines on Z.

Remark 6.4. As we will see in Section 8 (by Lemma 8.1), if a line on a smooth quintic surface X_5 is met by at least 26 other lines, then X_5 contains a fivetuplet of coplanar lines (including the given line). Thus no smooth surface in the family $\mathcal{Z}_5$ may contain more than 125 lines by Corollary 6.1.

7. Lines *vs* conics or twisted cubics

Before we can come to the proof of Theorem 1.2, we need a few more preparations, but now in a slightly different direction. Namely we want to get a rough idea of the number of lines on a quintic meeting a given (smooth) rational curve C – here either an irreducible conic or a twisted cubic. To this end, we start by following fairly closely the lines of [15].

At any point $P \in C$, we can compare the tangent plane to $X_5 = V(f) \subset \mathbb{P}^3$,

$$T_P = T_P X_5,$$

and the Hessian quadric $V_P \subset \mathbb{P}^3$ defined by (3.5). The intersection $T_P \cap V_P$ generically consists of the two 3-contact lines of X_5 at P; in particular, if some line $\ell \subset X_5$ contains P, then

$$\ell \subset T_P \cap V_P.$$

Depending on the chosen curve C, we now turn to the total space

$$Z = Z_C = \cup_{P \in C}(T_P \cap V_P) \subset \mathbb{P}^3. \tag{7.1}$$

7.1. Conic case

Proposition 7.1. *Assume that C is an irreducible conic. Then Z is a surface of degree 22 in $\mathbb{P}^3$. It does not contain X_5, but the surface Z contains any line in X_5 meeting C.*

Proof. The proof follows essentially word by word that of [15, Proposition 2.2] □

Corollary 7.2. *A conic on a smooth quintic surface $X_5 \subset \mathbb{P}^3$ meets at most 88 lines.*

Proof. If the conic C is geometrically reducible, then the claim follows from Theorem 1.1. Else consider the effective divisor

$$D = Z \cap X_5 \in |\mathcal{O}_{X_5}(22)|$$

and let m denote the multiplicity of C in D. Then, since all lines meeting C are contained in D, we obtain two inequalities

$$\#\{\text{lines on } X_5 \text{ meeting } C\} \leq \begin{cases} \deg(D - mC) = 110 - 2m \\ C.(D - mC) = 44 + 4m. \end{cases} \tag{7.2}$$

The estimate in Corollary 7.2 arises from $m = 11$. □

Remark 7.3. If there are lines on X_5 planar with the conic C, then the estimate for total number of lines meeting C can be improved. What's more important for us: the bound

$$C.\left(\sum_{i=1}^{M}\ell_i\right)\leq 88$$

still holds where the second divisor comprises (all) lines contained in X_5.

Corollary 7.2 and Remark 7.3 will prove quite useful in the proof of Theorem 1.2 as they facilitate simplifications of some of the arguments involved.

7.2. Twisted cubic case

Twisted cubic curves show a rather different behavior than conics – notably because they are not plane. We first investigate the surface Z from (7.1) arising from the twisted cubic C.

Proposition 7.4. *Assume that C is a twisted cubic on a smooth quintic $X_5\subset\mathbb{P}^3$. Then Z is a surface of degree 33 in $\mathbb{P}^3$. It does not contain X_5, but the surface Z contains any line in X_5 meeting C.*

Proof. By a linear transformation, we can assume C to be parametrized by

$$\begin{aligned}\varphi:\ \mathbb{P}^1 &\to C\\ [s,t]&\mapsto \left[t^3,s^3,t^2s,ts^2\right].\end{aligned}$$

The homogeneous ideal of C is thus generated by the three quadrics

$$q_1=x_1x_2-x_3x_4,\quad q_2=x_1x_4-x_3^2,\quad q_3=x_2x_3-x_4^2.$$

At the point $P=\varphi(s,t)$, the tangent plane to the quintic X_5 is thus given by a bihomogeneous polynomial

$$g\in K[s,t][x_1,\ldots,x_4]$$

which is linear in the x_i and of degree 12 in s,t. Meanwhile the quadratic form q defining V_P has degree 9 in s,t. Z is defined by the resultant of g and q with respect to t which thus has degree $33=2\cdot 12+1\cdot 9$.

To conclude, we note as in [15] that any component of Z is automatically ruled by lines. In particular, neither component can equal X_5. □

Contrary to Corollary 7.2, Proposition 7.4 alone does not give a serious estimate for the number of lines on X_5 intersecting the twisted cubic C. Arguing as in the proof of Corollary 7.2 with the divisor

$$D=Z\cap X_5\in|\mathcal{O}_{X_5}(33)|,\tag{7.3}$$

we see at least that a plentitude of adjacent lines forces C to have large multiplicity in D. The combination with the following result will prove quite useful in the sequel.

Proposition 7.5. *Assume that some line ℓ on the smooth quintic $X_5 \subset \mathbb{P}^3$ meets the twisted cubic C with multiplicity two. Then ℓ has multiplicity at least two in D,* i.e., $D - 2\ell > 0$.

Proof. Assume that ℓ and C intersect in two distinct point P_1, P_2. On top of the above normalization, we can assume that

$$I(\ell) = (x_3, x_4) \quad \text{and} \quad P_1 = [1, 0, 0, 0], \quad P_2 = [0, 1, 0, 0].$$

By assumption, $f \in I(C) \cap I(\ell)$, *i.e.*, there are cubic polynomials $c_i \in K[x_1, \ldots, x_4]$ such that

$$f = q_1 c_1 + q_2 c_2 + q_3 c_3 \quad \text{with} \quad c_1 \in (x_3, x_4).$$

In the end, it will suffice to consider polynomials modulo st (such that only the pure powers in s and in t remain); for this purpose, the following observation concerning the evaluation at $P = \varphi(s, t)$ will be useful:

$$c_1, q_2, q_3 \in (x_3, x_4) \implies st \mid c_1(P), q_2(P), q_3(P).$$

Moreover, the same holds true for the (iterated) partial derivates with respect to x_1 and x_2. With this in mind, it is easy to get a hand on the defining polynomials g of $T_P(S)$ and q of V_P. For instance,

$$g \equiv s^3 c_3(P) x_3 + t^3 c_2(P) x_4 \mod (st).$$

In other words, $g \in (x_3, x_4, st)$, and the same holds for q (as one easily checks by considering the double partials with respect to x_1, x_2).

We continue by computing the resultant of g and q via the Sylvester matrix with respect to s or t. The above argument shows that the first and the last column of this matrix has all (two) entries in (x_3, x_4). Hence the determinant of the matrix is in $(x_3, x_4)^2$, and the line ℓ has multiplicity at least two in D as claimed.

If the line ℓ meets C tangentially, say at $[1, 0, 0, 0]$, then $I(\ell) = (x_2, x_4)$, and the analogous argument shows that

$$g, q \in (x_2, x_4, s^2).$$

Hence the first two (or the last two) columns of the Sylvester matrix have entries in (x_2, x_4), and we conclude as before. □

8. Bounding the total number of lines on quintics

This section has a slightly different flavour than the previous sections of the paper as the methods are rather different. Our aim is to complete the proof of our second main result, Theorem 1.2. Recall from Corollary 6.1 that if a smooth quintic surface $X_5 \subset \mathbb{P}^3$ admits a hyperplane section splitting into five lines, then X_5 contains at

most 125 lines. In order to prove Theorem 1.2, it thus remains to study the other possible configurations of lines.

Recall that the locus of points P such that there exists a line that meets X_5 with multiplicity at least 4 in P is the support of a divisor in $\mathcal{O}_{X_5}(31)$ (see [4], [8, Section 8]). Following classical terminology, we call it the *flecnodal divisor* of X_5 and denote it by $\mathcal{F}$. Observe that each line on X_5 is a component of $\mathcal{F}$. A curve on X_5 is called flecnodal if and only if it is a component of the flecnodal divisor $\mathcal{F}$.

We record the following simple observation.

Lemma 8.1. *If a line $\ell \subset X_5$ is not contained in any hyperplane splitting into lines, then*

$$v(\ell) \leq 25.$$

Proof. This is a consequence of the Euler number formula [1, Proposition III.11.4] for the fibration (2.1) and the Euler number computation for singular curves (see [7, Corollary V.4.4.ii]). □

In order to start working towards the proof of Theorem 1.2, we make from now the following assumption:

Assumption 8.2. The smooth quintic X_5 contains $M > 127$ lines.

In particular, Corollary 6.1 implies that X_5 does not admit any hyperplane splitting into five lines. It thus follows from Lemma 8.1 that $v(\ell) \leq 25$ for any line on X_5. Numbering the lines on X_5 from ℓ_1 to ℓ_M, we introduce the following auxiliary effective divisor:

$$\mathcal{F}' := \mathcal{F} - \sum_{i=1}^{M} \ell_i, \quad \deg(\mathcal{F}') = 155 - M \leq 27. \tag{8.1}$$

Note that we can compute the intersection number of $\mathcal{F}'$ and any line $\ell \subset X_5$ as follows:

$$\ell.\mathcal{F}' = \ell.\left(\mathcal{F} - \sum_{i=1}^{M} \ell_i\right) = 31 - (v(\ell) - 3) = 34 - v(\ell) \geq 9, \tag{8.2}$$

where the last estimate follows from Lemma 8.1.

The overall idea of our approach goes as follows: For large M, the intersection number in (8.2) becomes too big relative to the degree of $\mathcal{F}'$. Indeed, for a single line, such a phenomenon might well occur, but not for all lines at the same time as we shall see soon.

For later reference, we compute the self-intersection number $(\mathcal{F}')^2$ in two ways. We know that $\mathcal{F}^2 = 5 \cdot 31^2$. Let us compare this intersection number with the decomposition (8.1). Clearly we have

$$\left(\sum_{i=1}^{M} \ell_i\right)^2 = \sum_{i=1}^{M} \left(\ell_i . \sum_{j=1}^{M} \ell_j\right) = \sum_{i=1}^{M} (v(\ell_i) - 3) = \sum_{i=1}^{M} v(\ell_i) - 3M$$

while

$$\sum_{i=1}^{M} \ell_i.\mathcal{F}' = \sum_{i=1}^{M}(34 - v(\ell_i)) = 34M - \sum_{i=1}^{M} v(\ell_i).$$

Together this yields, using Lemma 8.1,

$$\mathcal{F}^2 = (\mathcal{F}')^2 + 65M - \sum_{i=1}^{M} v(\ell_i) \geq (\mathcal{F}')^2 + 40M. \tag{8.3}$$

Assumption 8.2 thus gives

$$(\mathcal{F}')^2 \leq 5 \cdot 31^2 - 40M \leq -315. \tag{8.4}$$

Note that, if some lines have valency less than 25, then the bounds (8.3), (8.4) improve accordingly (as we exploit occasionally), and similarly for $M > 128$.

If $\mathcal{F}'$ does not contain any multiple components, then (8.4) directly leads to a contradiction as follows: Write

$$\mathcal{F}' = \sum_{j=1}^{r} C_i$$

for some distinct irreducible curves $C_i \subset X_5$. Since $K_{X_5} = H$, adjunction gives

$$C_i^2 \geq -2 - \deg(C_i). \tag{8.5}$$

Applied to the above decomposition of $\mathcal{F}'$, this gives

$$(\mathcal{F}')^2 = \left(\sum_{j=1}^{r} C_j\right)^2 \geq \sum_{j=1}^{r} C_j^2 \geq -2r - \deg \mathcal{F}' \geq -3\deg(\mathcal{F}') \geq -81.$$

This contradicts (8.4) by far, so $\mathcal{F}'$ has to admit multiple components. We shall now study the multiple components more precisely, but before going into the details, we eliminate the lines from much of what is to follow.

Lemma 8.3. *Any line contained in the support of $\mathcal{F}'$ contributes positively to $(\mathcal{F}')^2$.*

Proof. Write $\mathcal{F}' = \mathcal{F}'' + \mathcal{L}$, where $\mathcal{L}$ is the sum of lines contained in the support of $\mathcal{F}'$ (with multiplicities),

$$\mathcal{L} = \sum_{j=1}^{s} \ell^j,$$

and $\mathcal{F}''$ is an effective divisor on X_5 not containing any line in its support. We compute, using (8.2),

$$(\mathcal{F}')^2 = \mathcal{F}'.(\mathcal{F}'' + \mathcal{L}) = (\mathcal{F}'')^2 + \underbrace{\mathcal{F}''.\mathcal{L}}_{\geq 0} + \sum_{j=1}^{s} \underbrace{\ell^j.\mathcal{F}'}_{\geq 9}; \tag{8.6}$$

that is,

$$(\mathcal{F}')^2 \geq (\mathcal{F}'')^2 + 9\deg(\mathcal{L}) \tag{8.7}$$

as claimed. □

Thanks to Lemma 8.3, we can reduce our considerations for $\mathcal{F}'$ to satisfy (8.4) to an investigation of the non-linear components of $\mathcal{F}'$, *i.e.*, we shall often work with $\mathcal{F}''$. Sometimes we will even use linear components to our advantage, using the positive contribution in (8.7). In fact, this can be sharpened substantially:

Remark 8.4. More precisely, any line ℓ contained in the support of $\mathcal{F}'$ satisfies

$$9 \leq \ell.\mathcal{F}' = \underbrace{\ell.\mathcal{L}}_{\leq \deg(\mathcal{L})-4} + \ell.\mathcal{F}'' \implies \ell.\mathcal{F}'' \geq 13 - \deg(\mathcal{L}).$$

In particular, $\mathcal{L}.\mathcal{F}'' \geq \deg(\mathcal{L})(13 - \deg(\mathcal{L}))$. Plugging into (8.6) causes (8.7) to improve to

$$(\mathcal{F}')^2 \geq (\mathcal{F}'')^2 + \max\{9\deg(\mathcal{L}), \deg(\mathcal{L})(22 - \deg(\mathcal{L}))\}. \tag{8.8}$$

The next result will be crucial in determining the top multiplicity of the components of $\mathcal{F}'$. For later reference, we first note it in the general set-up.

Lemma 8.5. *Let D denote an effective divisor with $D^2 < 0$. Then the support of D contains a component C whose multiplicity N satisfies*

$$N \cdot C^2 \leq \epsilon \deg(C) \quad \textit{where} \quad \epsilon = \frac{D^2}{\deg(D)} < 0. \tag{8.9}$$

Proof. Write

$$D = \sum_i N_i C_i$$

for distinct irreducible curves C_i. Assume that (8.9) does not hold for either component of D. Then

$$D^2 \geq \sum_i N_i^2 C_i^2 > \sum_i N_i(\epsilon \deg(C_i)) = \epsilon \deg(D) = D^2,$$

giving the required contradiction. □

We shall use this lemma repeatedly. For our key case of $D = \mathcal{F}'$, we note the following consequence.

Lemma 8.6. *The support of $\mathcal{F}'$ contains a non-linear component C whose multiplicity N satisfies*

$$N \cdot C^2 < -11\deg(C). \tag{8.10}$$

Proof. By Lemma 8.3 and its proof, it suffices to consider the divisor $\mathcal{F}''$ without linear components of degree at most 27. Then apply Lemma 8.5 to $\mathcal{F}''$ and simplify. □

Remark 8.7. In special situations, this argument can be strengthened using (8.7), (8.8). Also we could insert any constant $\epsilon < 35/3$ instead of 11 in (8.10), but presently this would not have any impact on our arguments.

Lemma 8.8. *The support of $\mathcal{F}'$ contains an irreducible conic of multiplicity $N \geq 6$ or a twisted cubic curve of multiplicity $N \geq 7$.*

Proof. Obviously the two given cases satisfy the inequality (8.10) from Lemma 8.6. In comparison, the only remaining case in degree at most 3, a plane cubic curve $C \subset X_5$, has $C^2 = -3$ by adjunction, so (8.10) would give multiplicity at least 12, exceeding the degree of $\mathcal{F}'$.

For $\deg(C) \geq 4$, the inequality (8.5) eliminates all cases in combination with (8.10). □

Lemma 8.9. *Any conic has multiplicity at most* 9 *in $\mathcal{F}'$.*

Proof. For degree reasons, the multiplicity N of a conic Q in $\mathcal{F}'$ is at most 13. Recall from Corollary 7.2 that the valency of Q satisfies

$$v(Q) \leq 88,$$

so there certainly is a line ℓ off Q. But then $N \geq 10$ would imply

$$\ell.\mathcal{F}' = \ell.(\mathcal{F}' - NQ) \leq \deg(\mathcal{F}' - NQ) \leq 27 - 2N \leq 7,$$

contradicting (8.2). □

Before continuing, we record the following useful consequence:

Lemma 8.10. *Any line has multiplicity at most* 3 *in $\mathcal{F}'$.*

Proof. Let $\ell \subset X_5$ be of multiplicity N in $\mathcal{F}'$. First assume that $N \geq 5$. Then use (8.2) and compute

$$9 \leq \ell.\mathcal{F}' = -3N + \ell.(\mathcal{F}' - N\ell) \leq -3N + \deg(\mathcal{F}' - N\ell) \leq 27 - 4N.$$

The claim follows immediately. It remains to discuss the case $N = 4$. Write $\mathcal{F}' = 4\ell + \mathcal{R}$ with

$$\deg(\mathcal{R}) \leq 23 \quad \text{and} \quad \mathcal{R}^2 \leq -315 + 48 - 72 = -339$$

by (8.8). Then Lemma 8.5 implies that either $\mathcal{R}$ contains a line of multiplicity at least 5 (contradicting the above), or a conic of multiplicity $m \geq 8$. Hence we obtain

$$\mathcal{F}' = 4\ell + mQ + \mathcal{R}', \quad \deg(\mathcal{R}') \leq 23 - 2m \leq 7.$$

By Corollary 7.2 and Lemma 8.1, there is a line $\ell' \subset X_5$ which intersects neither ℓ nor Q. Hence $\ell'.\mathcal{F}' \leq \deg(\mathcal{R}') \leq 7$, contradicting (8.2). This completes the proof of Lemma 8.10. □

Remark 8.11. With a little more work, one can improve Lemma 8.10 to show that any line has multiplicity at most 2 in $\mathcal{F}'$, but we will not need this in what follows.

9. Proof of Theorem 1.2

We are now in the position to attack the proof of Theorem 1.2. To this end, we have to rule out all possible configurations with an N-fold irreducible conic Q supported on $\mathcal{F}'$ ($N = 6, \dots, 9$), or with an N-fold twisted cubic C ($N = 7, 8, 9$). We proceed by a case-by-case analysis of the residual effective divisor

$$\mathcal{R} = \mathcal{F}' - NQ, \deg(\mathcal{R}) \leq 27-2N, \quad \text{respectively} \quad \mathcal{R} = \mathcal{F}' - NC, \deg(\mathcal{R}) \leq 27-3N.$$

9.1. Multiple conic case

Following Lemma 8.8 and Lemma 8.9, we start by assuming that $\mathcal{F}$ contains an irreducible conic Q of multiplicity $N = 6, \dots, 9$ (and no conic of higher multiplicity). We proceed by a case-by-case analysis.

9.1.1. $N = 9$

Recall from Remark 7.3 that

$$Q.\sum_{i=1}^{M} \ell_i \leq 88.$$

Compared with $Q.\mathcal{F} = 62$, this yields

$$Q.\mathcal{R} \geq 10 \quad \text{and} \quad (\mathcal{F}')^2 \geq -324 + 180 + \mathcal{R}^2 = -144 + \mathcal{R}^2.$$

Hence (8.4) implies $\mathcal{R}^2 \leq -171$. One might like to continue disregarding the linear components as in Lemma 8.6, but here the positive contribution from linear components might already have been captured in $Q.\mathcal{R} \geq 10$, so we have to content ourselves with applying Lemma 8.5 to $\mathcal{R}$. It follows that there is a seven-fold line ℓ, but this contradicts Lemma 8.10.

9.1.2. $N = 8$

We deduce as in 9.1.1 that $Q.\mathcal{R} \geq 6$. Hence $\mathcal{R}^2 \leq -155$, and Lemma 8.5 shows as before that $\mathcal{R}$ contains a five-fold line which is ruled out by Lemma 8.10.

9.1.3. $N = 7$

Consider the residual divisor $\mathcal{R}$ with $\deg(\mathcal{R}) \leq 13$. As before, we infer $Q.\mathcal{R} \geq 2$ and $\mathcal{R}^2 \leq -147$. It follows from Lemma 8.5 that $\mathcal{R}$ contains either a 4-fold line ℓ (which is ruled out by Lemma 8.10) or a 6-fold conic $Q' \neq Q$. In the latter case, either $M = 129$ and the bound (8.4) improves to

$$(\mathcal{F}')^2 \leq -355$$

which is impossible to attain, or $\mathcal{F}' = 7Q + 6Q' + \ell$ for some line ℓ whose positive contribution by Remark 8.4 cannot be compensated for.

9.1.4. $N = 6$

The residual divisor $\mathcal{R}$ has degree $\deg(\mathcal{R}) \leq 15$ and $\mathcal{R}^2 \leq -171$. As in 9.1.3, Lemma 8.5 implies that $\mathcal{R}$ contains a line of multiplicity 4 (which again is excluded by Lemma 8.10) or another conic $Q' \neq Q$ of multiplicity $m \geq 6$. In the latter case, since $m = 7$ has already been covered in 9.1.3, it remains to realize that for $m = 6$, Lemma 8.5 applied to the residual divisor again leads to 3-fold line and subsequently to a contradiction by the positive contribution from (8.7).

9.2. Multiple twisted cubic case

Continuing the present line of arguments based on Lemma 8.8, we distinguish three cases depending on the multiplicity $N = 7, 8, 9$ of the twisted cubic C in $\mathcal{F}'$.

9.2.1. $N = 7$

Here $\mathcal{R}^2 \leq -70$ and $\deg(\mathcal{R}) \leq 6$, so Lemma 8.5 implies that $\mathcal{R}$ contains a 4-fold line. Hence Lemma 8.10 gives the required contradiction.

9.2.2. $N = 8$

In this case, $\deg(\mathcal{R}) \leq 3$ with no obvious further restrictions. We proceed by distinguishing how $\mathcal{R}$ decomposes.

If $\mathcal{R}$ contains a line, then its positive contribution following Remark 8.4 gives a contradiction.

If $\mathcal{R}$ is an irreducible conic, then $M = 129$, and the improved bound $\mathcal{F}'^2 \leq -355$ from (8.4) cannot be reached. Similarly, of course, if $\mathcal{R} = 0$.

Hence $\mathcal{R}$ is an irreducible cubic (plane or twisted) which thus meets no more than 98 lines (by looking at $\mathcal{R}.\mathcal{F}$). All the remaining lines (at least 30 in number) have to meet C with multiplicity two for (8.2) to hold. But then their valency drops by 7 to $v(\ell) = 18$. As this occurs for at least 30 lines, we get a big correction term of -210 to (8.4) which is impossible to beat.

9.2.3. $N = 9$

We have $\mathcal{F}' = 9C$ and $M = 128$. Then (8.2) implies that any line $\ell \subset X_5$ meets C, and for $C.\mathcal{F} = 93$ to hold, C has to intersect exactly 10 lines with multiplicity two, say $\ell_1 \ldots, \ell_{10}$. We turn to the effective divisor D from 7.2. By Proposition 7.5, there is a decomposition

$$D = \sum_{i=1}^{M} \ell_i + \sum_{j=1}^{10} \ell_j + mC + D'.$$

For degree reasons, this implies $m \leq 9$. In comparison, the intersection product reads

$$99 = C.D = 158 - 5m + C.D'$$

whence $m \geq 12$, contradiction.

9.3. Proof of Theorem 1.2

To conclude, let us wrap up the proof of Theorem 1.2. Assuming that $M > 127$, we deduced in Lemma 8.8 that $\mathcal{F}$ contains a conic with multiplicity $N \geq 6$, or a twisted cubic with multiplicity $N \geq 7$. On the other hand, $N \leq 9$ by Lemma 8.9 respectively for degree reasons. Then the considerations in Subsections 9.1, 9.2 successively ruled out all configurations which might have fitted (8.4), thus completing the proof of Theorem 1.2. □

Remark 9.1. By inspection, the results of the preceding four sections are totally geometric, *i.e.* they essentially only require that (2.1) has a smooth fiber, and that the flecnodal divisor does not degenerate.

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