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Completely Sidon sets in discrete groups

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Abstract. A subset of a discrete group G is called completely Sidon if its span in $C^*(G)$ is completely isomorphic to the operator space version of the space ℓ_1 (*i.e.*, ℓ_1 equipped with its maximal operator space structure). We recently proved a generalization to this context of Drury's classical union theorem for Sidon sets: completely Sidon sets are stable under finite unions. We give a different presentation of the proof emphasizing the "interpolation property" analogous to the one Drury discovered. In addition we prove the analogue of the Fatou-Zygmund property: any bounded Hermitian function on a symmetric completely Sidon set $\Lambda \subset G \setminus \{1\}$ extends to a positive definite function on G. In the final section, we give a completely isomorphic characterization of the closed span C_Λ of a completely Sidon set in $C^*(G)$: the dual (in the operator space sense) of C_Λ is exact if and only if Λ is completely Sidon. In particular, Λ is completely Sidon as soon as C_Λ is completely isomorphic (by an arbitrary isomorphism) to $\ell_1(\Lambda)$ equipped with its maximal operator space structure.

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In harmonic analysis (see [34]) a subset Λ of an Abelian discrete group G is called Sidon with constant C if for all finitely supported $a : \Lambda \to \mathbb{C}$ we have

$$\sum\nolimits_{n\in\Lambda} |a_n| \le C \| \sum\nolimits_{n\in\Lambda} a_n \gamma_n \|_{C(\hat{G})}$$

where $\hat{G}$ is the dual (compact) Abelian group, and where $\gamma_n : \hat{G} \to \mathbb{T}$ is the character on $\hat{G}$ associated to an element $n \in G$. Here $C(\hat{G})$ denotes the space of continuous functions on $\hat{G}$ equipped with the usual sup-norm. For instance, when $G = \mathbb{Z}$ we may view $\hat{G} = \mathbb{R}/\mathbb{Z}$ and $\gamma_n(t) = e^{2i\pi nt}$.

Equivalently, if $C_\Lambda \subset C(\hat{G})$ denotes the closed span of $\{\gamma_n \mid n \in \Lambda\}$ and (e_n) denotes the canonical basis of $\ell_1(\Lambda)$ the mapping $u : C_\Lambda \to \ell_1(\Lambda)$ defined by $u(\gamma_n) = e_n$ is an isomorphism with $\|u\| \le C$ (and trivially $\|u^{-1}\| \le 1$).

In the Abelian case the subject has a long and rich history for which we refer to [23,34,37,38]. The first period roughly 1960-1970 was driven by a major open problem: whether the union of two Sidon sets is a Sidon set. Eventually this was

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proved by Drury [16] using a beautiful convolution device. After this achievement, it was only natural to investigate the non-Abelian case. For that two options appear, either:

1. one replaces $\hat{G}$ by a compact non-Abelian group and Λ becomes a set of irreducible unitary representations on the latter compact group, or:
2. one replaces G by a discrete non-Abelian group.

We will not deal with case 1; in that case the union problem resisted generalization but was solved by Rider in 1975. The subject suffered from the disappointing discovery that the duals of most compact Lie groups do not contain infinite Sidon sets. We refer the reader to our recent survey [50] for more on this.

This paper is devoted to case 2. In this case, there were several attempts to generalize the Sidon set theory, notably by Picardello and Bożejko (see [9,44]), but no analogue of Drury's union theorem was found. The novelty of our approach is that while these authors defined Sidon sets using the Banach space structures of the relevant non-commutative operator algebras, we fully use their operator space structures. In particular, the Banach space $E = \ell_1(\Lambda)$ that enters the definition of a Sidon set has to be considered as an operator space, given together with an isometric embedding $E \subset A$ into a C^*-algebra A, or into $B(H)$ for some Hilbert space H.

By definition an operator space is a subspace $E \subset A$ (or $E \subset B(H)$). We may use a different A and a different embedding as long as it induces the same sequence of norms on all the spaces $M_n(E)$ ($n \geq 1$). Of course $M_n(A)$ is equipped with its unique C^*-norm, or equivalently the norm of the C^*-tensor product $M_n \otimes A$, and this induces a norm on the subspace $M_n(E)$. The theory of operator spaces is now well developed. The main novelty is that the bounded linear maps $u : E \to F$ between operator spaces are now replaced by the completely bounded (in short c.b.) ones and the norm $\|u\|$ is replaced by the cb-norm $\|u\|_{\mathrm{cb}}$. We say that u is a complete isomorphism if it is invertible and both u and u^{-1} are c.b. maps. See below for background on this. We refer to the books [18,47] for more information.

In the case of ℓ_1, there is a privileged operator space structure $\ell_1 \subset A$ that can be conveniently described using for A the C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group with countably infinitely many generators. Let (U_n) denote the unitaries in A corresponding to the free generators. The embedding $j : \ell_1 \subset A$ is defined by $j(e_n) = U_n$, where (e_n) is the canonical basis of ℓ_1. Similarly, given an arbitrary set Λ we may consider the group $\mathbb{F}_\Lambda$ freely generated by $(g_n)_{n\in\Lambda}$ and the corresponding unitaries $(U_n)_{n\in\Lambda}$ in $A = C^*(\mathbb{F}_\Lambda)$. We then define $j : \ell_1(\Lambda) \subset A$ again by $j(e_n) = U_n$ for $n \in \Lambda$. Following Blecher and Paulsen (see [47, Section 3] and [47, page 183]), we call this the maximal operator space structure on $\ell_1(\Lambda)$. Unless specified otherwise, we always assume $\ell_1(\Lambda)$ equipped with the latter. More explicitly we have for any C^*-algebra B (*e.g.* $B = M_N$) and any finitely supported $a : \Lambda \to B$

$$\left\| \sum\nolimits_\Lambda a_t \otimes U_t \right\| = \sup \left\{ \left\| \sum\nolimits_\Lambda a_t \otimes z_t \right\| \right\} \tag{1}$$

where the sup runs over all H and all functions $z : \Lambda \to B(H)$ such that $\sup_\Lambda \|z_t\| \le 1$.

Remark 1. By the Russo-Dye theorem, the supremum is unchanged if we restrict to z's with unitary values. Moreover, if we wish, we may (after translation by z_s^{-1}) restrict to z's with unitary values and such that $z_s = 1$ for a single fixed $s \in \Lambda$. In addition we may restrict to finite dimensional H's if we wish (see *e.g.* [47, page 155] for details).

In the case $B = \mathbb{C}$, we find

$$\left\| \sum\nolimits_\Lambda a_t \otimes U_t \right\| = \sum\nolimits_\Lambda |a_t|. \tag{2}$$

We now introduce the relevant generalization of Sidon sets.

Let $A \subset B(H)$ be a C^*-algebra. If $B \subset B(K)$ is any other C^*-algebra (for instance $B = M_N = B(K)$ when $\dim(K) = N$) and $x \in B \otimes A$ (algebraic tensor product) we denote by $\|x\|_{B \otimes_{\min} A}$ or more simply by $\|x\|_{\min}$ the norm of x in the minimal or spatial tensor product, *i.e.*, we set

$$\|x\|_{\min} = \|x : K \otimes_2 H \to K \otimes_2 H\|.$$

Moreover, we use the same definition when A, B are merely operator subspaces of $B(H), B(K)$. It is known that $\|x\|_{\min}$ does not depend on the choice of the completely isometric embeddings $A \subset B(H)$ and $B \subset B(K)$.

Let G be a discrete group. Let $U_G : G \to B(\mathcal{H})$ be the universal representation and let $C^*(G) \subset B(\mathcal{H})$ denote the C^*-algebra generated by U_G.

Given a subset $\Lambda \subset G$ we denote by $C_\Lambda \subset C^*(G)$ the operator space defined by

$$C_\Lambda = \overline{\operatorname{span}}[U_G(t) \mid t \in \Lambda].$$

Definition 2. We say that $\Lambda \subset G$ is completely Sidon if there is C such that for any $N \ge 1$ and any finitely supported $a : \Lambda \to M_N$

$$\left\| \sum\nolimits_{t \in \Lambda} a_t \otimes U_t \right\|_{M_N \otimes_{\min} C^*(\mathbb{F}_\Lambda)} \le C \left\| \sum\nolimits_{t \in \Lambda} a_t \otimes U_G(t) \right\|_{M_N \otimes_{\min} C^*(G)}.$$

More explicitly, this is the same as requiring

$$\sup \left\| \sum a_t \otimes u_t \right\|_{\min} \le C \left\| \sum a_t \otimes U_G(t) \right\|_{\min}, \tag{3}$$

where the sup runs over all families $(u_t)_{t \in \Lambda}$ of unitaries on an arbitrary Hilbert space H.

Equivalently, the linear map $u : C_\Lambda \to \ell_1(\Lambda)$ defined for $t \in \Lambda$ by $u(U_G(t)) = U_{\mathbb{F}_\Lambda}(g_t)$ is c.b. with $\|u\|_{\mathrm{cb}} \le C$. Then, since $\|u^{-1}\|_{\mathrm{cb}} \le 1$, the space C_Λ is completely isomorphic to $\ell_1(\Lambda)$ equipped with its maximal operator space structure.

The fundamental example is given by free sets, as follows.

Proposition 3. *Let $S \subset G$ be a free set, and let Λ be a translate of $S \cup \{1\}$. Then Λ is completely Sidon with $C = 1$. Conversely, any completely Sidon set with $C = 1$ is of this form.*

For the proof see Proposition 6.1 below.

We can now state our main results:

1. Completely Sidon sets are stable by finite unions.
2. Assume Λ completely Sidon, symmetric, $1 \notin \Lambda$ and assume for simplicity Λ without any element of order 2 (this case can also be handled), then the linear map $u : C_\Lambda \to C^*(\mathbb{F}_\Lambda)$ associated to the mapping $t \mapsto g_t$ extends to a completely positive (in short c.p.) map $\check{u} : C^*(G) \to C^*(\mathbb{F}_\Lambda)$.
3. If the operator space C_Λ is completely isomorphic to $\ell_1(\Lambda)$ via an arbitrary linear correspondence, or if the dual operator space C_Λ^* is exact, then Λ is completely Sidon.

Point 1 is the non-Abelian version of Drury's 1970 union theorem from [16]. Point 2 is analogous to the so-called "Fatou-Zygmund" property established by Drury in 1974 (see [17,37]), while point 3 is the analogue of the 1976 Varopoulos theorem from [56]. For emphasis, we should point out that a surprising dichotomy stems from it: for any infinite subset $\Lambda \subset G$ the space C_Λ^* is (roughly) either "very big" or "very small" in the operator space sense.

Points 1 and 2 answer questions raise by Bozejko in [9] (see Remark 1.3). The proof of Point 2 is similar to that of 1, but is better understood if one first runs through the proof of 1 as we do below. Moreover, the quantitative estimates we give in terms of the constant C may be of independent interest. Lastly 3 is new.

Remark 4. We should emphasize that the theory of completely Sidon sets does not contain the classical case, although it is very much parallel to it. Indeed, any group G that contains an infinite completely Sidon set must be non-amenable (and hence extremely non-commutative) because $C^*(G)$ cannot be exact. More precisely, if the set has at least n elements with completely Sidon constant $C < n/2\sqrt{n-1}$ then $C^*(G)$ is not exact (see [47, page 336]) and a fortiori G is not amenable. However, we do not know whether such a G must contain a copy of $\mathbb{F}_\infty$ (or equivalently $\mathbb{F}_2$).

Problem: By our main result, any finite union of translates of free sets is completely Sidon. Is the converse true? This fundamental question is analogous to a well-known open one for the classical Sidon sets (see [23, page 107]).

1. Notation and background

Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces, consider a map $u : E \to F$. For any $n \geq 1$, let $M_n(E)$ be the space of $n \times n$ matrices with entries in E. We

have $M_n(E) \subset M_n(B(H))$. We equip $M_n(E) \subset M_n(B(H))$ with the norm induced by $B(\ell_2^n(H)) \approx M_n(B(H))$ where $\ell_2^n(H)$ means $H \oplus H \oplus \cdots \oplus H$ (n times). We define $u_n : M_n(E) \longrightarrow M_n(F)$ by setting $u_n([a_{ij}]) = [u(a_{ij})]$. A map $u : E \to F$ is called completely bounded (in short c.b.) if $\sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)} < \infty$. Let

$$\|u\|_{\mathrm{cb}} = \sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)}.$$

We denote by $CB(E, F)$ the Banach space of all such maps equipped with the c.b. norm. Let $M_n(E)_+ = M_n(E) \cap M_n(B(H))_+$. We say that u is completely positive (c.p. in short) if u_n is positivity preserving, *i.e.*, $u_n(M_n(E)_+) \subset M_n(F)_+$ for any n. When E is an operator system (*i.e.*, E is a unital self-adjoint linear subspace) c.p. implies c.b. and $\|u\|_{\mathrm{cb}} = \|u\| = \|u(1)\|$. We denote by $CP(E, F)$ the set of c.p. maps.

Let A, B be C^*-algebras. We will denote by $D(A, B)$ the set of all "decomposable" maps $u : A \to B$, *i.e.*, the maps that are in the linear span of $CP(A, B)$. This means that $u \in D(A, B)$ iff there are $u_j \in CP(A, B)$ $(j = 1, 2, 3, 4)$ such that

$$u = u_1 - u_2 + i(u_3 - u_4).$$

We will repeatedly use the nice definition of the dec-norm of a linear map $u : A \to B$ between C^*-algebras given by Haagerup in [27], as follows. We set

$$\|u\|_{\mathrm{dec}} = \inf\{\max\{\|S_1\|, \|S_2\|\}\}, \tag{1.1}$$

where the infimum runs over all maps $S_1, S_2 \in CP(A, B)$ such that the map

$$V : x \to \begin{pmatrix} S_1(x) & u(x) \\ u(x^*)^* & S_2(x) \end{pmatrix} \tag{1.2}$$

is in $CP(A, M_2(B))$. This is equivalent to the simple minded choice of norm $\|u\| = \inf \sum_1^4 \|u_j\|$. When u is self-adjoint (*i.e.*, when $u(x^*) = u(x)^*$ for all $x \in A$) we have $\|u\|_{\mathrm{dec}} = \inf \|u_1 + u_2\|$ where the infimum runs over all the possible decompositions of u as $u = u_1 - u_2$ with u_1, u_2 c.p.

See [27] for the proofs of all the basic facts on decomposable maps, that are freely used throughout this note. In particular, we repeatedly use the fact that for any pair $v_j : A_j \to B_j$ $(j = 1, 2)$ of decomposable maps between C^*-algebras, the map $v_1 \otimes v_2$ on the algebraic tensor product uniquely extends to a map, still denoted by $v_1 \otimes v_2$, in $D(A_1 \otimes_{\max} A_2, B_1 \otimes_{\max} B_2)$ with

$$\|v_1 \otimes v_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2\|_{\mathrm{dec}} \leq \|v_1\|_{\mathrm{dec}} \|v_2\|_{\mathrm{dec}}. \tag{1.3}$$

Moreover, if v_1, v_2 are completely positive (c.p. in short) the resulting map $v_1 \otimes v_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$ is c.p. Here $A_1 \otimes_{\max} A_2$ stands for the C^*-algebra obtained by completing the algebraic tensor product $A_1 \otimes A_2$ with respect to the maximal C^*-norm (see *e.g.* [47, page 227]).

We also use from [27] that if $B = B(H)$ or if B is an injective C^*-algebra (which means the identity of B factors through $B(H)$ via c.p. maps) then for any C^*-algebra A we have $CB(A, B) = D(A, B)$ and for any $u \in CB(A, B)$

$$\|u\|_{\mathrm{cb}} = \|u\|_{\mathrm{dec}}. \tag{1.4}$$

See [18,47] for more background and references.

Let $\Lambda \subset G$ be a subset of a discrete group G.

Let U_G be the universal representation of G, and let $C^*(G)$ be the C^*-algebra generated by U_G.

Let λ_G be the left regular representation, and let $C^*_\lambda(G)$ be the C^*-algebra generated by λ_G. We denote by M_G the von Neumann algebra generated by λ_G.

The notation $(\delta_s)_{s\in G}$ is used mostly for the canonical basis of the group algebra $\mathbb{C}[G]$, and sometimes (abusively) for that of $\ell_2(G)$. As usual we view $\mathbb{C}[G]$ as a dense $*$-subalgebra of $C^*(G)$.

Proposition 1.1. *Assume we have an embedding $C^*(\mathbb{F}_\Lambda) \subset B(\mathcal{H})$. The following properties are all equivalent refomulations of Definition 2:*

(i) *The correspondence $t \mapsto g_t$ from Λ to the free generators of $\mathbb{F}_\Lambda$ extends to a c.b. linear map $u : C^*(G) \to B(\mathcal{H})$ with $\|u\|_{\mathrm{cb}} \le C$;*

(ii) *For any Hilbert space H, for any bounded mapping $z : \Lambda \to B(H)$ there is a bounded linear map $u_z : C^*(G) \to B(H)$ with $\|u_z\|_{\mathrm{cb}} \le C \sup_{t\in\Lambda} \|z(t)\|$ such that $u_z(U_G(t)) = z(t)$ for any $t \in \Lambda$.*

Proof. If Λ is completely Sidon then clearly (i) holds by the injectivity of $B(\mathcal{H})$, and conversely (i) obviously implies Λ completely Sidon.

By the injectivity of $B(H)$ for any z as in (ii) there is a linear $v_z : B(\mathcal{H}) \to B(H)$ extending the correspondence $U_{\mathbb{F}_\Lambda}(g_t) \to z(t)$ $(t \in \Lambda)$ with $\|v_z\|_{\mathrm{cb}} = \sup_{t\in\Lambda} \|z(t)\|$ (this expresses the fact that $\{g_t \mid t \in \Lambda\}$ is completely Sidon with constant 1). Then the composition $u_z = v_z u$ shows that (i) implies (ii). The converse is obvious. □

Remark 1.2. If Λ is asymmetric in the sense that $\Lambda \cap \Lambda^{-1} = \phi$, we show in Corollary 4.4 that the correspondence $\Lambda \ni t \mapsto g_t \in \mathbb{F}_\Lambda$ extends to a c.p. map $u : C^*(G) \to C^*(\mathbb{F}_\Lambda)$ but then we only obtain $\|u\|_{\mathrm{cb}} (= \|u\|) \le O(C^4)$.

Remark 1.3. In [9] Bożejko considers the property appearing in (ii) in Proposition 1.1 and he calls "w-operator Sidon" the sets with this property. He calls "operator Sidon" the sets $\Lambda \subset G$ satisfying $\Lambda \cap \Lambda^{-1} = \phi$ such that any $B(H)$-valued bounded function on Λ admits a positive definite extension on G, and proves that free sets (*i.e.*, $\{g_t \mid t \in \Lambda\}$ in $\mathbb{F}_\Lambda$) have this property. "Operator Sidon" is a priori stronger than "w-operator Sidon", but actually, we will show later on in this paper (see Theorem 4.1) that the two properties are equivalent. Bożejko also asked whether these sets are stable under union. We show this in Corollary 3.3. Our results suggest to revise the terminology: perhaps the term "operator Sidon" should be adopted instead of our "completely Sidon".

Remark 1.4. The following observation plays a crucial role in this paper. Let $\Gamma = \mathbb{F}_\Lambda$. Let $Q : C^*(\Gamma) \to M_\Gamma$ be the $*$-homomorphism associated to λ_Γ. Let $E \subset C^*(G)$ be an operator subspace. Then for any $u \in CB(E, C^*(\Gamma))$ there is $u^\dagger \in D(C^*(G), M_\Gamma)$ with $\|u^\dagger\|_{\mathrm{dec}} \le \|u\|_{\mathrm{cb}}$ such that $u^\dagger{}_{|E} = Qu$. Indeed, free groups satisfy Kirchberg's factorization property from [35]. In particular, by a well-known construction involving ultraproducts (see Th. 6.4.3 and Th. 6.2.7 in [13]), for some H the map Q factors through $B(H)$ via c.p. contractive maps $Q_1 : B(H) \to M_\Gamma$ and $Q_2 : C^*(\Gamma) \to B(H)$ so that $Q = Q_1 Q_2$. By the injectivity of $B(H)$ the composition $Q_2 u : E \to B(H)$ admits an extension $\tilde{Q_2 u} \in CB(C^*(G), B(H))$ with $\|\tilde{Q_2 u}\|_{\mathrm{cb}} \le \|Q_2 u\|_{\mathrm{cb}} \le \|u\|_{\mathrm{cb}}$. But by (1.4) $CB(C^*(G), B(H)) = D(C^*(G), B(H))$ isometrically. Therefore $\|\tilde{Q_2 u}\|_{\mathrm{dec}} \le \|u\|_{\mathrm{cb}}$. The mapping $u^\dagger = Q_1 \tilde{Q_2 u}$ has the announced properties. If we assume in addition that E is an operator system and that u is c.p. then we find $u^\dagger \in CP(C^*(G), M_\Gamma)$ with $\|u^\dagger\| = \|u^\dagger\|_{\mathrm{dec}} \le \|u\| = \|u\|_{\mathrm{cb}}$.

In particular, if $\Lambda \subset G$ is a completely Sidon set with constant C, let $E \subset C^*(G)$ be the span of $\{U_G(t) \mid t \in \Lambda\}$. We may apply the preceding observation to the linear mapping u defined by $u(U_G(t)) = U_\Gamma(t)$ $(t \in \Lambda)$. We find $\mathcal{U} \in D(C^*(G), M_\Gamma)$ such that $\mathcal{U}(U_G(t)) = \lambda_\Gamma(g_t)$ for all $t \in \Lambda$ with $\|\mathcal{U}\|_{\mathrm{dec}} \le \|u\|_{\mathrm{cb}} \le C$. We will show below (see Corollary 2.9) that conversely the existence of such a $\mathcal{U}$ implies that $\Lambda \subset G$ is completely Sidon.

Remark 1.5. Let A be a unital C^*-algebra. By [33] any $a \in A$ with $\|a\| < 1 - 2/n$ can be written as an average of n unitaries in A.

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2. Operator valued harmonic analysis

Let G be a discrete group. Let $\varphi : G \to A$ be a function with values in a C^*-algebra. Let $u_\varphi : \mathbb{C}[G] \to A$ be the linear map extending φ. We denote respectively by

$$B(G, A), \quad CP(G, A), \quad CB(G, A), \quad D(G, A)$$

the set of those φ such that u_φ extends to a map $u_\varphi : C^*(G) \to A$ respectively in

$$B(C^*(G), A), \quad CP(C^*(G), A), \quad CB(C^*(G), A), \quad D(C^*(G), A)$$

and we set

$$\|\varphi\|_{B(G,A)} = \|u_\varphi\|, \quad \|\varphi\|_{CB(G,A)} = \|u_\varphi\|_{\mathrm{cb}}, \quad \|\varphi\|_{D(G,A)} = \|u_\varphi\|_{\mathrm{dec}}. \tag{2.1}$$

By (1.4), when $A = B(H)$ or when A is injective then $CB(G, A) = D(G, A)$ and $\|\varphi\|_{CB(G,A)} = \|\varphi\|_{D(G,A)}$, but in general we only have $D(G, A) \subset CB(G, A)$ with $\|\varphi\|_{CB(G,A)} \leq \|\varphi\|_{D(G,A)}$ and the inclusion is strict.

When $A = \mathbb{C}$ we have $B(G, \mathbb{C}) = CB(G, \mathbb{C}) = D(G, \mathbb{C})$ isometrically and we recover the non-commutative analogue of the classical "Fourier-Stieltjes algebra" $B(G)$ (see, *e.g.*, [19] or [22, page 3]), which can be identified isometrically with $C^*(G)^*$: we have $\varphi \in B(G) = B(G, \mathbb{C})$ iff there is a unitary representation $\pi : G \to B(H)$ and vectors $\xi, \eta \in H$ such that $\varphi(.) = \langle \eta, \pi(.)\xi \rangle$ and $\|\varphi\|_{B(G)} = \inf\{\|\eta\|\|\xi\|\}$ where the infimum (actually the minimum is attained) runs over all possible such representations of φ.

By the factorization of c.b. maps (see *e.g.* [18,43]) the case when $A = B(H)$ is entirely analogous: in that case $\varphi \in CB(G, B(H))$ if and only if there are $\hat{H}$, a unitary representation $\pi : G \to B(\hat{H})$ and operators $\xi, \eta \in B(H, \hat{H})$ such that $\varphi(.) = \eta^* \pi(.)\xi$ and

$$\|\varphi\|_{CB(G,A)} = \inf\{\|\eta\|\|\xi\|\}, \tag{2.2}$$

where the infimum (actually a minimum) runs over all possible such representations of φ.

With this notation we can immediately reformulate Proposition 1.1 like this:

Proposition 2.1. *A subset $\Lambda \subset G$ is a completely Sidon set with constant C iff for any H and any $z : \Lambda \to B(H)$ such that $\sup_\Lambda \|z\| \leq 1$ there is $\varphi \in CB(G, B(H))$ with $\|\varphi\|_{CB(G,B(H))} \leq C$ such that $\varphi_{|\Lambda} = z$. Moreover, for the latter to hold it suffices that it holds for any finite dimensional H.*

The next lemma is a simple refinement of the last statement. The proof is based on a specific "extremal" property of the norm in (1).

Lemma 2.2. *Let $0 < \varepsilon < 1$. Let H be a Hilbert space and $c > 0$ a constant. Assume that for any $z : \Lambda \to B(H)$ with $\sup_\Lambda \|z\| \leq 1$ there is $\varphi_0 \in CB(G, B(H))$ with $\|\varphi_0\|_{CB(G,B(H))} \leq c$ such that $\sup_\Lambda \|z - \varphi_0\| \leq \varepsilon$. Then for any $z : \Lambda \to B(H)$ with $\sup_\Lambda \|z\| \leq 1$ there is $\varphi \in CB(G, B(H))$ such that $\varphi_{|\Lambda} = z$ with $\|\varphi\|_{CB(G,B(H))} \leq c/(1 - \varepsilon)$.*

Proof. Applying the assumption to the function $(\varphi_0 - z)/\varepsilon$ we find $\varphi_1 \in CB(G, B(H))$ with $\|\varphi_1\|_{CB(G,B(H))} \leq c$ such that $\sup_\Lambda \|z - \varphi_0 - \varepsilon\varphi_1\| \leq \varepsilon^2$. Repeating this step, we obtain φ_j with $\|\varphi_j\|_{CB(G,B(H))} \leq c$ such that $\sup_\Lambda \|z - \varphi_0 - \cdots - \varepsilon^j \varphi_j\| \leq \varepsilon^{j+1}$. Then $\varphi = \sum_0^\infty \varepsilon^j \varphi_j$ gives us the desired function. □

Remark 2.3 (On completely positive definite functions). We will say (following [43]) that $\varphi : G \to A$ is completely positive definite if for any finite subset $\{t_1, ..., t_n\} \subset G$ we have $[\varphi(t_i^{-1} t_j)] \in M_n(A)_+$. By classical results (due to Naimark, see [43, page 51]) $\varphi \in CP(G, A)$ iff φ is completely positive definite. Assuming $A \subset B(H)$ φ is completely positive definite iff there are $\hat{H}, \pi : G \to B(\hat{H})$

and $\xi \in B(H, \hat{H})$ such that $\varphi(.) = \xi^* \pi(.)\xi$. When $A = B(H)$, by a polarization argument (2.2) shows that any $\varphi \in CB(G, A)$ can be written as a linear combination $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ with $\varphi_j \in CP(G, A)$ for all $j = 1, ..., 4$.

The spaces $CB(G, A)$ and $D(G, A)$ can also be viewed as spaces of multipliers. To any $\varphi : G \to A$ we associate a "multiplier" $M_\varphi : \mathbb{C}[G] \to C^*(G) \otimes_{\min} A$ that takes $t \in G$ to $U_G(t) \otimes \varphi(t)$.

Proposition 2.4. *The multiplier M_φ extends to a c.b. (respectively decomposable) map from $C^*(G)$ to $C^*(G) \otimes_{\min} A$ (respectively $C^*(G) \otimes_{\max} A$) iff $\varphi \in CB(G, A)$ (respectively $\varphi \in D(G, A)$), and we have*

$$\|\varphi\|_{CB(G,A)} = \|M_\varphi : C^*(G) \to C^*(G) \otimes_{\min} A\|_{\mathrm{cb}}$$

$$\begin{aligned}\|\varphi\|_{D(G,A)} &= \|M_\varphi : C^*(G) \to C^*(G) \otimes_{\max} A\|_{\mathrm{dec}} \\ &= \|M_\varphi : C^*(G) \to C^*(G) \otimes_{\min} A\|_{\mathrm{dec}}.\end{aligned}$$

Moreover, $\varphi \in CP(G, A)$ iff M_φ extends to a c.p. map from $C^*(G)$ to $C^*(G) \otimes_{\max} A$, or equivalently a c.p. map from $C^*(G)$ to $C^*(G) \otimes_{\min} A$.

Proof. Let $\pi_1 : G \to \mathbb{C} = B(\mathbb{C})$ be the trivial representation and let $u_1 : C^*(G) \to \mathbb{C} = B(\mathbb{C})$ be the associate $*$-homomorphism. Note that $u_\varphi = (u_1 \otimes Id_A)M_\varphi$. This shows that if M_φ is either c.b., c.p. or decomposable with values in $C^*(G) \otimes_{\min} A$ then the same is true for u_φ. Conversely, if $\|u_\varphi\|_{\mathrm{cb}} \leq 1$ then $\|Id_{C^*(G)} \otimes u_\varphi : C^*(G) \otimes_{\min} C^*(G) \to C^*(G) \otimes_{\min} A\|_{\mathrm{cb}} \leq 1$. Let $J_{\min} : C^*(G) \to C^*(G) \otimes_{\min} C^*(G)$ be the diagonal embedding taking $t \in G$ to $(t, t) \in G \times G$ (corresponding to $U_G \simeq U_G \otimes U_G$ as representations on G). Then $M_\varphi = (Id_{C^*(G)} \otimes u_\varphi)J_{\min}$ and hence $\|M_\varphi : C^*(G) \to C^*(G) \otimes_{\min} A\|_{\mathrm{cb}} \leq 1$. Similarly, u_φ c.p. implies that $M_\varphi : C^*(G) \to C^*(G) \otimes_{\min} A$ is c.p.

Assume $\|u_\varphi\|_{\mathrm{dec}} \leq 1$. Then by (1.3) $\|Id_{C^*(G)} \otimes u_\varphi : C^*(G) \otimes_{\max} C^*(G) \to C^*(G) \otimes_{\max} A\|_{\mathrm{dec}} \leq 1$. Let $J_{\max} : C^*(G) \to C^*(G) \otimes_{\max} C^*(G)$ be the analogous diagonal embedding so that $M_\varphi = (Id_{C^*(G)} \otimes u_\varphi)J_{\max}$. It follows that $\|M_\varphi : C^*(G) \to C^*(G) \otimes_{\max} A\|_{\mathrm{dec}} \leq 1$. A fortiori, composing with the $*$-homomorphism $C^*(G) \otimes_{\max} A \to C^*(G) \otimes_{\min} A$ we have $\|M_\varphi : C^*(G) \to C^*(G) \otimes_{\min} A\|_{\mathrm{dec}} \leq 1$. □

Remark 2.5. By a classical result (see [19]) $B(G)$ (which is isometrically the same as $CB(G, \mathbb{C})$ or $D(G, \mathbb{C})$) is a Banach algebra for the pointwise product. In the operator valued case there are *two distinct analogues* of this fact, as follows. Let A_j be C^*-algebras ($j = 1, 2$). Let $\varphi_j \in CB(G, A_j)$ (respectively $\varphi_j \in D(G, A_j)$). Then the function $\varphi_1 \otimes \varphi_2 : G \to A_1 \otimes A_2$ is in $CB(G, A_1 \otimes_{\min} A_2)$ (respectively $D(G, A_1 \otimes_{\max} A_2)$ with norm

$$\|\varphi_1 \otimes \varphi_2\|_{CB(G, A_1 \otimes_{\min} A_2)} \leq \|\varphi_1\|_{CB(G,A_1)} \|\varphi_2\|_{CB(G,A_2)}$$

$$\text{(respectively } \|\varphi_1 \otimes \varphi_2\|_{D(G, A_1 \otimes_{\max} A_2)} \leq \|\varphi_1\|_{D(G,A_1)} \|\varphi_2\|_{D(G,A_2)}.\text{)}$$

To check this it suffices to observe that $u_{\varphi_1\otimes\varphi_2} = (u_{\varphi_1} \otimes u_{\varphi_2})J_{\min}$ (respectively $u_{\varphi_1\otimes\varphi_2} = (u_{\varphi_1} \otimes u_{\varphi_2})J_{\max}$). Moreover, in both cases $\varphi_1 \otimes \varphi_2$ is completely positive definite if each φ_1, φ_2 is so.

We now investigate the converse direction: how to obtain a multiplier from a linear mapping.

Remark 2.6. We have an embedding $G \to G \times G$ as a diagonal subgroup $\Delta_G \subset G \times G$. In that case it is well-known (see *e.g.* [47, page 154]) that we have a c.p. projection from $C^*(G \times G)$ onto the closed span of the subgroup Δ_G in $C^*(G \times G)$. It follows that the map $P_{\max} : C^*(G) \otimes_{\max} C^*(G) \to C^*(G)$ defined by $P_{\max}(U_G(s) \otimes U_G(t)) = U_G(t)$ if $s = t$ and $= 0$ otherwise is a unital c.p. map such that $\|P_{\max}\| = \|P_{\max}\|_{\mathrm{dec}} = 1$. Moreover, obviously $P_{\max}J_{\max} = Id_{C^*(G)}$. Therefore $J_{\max}P_{\max}$ is a unital c.p. projection (a conditional expectation) from $C^*(G) \otimes_{\max} C^*(G)$ to $J_{\max}(C^*(G)) \simeq C^*(\Delta_G)$.

For any $t \in G$ we denote by $f_t^G \in M_G^*$ the functional defined by

$$f_t^G(x) = \langle \delta_t, x\delta_e \rangle.$$

Note that (f_t^G) is biorthogonal to $(\lambda_G(t))$.

The next result, essentially from [47, page 150], is a refinement of Remark 2.6, that illustrates the usefulness of the Fell absorption principle. The latter says that for any unitary representation π on G the representation $\lambda_G \otimes \pi$ is unitarily equivalent to $\lambda_G \otimes I$ (see *e.g.* [47, page 149]).

Theorem 2.7. *We have an isometric (C^*-algebraic) embedding*

$$J_G : \ C^*(G) \subset M_G \otimes_{\max} M_G$$

taking $U_G(t)$ to $\lambda_G(t) \otimes \lambda_G(t)$ $(t \in G)$, and a completely contractive c.p. mapping

$$P_G : \ M_G \otimes_{\max} M_G \to C^*(G)$$

such that

$$Id_{C^*(G)} = P_G J_G.$$

Moreover, $\forall a, b \in M_G$, $a = \sum_{t\in G} a(t)\lambda_G(t)$, $b = \sum_{t\in G} b(t)\lambda_G(t)$ we have (absolutely convergent series)

$$P_G(a \otimes b) = \sum\nolimits_{t\in G} a(t)b(t)U_G(t).$$

We illustrate this by the following diagram: we have $J_G = (\Upsilon_G \otimes \Upsilon_G)J_{\max}$:

$$\begin{array}{ccc} C^*(G) \otimes_{\max} C^*(G) & \xrightarrow{\Upsilon_G\otimes\Upsilon_G} & M_G \otimes_{\max} M_G \\ {\scriptstyle J_{\max}}\uparrow & & \downarrow{\scriptstyle P_G} \\ C^*(G) & \xrightarrow{Id_{C^*(G)}} & C^*(G) \end{array}$$

where $J_{\max} : C^*(G) \to C^*(G) \otimes_{\max} C^*(G)$ (as before) and $\Upsilon_G : C^*(G) \to M_G$ are the $*$-homomorphisms determined by

$$\forall t \in G \quad J_{\max}(U_G(t)) = U_G(t) \otimes U_G(t) \text{ and } \Upsilon_G(U_G(t)) = \lambda_G(t). \tag{2.3}$$

Proof. Let $x \in M_G \otimes M_G$ (algebraic tensor product). For $s, t \in G$ let $x(s,t) = (f_s^G \otimes f_t^G)(x)$. Note that $(a \otimes b)(s,t) = f_s^G(a) f_t^G(b)$ and $(\sum_s |f_s^G(y)|^2)^{1/2} \leq \|y\|_{M_G}$ for any $y \in M_G$. Therefore $\sum_t |(a \otimes b)(t,t)| \leq \|a\|_{M_G} \|b\|_{M_G}$. This shows that $\sum_t |x(t,t)| < \infty$ for any $x \in M_G \otimes M_G$.
Note for future reference that for any $b \in M_G$

$$P_G(\lambda_G(t) \otimes b) = f_t^G(b) U_G(t). \tag{2.4}$$

We will show the following claim:

$$\left\| \sum\nolimits_t x(t,t) U_G(t) \right\|_{C^*(G)} \leq \|x\|_{M_G \otimes_{\max} M_G}. \tag{2.5}$$

Then we set $P_G(x) = \sum_t x(t,t) U_G(t)$. This implies the result. Indeed, in the converse direction we have obviously

$$\left\| \sum x(t,t) \lambda_G(t) \otimes \lambda_G(t) \right\|_{\max} \leq \left\| \sum x(t,t) U_G(t) \right\|,$$

and hence (2.5) implies at the same time that J_G defines an isometric $*$-homomorphism and that the natural ("diagonal") projection onto $J_G(C^*(G))$ is a contractive map (actually a conditional expectation). The proof of the claim will actually show that P_G is c.p. We now prove this claim. Let $\pi : G \to B(H)$ be a unitary representation of G. As usual we denote by ρ_G the right regular representation taking any $t \in G$ to the unitary of right translation by t^{-1}. We introduce a pair of commuting representations (π_1, π_2) on $\ell_2(G) \otimes_2 H$ as follows:

$$\pi_1(\lambda_G(t)) = \lambda_G(t) \otimes \pi(t) \quad \text{and} \quad \pi_2(\lambda_G(t)) = \rho_G(t) \otimes I.$$

Note that both π_1 and π_2 extend to normal isometric representations on M_G. For π_1 this follows from the Fell absorption principle. For π_2, it follows from the fact that $\rho_G \simeq \lambda_G$ (indeed if $W : \ell_2(G) \to \ell_2(G)$ is the unitary taking δ_t to $\delta_{t^{-1}}$, then $W^* \lambda_G(\cdot) W = \rho_G(\cdot)$).

We denote by $\pi_1.\pi_2 : M_G \otimes M_G \to B(\ell_2(G) \otimes_2 H)$ the linear map (actually a $*$-homomorphism) defined on finite sums of rank 1 tensors by $(\pi_1.\pi_2)(\sum a_j \otimes b_j) = \sum \pi_1(a_j)\pi_2(b_j)$.

Since π_1 and π_2 have commuting ranges, we have

$$\|(\pi_1.\pi_2)(x)\|_{B(\ell_2(G) \otimes_2 H)} \leq \|x\|_{M_G \otimes_{\max} M_G}, \tag{2.6}$$

hence compressing the left-hand side to $K = \delta_e \otimes H \subset \ell_2(G) \otimes_2 H$, we obtain (note that $\langle \delta_e, \lambda_G(s) \rho_G(t) \delta_e \rangle = 1$ if $s = t$ and zero otherwise)

$$\sum\nolimits_t x(t,t) \pi(t) = P_K (\pi_1.\pi_2)(x)_{|K}$$

and hence

$$\left\| \sum\nolimits_t x(t,t)\pi(t) \right\|_{B(H)} \leq \|x\|_{M_G \otimes_{\max} M_G}. \tag{2.7}$$

Finally, taking the supremum over π, we obtain the announced claim (2.5). This argument shows that P_G is c.p. and $\|P_G\|_{\mathrm{cb}} \leq 1$. □

Remark 2.8. Let us denote $\bar{A}$ the complex conjugate of a C^*-algebra A, *i.e.*, A equipped with scalar multiplication defined for $\alpha \in \mathbb{C}, a \in A$ by: $\alpha \bar{a} = \overline{\bar{\alpha} a}$, where $\bar{a}$ denotes a viewed as an element of $\bar{A}$. Note that the correspondence $U_G(t) \mapsto \overline{U_G(t)}$ (respectively $\lambda_G(t) \mapsto \overline{\lambda_G(t)}$) extends to a $\mathbb{C}$-linear isomorphism from $C^*(G)$ to $\overline{C^*(G)}$ (respectively from M_G to $\overline{M_G}$). Therefore the following variant of Theorem 2.7 also holds: *There is an embedding $j_G : C^*(G) \to \overline{M_G} \otimes_{\max} M_G$ that takes $t \in G$ to $\overline{\lambda_G(t)} \otimes \lambda_G(t)$ and a contractive c.p. map $p_G : \overline{M_G} \otimes_{\max} M_G \to C^*(G)$ such that $p_G(\overline{\lambda_G(t)} \otimes \lambda_G(s)) = 0$ for $t \neq s$ satisfying $p_G j_G = Id_{C^*(G)}$.*

Corollary 2.9. *Let $\Gamma = \mathbb{F}_\Lambda$. A subset $\Lambda \subset G$ is completely Sidon iff there is a $\mathcal{U}$ in $D(C^*(G), M_\Gamma)$ such that $\mathcal{U}(U_G(t)) = \lambda_\Gamma(g_t)$ for all $t \in \Lambda$. In that case, the Sidon constant is at most $\|\mathcal{U}\|_{\mathrm{dec}}^2$.*

Proof. Assume there is such a $\mathcal{U}$. Let $J_{\max}$ be as in (2.3). Consider the mapping

$$J = P_\Gamma(\mathcal{U} \otimes \mathcal{U}) J_{\max}.$$

Clearly $J(U_G(t)) = U_\Gamma(g_t)$ for all $t \in \Lambda$. By Theorem 2.7 and (1.3) we have

$$\|J : C^*(G) \to C^*(\Gamma)\|_{\mathrm{dec}} \leq \|\mathcal{U}\|_{\mathrm{dec}}^2.$$

A fortiori $\|J\|_{\mathrm{cb}} \leq \|\mathcal{U}\|_{\mathrm{dec}}^2$. By Proposition 1.1 Λ is completely Sidon with constant $\|\mathcal{U}\|_{\mathrm{dec}}^2$.

For the converse, see Remark 1.4. □

Proposition 2.10. *Let $u \in D(C^*(G), M_G \otimes_{\max} A)$ with $\|u\|_{\mathrm{dec}} \leq 1$. We define $\varphi_u : G \to A$ by*

$$\varphi_u(t) = (f_t^G \otimes Id_A) u(U_G(t)).$$

Then $\|\varphi_u\|_{D(G,A)} \leq 1$. If u is c.p. then $\varphi_u \in CP(G, A)$.

Moreover, if there is $\varphi : G \to A$ such that $u(U_G(t)) = \lambda_G(t) \otimes \varphi(t)$ for all $t \in G$, then $\varphi_u = \varphi$.

Proof. Let $u_\lambda : C^*(G) \to M_G$ be the $*$-homomorphism taking $t \in G$ to $\lambda_G(t)$. By (1.3)

$$\|u_\lambda \otimes u : C^*(G) \otimes_{\max} C^*(G) \to M_G \otimes_{\max} M_G \otimes_{\max} A\|_{\mathrm{dec}} \leq 1.$$

Let $v = (P_G \otimes Id_A)(u_\lambda \otimes u) J_G$. Then $v(U_G(t)) = U_G(t) \otimes \varphi_u(t)$ by (2.4) and $\|v\|_{\mathrm{dec}} \leq 1$. With u_1 associated as above to the trivial representation $u_1 v(U_G(t)) = \varphi_u(t)$ and hence $\|\varphi_u\|_{D(G,A)} = \|u_1 v\|_{\mathrm{dec}} \leq 1$. If u is c.p. so is $u_1 v$ and $\varphi_u \in CP(G, A)$. The last assertion is immediate. □

Let Γ be another discrete group. Let $T \in D(C^*(G), M_\Gamma)$. Let $T(\gamma, s)$ be the associated "matrix" defined by

$$T(\gamma, s) = f_\gamma^\Gamma(T(U_G(s))) \tag{2.8}$$

and determined by the identity $T(s) = \sum_\gamma T(\gamma, s)\lambda_\Gamma(\gamma)$, where the convergence is in $L_2(\tau_\Gamma)$. Note

$$\sup_{s\in G}\left(\sum\nolimits_{\gamma\in\Gamma} |T(\gamma, s)|^2\right)^{1/2} \le \|T\|. \tag{2.9}$$

We will use the following special case of Proposition 2.10.

Lemma 2.11. *Let $v \in D(C^*(G), C^*(G))$. Let $T_v = u_\lambda v \in D(C^*(G), M_G)$ and let $T_v(t, s)$ be the associated matrix as in* (2.8). *Let $v^\bullet : G \to \mathbb{C}$ be the function defined by*

$$v^\bullet(t) = T_v(t, t).$$

Then $v^\bullet \in B(G)$ and

$$\|v^\bullet\|_{B(G)} = \|v^\bullet\|_{CB(G,\mathbb{C})} = \|v^\bullet\|_{D(G)} \le \|v\|_{\mathrm{dec}}.$$

Proof. We apply Proposition 2.10 with $A = \mathbb{C}$ and $u = u_\lambda v$. Then $\varphi_u = v^\bullet$ and $\|v^\bullet\|_{D(G)} \le \|u_\lambda v\|_{\mathrm{dec}} \le \|v\|_{\mathrm{dec}}$. The isometric identities $CB(G, \mathbb{C}) = D(G, \mathbb{C}) = B(G, \mathbb{C})$ give the rest. □

Remark 2.12. Let A be a C^*-algebra, let $v \in D(C^*(G), C^*(G) \otimes_{\max} A)$ and let $u = (u_\lambda \otimes Id_A)v$. We will again denote $v^\bullet = \varphi_u$ where $\varphi_u : G \to A$ is the function defined in Proposition 2.10. We then have $\|v^\bullet\|_{D(G,A)} \le \|v\|_{\mathrm{dec}}$. Moreover, $v^\bullet \in P(G, A)$ if v is c.p.

More generally we will use the following variant of Lemma 2.11.

Lemma 2.13. *Let Γ be another discrete group. Assume that there is a group morphism $q : \Gamma \to G$. Let $T \in D(C^*(G), C^*(\Gamma))$ such that there is a scalar matrix $[T(\gamma, s)]$ $(\gamma \in \Gamma, s \in G)$ satisfying*

$$\forall s \in G \quad \sum\nolimits_{\gamma\in\Gamma} |T(\gamma, s)| < \infty \text{ and } T(U_G(s)) = \sum\nolimits_{\gamma\in\Gamma} T(\gamma, s)U_\Gamma(\gamma).$$

Let $\Theta : C^*(G) \to C^*(\Gamma)$ be defined by

$$\Theta(U_G(s)) = \sum\nolimits_{\gamma\in\Gamma, q(\gamma)=s} T(\gamma, s)U_\Gamma(\gamma).$$

Then $\Theta \in D(C^*(G), C^*(\Gamma))$ with $\|\Theta\|_{\mathrm{dec}} \le \|T\|_{\mathrm{dec}}$. Moreover, Θ is c.p. if T is c.p.

Proof. Let $\hat{q} : C^*(\Gamma) \to C^*(G)$ denote the $*$-homomorphism associated to $q : \Gamma \to G$. Let $T_1 = J_\Gamma T : C^*(G) \to C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$, and $v = (\hat{q} \otimes Id_{C^*(\Gamma)})T_1$. Clearly $v \in D(C^*(G), C^*(G) \otimes_{\max} C^*(\Gamma))$ with $\|v\|_{\mathrm{dec}} \le \|T\|_{\mathrm{dec}}$. Let $\Psi = v^\bullet : G \to C^*(\Gamma)$ be the function defined in Remark 2.12 and let $\Theta : C^*(G) \to C^*(\Gamma)$ be the linear map associated to Ψ. Then the latter implies $\|\Theta\|_{\mathrm{dec}} = \|\Psi\|_{D(G,C^*(\Gamma))} \le \|T\|_{\mathrm{dec}}$. □

Remark 2.14. Let $J_{\max} : C^*(G) \to C^*(G) \otimes_{\max} C^*(G)$ and $P_{\max} : C^*(G) \otimes_{\max} C^*(G) \to C^*(G)$ be as before. Let $\chi_G = \Upsilon_G \otimes \Upsilon_G : C^*(G) \otimes_{\max} C^*(G) \to M_G \otimes_{\max} M_G$ with Υ_G as in (2.3). Then, recapitulating, we have

$$P_{\max} = P_G \chi_G, \quad J_G = \chi_G J_{\max} \text{ and } P_{\max} J_{\max} = P_G J_G = Id_{C^*(G)}.$$

3. Interpolation

We start by an interpolation theorem that can be viewed as a non-commutative Drury trick.

Theorem 3.1. *Let $\Lambda \subset G$ be a completely Sidon set with constant C. Let $w(\varepsilon) = C^2/\varepsilon$ for $\varepsilon > 0$. For any $0 \le \varepsilon \le 1$ there is a function $\psi_\varepsilon \in B(G)$ with $\|\psi_\varepsilon\|_{B(G)} \le w(\varepsilon)$ such that $\psi_\varepsilon(s) = 1$ for any $s \in \Lambda$ and $|\psi_\varepsilon(s)| \le C^2\varepsilon$ for any $s \notin \Lambda$.*

More generally, for any $0 \le \varepsilon \le 1$ and any function $z : \Lambda \to A$ with values in a unital C^*-algebra A with $\sup_\Lambda \|z\| < 1$ there is $\psi_{\varepsilon,z} \in D(G, A)$ with $\|\psi_{\varepsilon,z}\|_{D(G,A)} \le w(\varepsilon)$ such that

$$\psi_{\varepsilon,z|\Lambda} = z \text{ and } \sup_{G\setminus\Lambda} \|\psi_{\varepsilon,z}\|_A \le C^2\varepsilon.$$

Outline of proof. The first step is the special case when $G = \mathbb{F}_\Lambda$ for the set $\tilde{\Lambda} \subset \mathbb{F}_\Lambda$ formed of the free generators indexed by Λ (see Lemma 3.6). The second step (Lemma 3.9) establishes a strong link between the set Λ and the set $\tilde{\Lambda}$. We will then complete the proof (after Remark 3.11) by transplanting the case of $\tilde{\Lambda} \subset \mathbb{F}_\Lambda$ to that of $\Lambda \subset G$. □

Remark 3.2. Note that when $A = B(H)$, if we settle for a weaker estimate, the first part implies the second one. Indeed, let $z : \Lambda \to B(H)$ with $\sup_\Lambda \|z\| \le 1$ and let $\varphi \in CB(G, B(H))$ with $\|\varphi\|_{CB(G,B(H))} \le C$ extending z as in Proposition 2.1. Then the function $\psi_{\varepsilon,z} = \varphi\psi_\varepsilon$ satisfies $\psi_{\varepsilon,z|\Lambda} = z$, $\|\psi_{\varepsilon,z}\|_{D(G,B(H))} \le Cw(\varepsilon)$ and $\sup_{G\setminus\Lambda} \|\psi_{\varepsilon,z}\|_{B(H)} \le C^3\varepsilon$.

Using this statement, the following is immediate by well-known arguments.

Corollary 3.3. *The union of two completely Sidon sets is completely Sidon.*

Proof. Fix $0 < \varepsilon < 1$. Let Λ_1, Λ_2 be completely Sidon sets in G with respective constants C_1, C_2 and let $\Lambda = \Lambda_1 \cup \Lambda_2$. We may and do assume Λ_1, Λ_2 disjoint.

Let $z : \Lambda \to B(H)$ with $\sup_\Lambda \|z\| \le 1$. By Theorem 3.1 (recalling (1.4)) there are $\varphi_j \in CB(G, B(H))$ with $\|\varphi_j\|_{CB(G,B(H))} \le C_j^4/\varepsilon$ such that $\varphi_j = z$ on Λ_j and $\sup_{G\setminus\Lambda_j} \|\varphi_j\| \le \varepsilon$ for both $j = 1, 2$. Then $\varphi = \varphi_1 + \varphi_2$ satisfies $\|\varphi\|_{CB(G,B(H))} \le (C_1^4 + C_2^4)/\varepsilon$ and $\sup_\Lambda \|\varphi - z\| \le \varepsilon$. By Proposition 2.1 this shows that Λ is completely Sidon with constant $(C_1^4 + C_2^4)/(\varepsilon(1-\varepsilon))$. □

Remark 3.4 (Can the estimates be improved ?). Actually as the proof below shows, we can use for w any function w such that Theorem 3.1 holds when $\Lambda = \{g_t \mid t \in \Lambda\} \subset \mathbb{F}_\Lambda$. Given the spectrum of the Haagerup multiplier $h_\varepsilon(t) = \varepsilon^{|t|}$ appearing below (that generalizes Riesz products to the non-commutative case) we may apply an argument due to Méla [39, Lemme 3] for which we refer for more details to [51, Remark 1.16] that implies that Theorem 3.1 holds for a better w, namely for $w(\varepsilon) = C^2 c_1 \log(2/\varepsilon)$ for some numerical constant $c_1 > 0$ (instead of $w(\varepsilon) = C^2/\varepsilon$). In the preceding corollary, assuming $C = \max\{C_1, C_2\}$ large, this leads to $\Lambda = \Lambda_1 \cup \Lambda_2$ completely Sidon with a constant $C(\Lambda) = O(C^2 \log C)$. This same estimate has been known for Sidon sets since Méla's work. However, it seems to be still open whether there is a better estimate than $O(C^2 \log C)$. The same question arises of course for completely Sidon sets. In particular, although unlikely to be true, it seems that an estimate $C(\Lambda) = O(C)$ is not ruled out.

We will use the following variant of Haagerup's well-known theorem from [26]. This plays the role of the Riesz products used in Drury's original argument (see Remark 3.13).

Theorem 3.5. *For any $0 \le \varepsilon \le 1$ there is a function $f_\varepsilon : \mathbb{F}_\Lambda \to \mathbb{C}$ in $B(\mathbb{F}_\Lambda)$ with $\|f_\varepsilon\|_{B(\mathbb{F}_\Lambda)} \le 1/\varepsilon$ such that*

$$\forall t \in \Lambda \quad f_\varepsilon(g_t) = 1 \text{ and } \forall \gamma \notin \{g_t \mid t \in \Lambda\} \quad |f_\varepsilon(\gamma)| \le \varepsilon.$$

Proof. Haagerup's theorem produces a unital c.p. map associated to the multiplier operator for the function $h_\varepsilon : t \mapsto \varepsilon^{|t|}$. the latter is in $B(\mathbb{F}_\Lambda)$ with norm 1. For any fixed $z \in \mathbb{T}$, let $\chi_z(t) = z^{n(t)}$ ($n(t) \in \mathbb{Z}$) where $t \mapsto z^{n(t)}$ is the group morphism on $\mathbb{F}_\Lambda$ taking all the generators to z (and hence their inverses to z^{-1}). Clearly χ_z has norm 1 in $B(\mathbb{F}_\Lambda)$. Therefore the function

$$f_\varepsilon(t) = (1/\varepsilon) h_\varepsilon(t) \int \bar{z} \chi_z(t) dm(z),$$

where m is normalized Haar measure on $\mathbb{T}$, satisfies by Jensen $\|f_\varepsilon(t)\|_{B(\mathbb{F}_\Lambda)} \le 1/\varepsilon$, $f_\varepsilon(1) = f_\varepsilon(g_t^{-1}) = 0$, $f_\varepsilon(g_t) = 1$ and $|f_\varepsilon(t)| \le \varepsilon$ whenever $|t| > 1$. All the announced properties are now easy to check. □

Lemma 3.6. *The set $\tilde{\Lambda} = \{g_t \mid t \in \Lambda\} \subset \mathbb{F}_\Lambda$ satisfies the properties in Theorem 3.1 with $C = 1$.*

Proof. Let $z : \Lambda \to U(A)$. There is a unitary representation $\pi_z : \mathbb{F}_\Lambda \to U(A)$ such that $\pi_z(g_t) = z(t)$ for any $t \in \Lambda$. Let $\psi_{\varepsilon,z}(t) = f_\varepsilon(t)\pi_z(t)$ (*i.e.*, the pointwise product). Then $\psi_{\varepsilon,z}$ extends z, $\|\psi_{\varepsilon,z}(t)\| \le \varepsilon$ if $t \notin \Lambda$ and we claim that $\|\psi_{\varepsilon,z}\|_{D(\mathbb{F}_\Lambda,A)} \le 1/\varepsilon$. Indeed, let $u_{\pi_z} : C^*(\mathbb{F}_\Lambda) \to A$ be the associated $*$-homomorphism. Clearly $\|u_{\pi_z}\|_{\mathrm{dec}} = 1$ (see the proof of Proposition 3). Let $M_{f_\varepsilon} : C^*(\mathbb{F}_\Lambda) \to C^*(\mathbb{F}_\Lambda)$ be the multiplier by f_ε. Then $\|M_{f_\varepsilon}\|_{\mathrm{dec}} \le 1/\varepsilon$ (see Proposition 2.4). Therefore $u_{\pi_z}M_{f_\varepsilon} \in D(C^*(\mathbb{F}_\Lambda), C^*(\mathbb{F}_\Lambda))$ with $\|u_{\pi_z}M_{f_\varepsilon}\|_{\mathrm{dec}} \le 1/\varepsilon$. Since $u_{\pi_z}M_{f_\varepsilon}$ is the linear map associated to the function $\psi_{\varepsilon,z}$ the claim follows. This completes the proof in case z takes its values in $U(A)$. Using Remark 1.5 one easily extends this to the case when $\sup_\Lambda \|z\| < 1$. □

Let Γ be another discrete group.
Let $T_1, T_2 \in D(C^*(G), M_\Gamma)$.
Let $T_1 \sharp T_2 : \mathbb{C}[G] \to \ell_1(\Gamma)$ be defined by

$$[T_1 \sharp T_2](\delta_s) = \sum\nolimits_{\gamma\in\Gamma} T_1(\gamma, s)T_2(\gamma, s)\, e_\gamma,$$

where (δ_s) is the natural basis of $\mathbb{C}[G]$ and (e_γ) the canonical basis of $\ell_1(\Gamma)$. Note that by (2.9) the last sum is absolutely convergent. Since $\ell_1(\Gamma) \subset C^*(\Gamma)$ (in the usual way) we may view $T_1 \sharp T_2$ as a map with values in $C^*(\Gamma)$. Then we set equivalently

$$[T_1 \sharp T_2](\delta_s) = \sum\nolimits_{\gamma\in\Gamma} T_1(\gamma, s)T_2(\gamma, s)U_\Gamma(\gamma). \tag{3.1}$$

Proposition 3.7. *For any $T_1, T_2 \in D(C^*(G), M_\Gamma)$, the mapping $T_1 \sharp T_2$ extends to a decomposable map still denoted (abusively) by $T_1 \sharp T_2$ in $D(C^*(G), C^*(\Gamma))$ such that*

$$\|T_1 \sharp T_2\|_{\mathrm{dec}} \le \|T_1\|_{\mathrm{dec}}\|T_2\|_{\mathrm{dec}}. \tag{3.2}$$

Proof. Just observe

$$(T_1 \sharp T_2) = P_\Gamma(T_1 \otimes T_2)J_G : C^*(G) \to C^*(\Gamma),$$

and use (1.3). □

Remark 3.8. Assume that there is a morphism $q : \Gamma \to G$ onto G so that G is a quotient of Γ. Let $\hat{q} : C^*(\Gamma) \to C^*(G)$ be defined by

$$\hat{q}(U_\Gamma(\gamma)) = U_G(q(\gamma)).$$

Then $\hat{q}$ is a $*$-homomorphism. A fortiori it is a c.p. contractive mapping and hence $\|\hat{q}\|_{\mathrm{dec}} = 1$.

Let $T \in D(C^*(G), C^*(\Gamma))$ such that $T(U_G(s)) = \sum_{\gamma\in\Gamma} T(\gamma, s)U_\Gamma(\gamma)$ with $\sum_{\gamma\in\Gamma} |T(\gamma, s)| < \infty$ for all $s \in G$. Let $v = \hat{q}T : C^*(G) \to C^*(G)$. Note that $v(U_G(s)) = \sum_{s'\in G} \sum_{\gamma\in\Gamma, q(\gamma)=s'} T(\gamma, s)U_G(s')$, and hence

$$T_v(s', s) = \sum\nolimits_{\gamma\in\Gamma, q(\gamma)=s'} T(\gamma, s)$$

and $\|v : C^*(G) \to C^*(G)\|_{\mathrm{dec}} \le \|T : C^*(G) \to C^*(\Gamma)\|_{\mathrm{dec}}$. By Lemma 2.11 we have

$$\|v^\bullet\|_{B(G)} \le \|T\|_{\mathrm{dec}}, \tag{3.3}$$

and

$$v^\bullet(s) = \sum\nolimits_{\gamma\in\Gamma, q(\gamma)=s} T(\gamma, s). \tag{3.4}$$

This brings us to the second step of the proof of Theorem 3.1, as follows:

Lemma 3.9. *Let $\Lambda \subset G$ be a subset generating G. Let $\Gamma = \mathbb{F}_\Lambda$. Let $q : \Gamma \to G$ be the quotient morphism taking g_t to t. If $\Lambda \subset G$ is completely Sidon with constant C, there is a scalar "matrix" $T(\gamma, s)$ such that*

$$\sup\nolimits_{s\in G} \sum\nolimits_{\gamma\in\Gamma} |T(\gamma, s)| \le C^2, \tag{3.5}$$

and such that the corresponding operator $T : C^*(G) \to C^*(\Gamma)$ satisfies

$$\forall t \in \Lambda \quad T(U_G(t)) = U_\Gamma(g_t).$$

Moreover, the map $\Theta : C^*(G) \to C^*(\Gamma)$ defined by

$$\Theta(U_G(s)) = \sum\nolimits_{\gamma\in\Gamma, q(\gamma)=s} T(\gamma, s)U_\Gamma(\gamma)$$

is in $D(C^*(G), C^*(\Gamma))$ with $\|\Theta\|_{\mathrm{dec}} \le C^2$.

Proof. By Remark 1.4, there is a map $\mathcal{U} : C^*(G) \to M_\Gamma$ with $\|\mathcal{U}\|_{\mathrm{dec}} \le C$ such that $\mathcal{U}(U_G(t)) = \lambda_\Gamma(g_t)$ for all $t \in \Lambda$. Now let $T = \mathcal{U}\sharp\mathcal{U}$. Then (3.5) follows by (2.9) and (3.1). By (3.2) $\|T\|_{\mathrm{dec}} \le C^2$. The second part then follows from Lemma 2.13. □

Remark 3.10. By Remark 2.8 using $\overline{\mathcal{U}}\sharp\mathcal{U}$ we can in addition obtain $T(\gamma, s) \ge 0$ for all γ, s.

Remark 3.11. Let $\Psi : G \to C^*(\Gamma)$ be the function associated to Θ, *i.e.*

$$\forall s \in G \quad \Psi(s) = \sum\nolimits_{q(\gamma)=s} T(\gamma, s)U_\Gamma(\gamma).$$

Then $\Psi \in D(G, C^*(\Gamma))$ with $\|\Psi\|_{D(G,C^*(\Gamma))} \le C^2$ and $\Psi(t) = U_\Gamma(g_t)$ for any $t \in \Lambda$.

Proof of Theorem 3.1. We may assume w.l.o.g. that G is the group generated by Λ. We apply Lemma 3.9 and (3.5) to transplant the result of Lemma 3.6 from $\mathbb{F}_\Lambda$ to G. Recall $\Gamma = \mathbb{F}_\Lambda$. Fix $0 \le \varepsilon < 1$. Let $z : \Lambda \to A$ such that $\sup_\Lambda \|z\| < 1$. Let $z' : \tilde{\Lambda} \to A$ be the transplanted copy of z defined by $z'(g_t) = z(t)$ for any $t \in \Lambda$. Of course $\sup_{\tilde{\Lambda}} \|z'\| < 1$. By Lemma 3.6 there is $\psi'_{\varepsilon,z} : \Gamma \to A$ with $\|\psi'_{\varepsilon,z}\|_{D(\Gamma,A)} \le 1/\varepsilon$ extending z' and such that $\|\psi'_{\varepsilon,z}(\gamma)\| \le \varepsilon$ if $\gamma \notin \tilde{\Lambda}$. Let $u_{\varepsilon,z} : C^*(\Gamma) \to A$ be the linear map associated to $\psi'_{\varepsilon,z}$ (*i.e.*, $u_{\varepsilon,z}$ is $u_{\psi'_{\varepsilon,z}}$ is the

sense of (2.1)). Let Ψ be associated to $\Theta : C^*(G) \to C^*(\Gamma)$ as in Remark 3.11 so that $\|\Psi\|_{D(G,C^*(\Gamma))} = \|\Theta\|_{D(C^*(G),C^*(\Gamma))}$. We then set

$$\psi_{\varepsilon,z} = u_{\varepsilon,z}(\Psi), \tag{3.6}$$

so that $u_{\varepsilon,z}\Theta$ is the linear map associated to $\psi_{\varepsilon,z}$. Thus

$$\begin{aligned}\|\psi_{\varepsilon,z}\|_{D(G,A)} &\le \|\Theta\|_{D(C^*(G),C^*(\Gamma))}\|u_{\varepsilon,z}\|_{D(C^*(\Gamma),A)}\\ &= \|\Psi\|_{D(G,C^*(\Gamma))}\|\psi'_{\varepsilon,z}\|_{D(\Gamma,A)} \le C^2/\varepsilon.\end{aligned}$$

Equivalently (3.6) means that for any $s \in G$ we have

$$\psi_{\varepsilon,z}(s) = \sum\nolimits_{q(\gamma)=s} T(\gamma, s)\psi'_{\varepsilon,z}(\gamma).$$

Observe that if $s \notin \Lambda$ and $q(\gamma) = s$ then necessarily $\gamma \notin \{g_t \mid t \in \Lambda\}$ and hence (3.5) gives us $\|\psi_{\varepsilon,z}(s)\| \le C^2\varepsilon$. Moreover for any $t \in \Lambda$ we have $\psi_{\varepsilon,z}(t) = u_{\varepsilon,z}(\Psi(t)) = \psi'_{\varepsilon,z}(g_t) = z'(g_t) = z(t)$. So the second (and more general) part of Theorem 3.1 follows. □

Remark 3.12. Let $|s|_\Lambda$ denote the length of an element $s \in G$ with respect to the generating set Λ, *i.e.*, $|s|_\Lambda = \inf\{|t| \mid t \in \mathbb{F}_\Lambda,\ q(t) = s\}$. In the preceding proof we find

$$|\psi_\varepsilon(s)| \le C^2\varepsilon^{|s|_\Lambda - 1} \text{ and } \|\psi_{\varepsilon,z}(s)\| \le C^2\varepsilon^{|s|_\Lambda - 1}.$$

Remark 3.13. If one replaces the free group by the free Abelian group $\Gamma^a = \mathbb{Z}^{(\Lambda)}$ the proof becomes quite similar to Drury's original one, but reformulated in operator theoretic terms. The group Γ^a is generated by generators $(g_t^a)_{t\in\Lambda}$ that are free except that they mutually commute. In this case M_{Γ^a} is an injective von Neumann algebra. Thus we have a mapping $v \in D(C^*(G), M_{\Gamma^a})$ as in Corollary 2.9 where now the g_t's are replaced by the generators g_t^a of Γ^a. When the group G is Abelian we again have a quotient map $q^a : \Gamma^a \to G$ such that $q^a(g_t^a) = t$ for all $t \in \Lambda$. The analogue of f_ε is then the Fourier transform of a probability measure on the compact group $\hat{G} = \mathbb{T}^\Lambda$, namely the Riesz product $\prod_{t\in\Lambda}(1 + \varepsilon(z_t + \bar{z}_t))$ where $z_t : \mathbb{T}^\Lambda \to \mathbb{T}$ is the t-th coordinate. This is defined only for $|\varepsilon| \le 1/2$ but one can use equally well whenever $|\varepsilon| \le 1$ the Riesz product based on the Poisson kernel:

$$\prod\nolimits_{t\in\Lambda}\left(\sum\nolimits_{n\in\mathbb{Z}} \varepsilon^{|n|} z_t^n\right).$$

Its Fourier transform is the exact analogue of f_ε on Γ^a.

See [24, Chapter 7] and [32, Chapter V] for more on Riesz products and their generalizations.

See [8,10,15,22] for generalizations of Haagerup's result (concerning the function h_ε) to free products of groups and [4] for free products of c.p. maps on C^*-algebras.

4. The Fatou-Zygmund property

We now turn to the Fatou-Zygmund (FZ in short) property. Recall $P(G)$ is the set of positive definite complex valued functions on G. The multiplier operator M_f associated to a function $f \in B(G)$ is c.p. on $C^*(G)$ iff $f \in P(G)$ and we have $\|M_f\| = \|M_f\|_{\mathrm{dec}} = f(1)$ for any $f \in P(G)$.

Theorem 4.1. *Let $\Lambda \subset G \setminus \{1\}$ be a symmetric completely Sidon set. Any bounded Hermitian function $\varphi : \Lambda \to \mathbb{C}$ admits an extension $\tilde{\varphi} \in P(G)$. More generally, there is a constant C' such that for any unital C^*-algebra A, any bounded Hermitian function $\varphi : \Lambda \to A$ admits an extension $\tilde{\varphi} \in CP(G, A)$ satisfying*

$$\|\tilde{\varphi}(1)\| \leq C' \sup_{t\in\Lambda} \|\varphi(t)\|,$$

and moreover $\tilde{\varphi}(1) = 1_A\|\tilde{\varphi}(1)\|$.

The structure of the proof follows Drury's idea in [17], but we again use decomposable maps as above, and harmonic analysis on the free group instead of the free Abelian one.

The key Lemma is parallel to the one in [17]. It is convenient to formulate it directly for positive definite functions with values in a unital C^*-algebra A.

Lemma 4.2 (Key Lemma). *Let $\Lambda \subset G \setminus \{1\}$ be a symmetric completely Sidon set with constant C. Let A be a unital C^*-algebra. Let $\varphi : \Lambda \to A$ be a Hermitian function (*i.e., *we assume $\varphi(t^{-1}) = \varphi(t)^*$ for any $t \in \Lambda$) with $\sup_\Lambda \|\varphi\| < 1$. For any $0 \leq \varepsilon \leq 1$ there is $\Phi_\varepsilon \in P(G, A)$ with*

$$\|\Phi_\varepsilon\|_{CB(G,A)} = \|\Phi_\varepsilon(1)\| \leq 4C^2 \text{ and } \sup_{s\in\Lambda} \|\Phi_\varepsilon(s) - \varepsilon\varphi(s)\| \leq 4C^2\varepsilon^2.$$

Proof. For simplicity we give the proof assuming that Λ does not contain elements such that $t = t^{-1}$. Let $\Lambda_1 \subset \Lambda$ be such that Λ is the disjoint union of Λ_1 and $\Lambda_1^{-1} = \{t^{-1} \mid t \in \Lambda_1\}$. We will work with the free group $\Gamma = \mathbb{F}_{\Lambda_1}$ instead of $\mathbb{F}_\Lambda$. As before we set $q(g_t) = t$ for all $t \in \Lambda_1$.

Then we consider the self-adjoint operator space E spanned by $\{U_G(t) \mid t \in \Lambda\}$. Let $u : E \to M_\Gamma$ be the linear mapping defined by $u(U_G(t)) = \lambda_\Gamma(g_t)$ and $u(U_G(t^{-1})) = \lambda_\Gamma(g_t)^{-1}$ for $t \in \Lambda_1$. Note that u is self-adjoint in the sense that $u = u_*$ where $u_*(x) = u(x^*)^*$ for all $x \in E$. By Remark 1.4, since Λ is completely Sidon with constant C, u is the restriction to E of a mapping $T \in D(C^*(G), M_\Gamma)$ with $\|T\|_{\mathrm{dec}} \leq C$. Replacing T by $1/2(T + T_*)$ we may assume that T is self-adjoint. Then (see [27]) we have a decomposition $T = T^+ - T^-$ where $T^\pm \in CP(C^*(G), M_\Gamma)$ with

$$\max\{\|T^+\|, \|T^-\|\} \leq \|T^+ + T^-\| \leq \|T\|_{\mathrm{dec}}. \tag{4.1}$$

We have

$$T \sharp T = a - b$$

with

$$a = T^+ \sharp T^+ + T^- \sharp T^- \text{ and } b = T^+ \sharp T^- + T^- \sharp T^+.$$

Note that $a, b \in CP(C^*(G), C^*(\Gamma))$.

Fix $0 \le \varepsilon \le 1$. Let $h_\varepsilon : C^*(\Gamma) \to C^*(\Gamma)$ be as before the Haagerup c.p. multiplier defined on $\mathbb{F}_\Lambda$ by (see [26]) $h_\varepsilon(t) = \varepsilon^{|t|}$. Note that both h_ε and $h_{-\varepsilon}$ are in $P(\mathbb{F}_\Lambda)$ (indeed, $h_{-\varepsilon}(t) = h_\varepsilon(t)\chi_{-1}(t)$).

The function φ' defined on the words of length 1 by $\varphi'(g_t^\pm) = \varphi(t^\pm)$ is Hermitian. By Haagerup's [26] and the operator valued version in [9] (see Remark 1.3), there is a positive definite function $f \in P(\Gamma, A)$ extending φ' such that $f(1) = 1$ and $f(g_t) = \varphi(t)$ (and $f(g_t^{-1}) = \varphi(t^{-1})$) for all $t \in \Lambda_1$. Indeed, this is precisely the FZ-property of the free group $\Gamma = \mathbb{F}_{\Lambda_1}$. (See [4] for a generalization of this to c.p. maps on free products.) Let $M_f : C^*(\Gamma) \to C^*(\Gamma) \otimes_{\max} A$ be the associated "multiplier" taking $U_\Gamma(t)$ to $U_\Gamma(t) \otimes f(t)$. Clearly $M_f \in CP(C^*(\Gamma), C^*(\Gamma) \otimes_{\max} A)$ and $\|M_f\| = \|M_f(1)\| = 1$.

We now introduce for any $0 \le \varepsilon \le 1$

$$Y_\varepsilon = (\hat{q} M_{h_\varepsilon} \otimes Id_A) M_f a + (\hat{q} M_{h_{-\varepsilon}} \otimes Id_A) M_f b.$$

Clearly $Y_\varepsilon \in CP(C^*(G), C^*(G) \otimes_{\max} A)$. Let $\Phi_\varepsilon = Y_\varepsilon^\bullet$ in the sense of Remark 2.12. Since Y_ε is c.p. we know that $\Phi_\varepsilon \in CP(G, A)$. Moreover, by (4.1)

$$\begin{aligned}\|\Phi_\varepsilon\|_{CB(G,A)} = \|\Phi_\varepsilon(1)\| &\le \|a\| + \|b\| \le \|T^+\|^2 + \|T^-\|^2 + 2\|T^+\|\|T^-\| \\ &\le 4\|T\|_{\mathrm{dec}}^2 \le 4C^2.\end{aligned}$$

We now compute $\Phi_\varepsilon(s)$ for $s \in \Lambda$. We have

$$\begin{aligned}\Phi_\varepsilon(s) = &\sum\nolimits_{\gamma \in \Gamma, q(\gamma) = s} h_\varepsilon(\gamma) f(\gamma)(T^+(\gamma, s)^2 + T^-(\gamma, s)^2) \\ &+ h_{-\varepsilon}(\gamma) f(\gamma)(2T^+(\gamma, s)T^-(\gamma, s)).\end{aligned}$$

We can write (recall $s \neq 1$ and hence $q(\gamma) = s$ implies $|\gamma| \ge 1$)

$$\Phi_\varepsilon(s) = I(s) + E(s),$$

where

$$\begin{aligned}I(s) = &\sum\nolimits_{\gamma \in \Gamma, q(\gamma) = s, |\gamma| = 1} h_\varepsilon(\gamma) f(\gamma)(T^+(\gamma, s)^2 + T^-(\gamma, s)^2) \\ &+ h_{-\varepsilon}(\gamma) f(\gamma)(2T^+(\gamma, s)T^-(\gamma, s)),\end{aligned}$$

and the "error term" $E(s)$ is

$$\begin{aligned}E(s) = &\sum\nolimits_{\gamma \in \Gamma, q(\gamma) = s, |\gamma| > 1} h_\varepsilon(\gamma) f(\gamma)(T^+(\gamma, s)^2 + T^-(\gamma, s)^2) \\ &+ h_{-\varepsilon}(\gamma) f(\gamma)(2T^+(\gamma, s)T^-(\gamma, s)).\end{aligned}$$

Fix $s \in \Lambda_1$. If $|\gamma| = 1$ and $q(\gamma) = s$ we must have $\gamma = g_s$, $h_\varepsilon(\gamma) = \varepsilon$ and $h_{-\varepsilon}(\gamma) = -\varepsilon$ and $f(\gamma) = \varphi(q(\gamma)) = \varphi(s)$ so we recover

$$\begin{aligned} I(s) &= \varepsilon\varphi(s)[(T^+(g_s, s)^2 + T^-(g_s, s)^2) - (2T^+(g_s, s)T^-(g_s, s))] \\ &= \varepsilon\varphi(s)T(g_s, s)^2, \end{aligned}$$

and since $T(\gamma, s) = 1_{\gamma = g_s}$ we obtain for $s \in \Lambda_1$

$$I(s) = \varepsilon\varphi(s).$$

Similarly, $I(s^{-1}) = \varepsilon\varphi(s^{-1}) = \varepsilon\varphi(s)^*$.

It remains to estimate the error: Note that if $|\gamma| > 1$ we have $|h_{\pm\varepsilon}(\gamma)| \leq \varepsilon^2$ and hence by (2.9)

$$\begin{aligned} \|E(s)\| &\leq \varepsilon^2 \sum\nolimits_{\gamma\in\Gamma} |T^+(\gamma, s)^2 + T^-(\gamma, s)^2| + 2|T^+(\gamma, s)T^-(\gamma, s)| \\ &\leq \varepsilon^2 \sum\nolimits_{\gamma\in\Gamma} (|T^+(\gamma, s)| + |T^-(\gamma, s)|)^2 \\ &\leq \varepsilon^2(\|T^+\| + \|T^-\|)^2 \leq 4\varepsilon^2\|T\|_{\mathrm{dec}}^2 \leq 4\varepsilon^2 C^2. \end{aligned}$$

This completes the proof of the lemma, assuming Λ has no element of order 2. Otherwise let $\Lambda_2 \subset \Lambda$ be the set of such elements. We then replace $\mathbb{F}_{\Lambda_1}$ with $\Gamma = \mathbb{F}_{\Lambda_1} * (*_{t\in\Lambda_2}\mathbb{Z}_2)$. We leave the details to the reader. □

Remark 4.3. Let $\varphi_0 : G \to \mathbb{C}$ be such that $\varphi_0(t) = 1$ if $t = 1$ (unit of G) and $\varphi_0(t) = 0$ otherwise. Clearly $\varphi_0 \in P(G)$ (indeed $\varphi_0(t) = \langle\delta_1, \lambda_G(t)\delta_1\rangle$). Let $\varphi \in CP(G, A)$. Then $\varphi(1) \in A_+$ and hence $0 \leq \varphi(1) \leq \|\varphi(1)\|1_A$. Let $\psi(t) = \varphi(t) + \varphi_0(t)(\|\varphi(1)\|1_A - \varphi(1))$. Then $\psi \in CP(G, A)$, $\psi(1) = \|\varphi(1)\|1_A$ and $\psi(t) = \varphi(t)$ for all $t \neq 1$. Equivalently, if we are given $V \in CP(C^*(G), A)$ then there is $W \in CP(C^*(G), A)$ such that $W(1) = \|V(1)\|1_A$ and $W(U_G(t)) = V(U_G(t))$ for all $t \neq 1$.

Proof of Theorem 4.1. The theorem follows from the key Lemma 4.2 by a routine iteration argument (note that $\Phi_\varepsilon - \varepsilon\varphi$ is Hermitian), exactly as in [17]. For the last assertion we use Remark 4.3. □

The proof gives an estimate of the form $C' \leq cC^4$ where C is the completely Sidon constant and c a numerical constant, to be compared with Remark 3.4.

Corollary 4.4. *Assume for simplicity that $\Lambda \subset G \setminus \{1\}$ is symmetric, and is the disjoint union of Λ_1 and Λ_1^{-1} as before (in particular it has no element of order 2). Let $E_\Lambda \subset C^*(G)$ be the operator system generated by Λ and $\{1\}$. The following are equivalent:*

(i) *Λ is completely Sidon;*

(ii) *There is a completely positive linear map* $V : C^*(G) \to C^*(\mathbb{F}_{\Lambda_1})$ *such that*

$$\forall t \in \Lambda_1 \quad V(U_G(t)) = U_{\mathbb{F}_{\Lambda_1}}(g_t) \quad V(U_G(t^{-1})) = U_{\mathbb{F}_{\Lambda_1}}(g_t^{-1});$$

(iii) *There is* $\delta > 0$ *such that the (unital) mapping* $S_\delta : E_\Lambda \to C^*(\mathbb{F}_{\Lambda_1})$ *defined by*

$$S_\delta(1) = 1 \text{ and } \forall t \in \Lambda_1 \; S_\delta(U_G(t)) = \delta U_{\mathbb{F}_{\Lambda_1}}(g_t) \; S_\delta(U_G(t^{-1})) = \delta U_{\mathbb{F}_{\Lambda_1}}(g_t^{-1}),$$

is c.p.;

(iv) *There is* $\beta > 0$ *such that* S_β *admits a c.p. extension* $\tilde{S}_\beta : C^*(G) \to C^*(\mathbb{F}_{\Lambda_1})$.

Moreover, the relationships between the Sidon constant and δ *are* $C \le 1/\delta \le cC^4$, *and* $\beta \ge \delta^2$.

Proof. Assume (i). Let $A = C^*(\mathbb{F}_{\Lambda_1})$. Define $\varphi : \Lambda \to A$ by $\varphi(t) = g_t, \varphi(t^{-1}) = g_t^{-1}$ for $t \in \Lambda_1$. By Theorem 4.1 there is a c.p. mapping $V : C^*(G) \to A$ extending $U_G(t) \mapsto \varphi(t)$. This proves (i) $\Rightarrow$ (ii). Assume (ii). Let $\delta = \|V(1)\|^{-1}$. By Remark 4.3 there is $W \in CP(C^*(G), C^*(\Gamma))$ such that $W(1) = (1/\delta)1$ and $\forall t \in \Lambda_1 \quad W(U_G(t)) = U_{\mathbb{F}_{\Lambda_1}}(g_t) \quad W(U_G(t^{-1})) = U_{\mathbb{F}_{\Lambda_1}}(g_t^{-1})$. Then the restriction S_δ of δW to E_Λ satisfies (iii).

Assume (iii) or (iv). Then (i) follows because $\|S_\delta\|_{\mathrm{cb}} = 1$. Also (iv) trivially implies (iii).

Assume (iii). Let $\Gamma = \mathbb{F}_{\Lambda_1}$. By Remark 1.4 S_δ extends to a c.p. map $\mathcal{U} : C^*(G) \to M_\Gamma$. Now consider $S = \mathcal{U}\sharp\mathcal{U}$. Then S is c.p. and extends S_{δ^2}. Thus (iii) implies (iv).

The relationships between the constants can be traced back easily from the proof. □

Remark 4.5. All the preceding can be developed in parallel for the free Abelian group. The last statement gives an apparently new fact (or rather, say, a new reformulation of the FZ property) in the commutative case. We state it for emphasis because it seems interesting. Let G be a discrete commutative group. Assume for simplicity that $\Lambda \subset G \setminus \{0\}$ has no element of order 2 and is the (symmetric) disjoint union of Λ_1 and Λ_1^{-1} as before. Let Γ_1 be the free Abelian group $\mathbb{Z}^{(\Lambda_1)}$. Note $C^*(\Gamma_1) \simeq C(\mathbb{T}^{\Lambda_1})$. Then Λ is Sidon iff there is $\delta > 0$ such that the mapping

$$S_\delta : E_\Lambda \to C^*(\Gamma_1) \simeq C(\mathbb{T}^{\Lambda_1})$$

defined as above but with $\mathbb{Z}^{(\Lambda_1)}$ in place of $\mathbb{F}_{\Lambda_1}$ is positive. Note that in the commutative case positive implies c.p.

5. Characterizations by operator space properties

Let $\Lambda \subset G$ be a subset and let $C_\Lambda \subset C^*(G)$ be its closed linear span. In the classical setting, when G is a commutative discrete group, Varopoulos [56] proved

that Λ is Sidon as soon as C_Λ is isomorphic to $\ell_1(\Lambda)$ as a Banach space (via an arbitrary isomorphism). Shortly after that, the author and independently Kwapień and Pełczyński proved that it suffices to assume that C_Λ is of cotype 2. This was refined by Bourgain and Milman [3] who showed that Λ is Sidon if (and only if) C_Λ is of finite cotype. It is natural to try to prove analogues of these results for a general discrete group G. The next statement shows that if C_Λ is *completely* isomorphic to $\ell_1(\Lambda)$ (equipped with its maximal operator space structure) then Λ is completely Sidon. Indeed, the dual operator space C_Λ^* is then completely isomorphic to $\ell_\infty(\Lambda)$, and the latter is exact with constant 1.

We recall that an operator space (o.s. in short) $X \subset B(H)$ is called exact if there is a constant C such that for any finite dimensional subspace $E \subset X$ there is an integer N, a subspace $\tilde{E} \subset M_N$ and an isomorphism $u : E \to \tilde{E}$ such that $\|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}} \le C$. The smallest constant C for which this holds is denoted by $\mathrm{ex}(X)$.

The dual o.s. of an o.s. $X \subset B(H)$ is characterized by the existence of an isometric embedding $X^* \subset B(H)$ such that the natural norms on the spaces $M_n(X^*)$ and $CB(X, M_n)$ coincide. See [47, Section 2.3] for more on this.

Theorem 5.1. *If C_Λ^* is an exact operator space, then Λ is completely Sidon with constant* $4\mathrm{ex}(C_\Lambda^*)^2$. *Conversely, if Λ is completely Sidon with constant C then then* $\mathrm{ex}(C_\Lambda^*) \le C$.

Proof. The converse part is clear because $\ell_\infty(\Lambda) = \ell_1(\Lambda)^*$ is exact with $\mathrm{ex}(\ell_\infty(\Lambda)) = 1$.

Assume that C_Λ^* is exact. Let $\alpha \subset \Lambda$ be a finite subset. Consider the mapping $T_0 : C_\Lambda \to C_\lambda^*(\mathbb{F}_\Lambda)$ defined by $T_0(t) = \lambda_{\mathbb{F}_\Lambda}(g_t)$ for $t \in \alpha$ and $T_0(t) = 0$ for $t \notin \alpha$. Let us denote by $\varphi_t \in (C^*(G))^*$ the functional biorthogonal to the natural system, *i.e.*, $\varphi_t(U_G(s)) = \delta_t(s)$.

Let $a : G \to M_N$ be a finitely supported M_N-valued function ($N \ge 1$). We have then by elementary arguments

$$\begin{aligned}\left\|\sum a(t) \otimes U_G(t)\right\| &\ge \left\|\sum a(t) \otimes \lambda_G(t)\right\| \\ &\ge \max\left\{\left\|\sum a(t)^* a(t)\right\|^{1/2}, \left\|\sum a(t) a(t)^*\right\|^{1/2}\right\}. \end{aligned} \tag{5.1}$$

By a well-known inequality with roots in Haagerup's [26] (see [47, page 188]) (5.1) implies

$$\left\|\sum a(t) \otimes U_G(t)\right\| \ge (1/2)\left\|\sum a(t) \otimes \lambda_{\mathbb{F}_\Lambda}(g_t)\right\| \tag{5.2}$$

and hence $\|T_0\|_{\mathrm{cb}} \le 2$. Equivalently this means that the tensor

$$\mathbf{T}_0 = \sum\nolimits_{t\in\alpha} \varphi_t \otimes \lambda_{\mathbb{F}_\Lambda}(g_t) \in (C_\Lambda)^* \otimes C_\lambda^*(\mathbb{F}_\Lambda)$$

satisfies

$$\|\mathbf{T}_0\|_{\min} = \|T_0 : C_\Lambda \to C_\lambda^*(\mathbb{F}_\Lambda)\|_{\mathrm{cb}} \le 2. \tag{5.3}$$

Let $\varepsilon > 0$. Assume $|\alpha| = n$ and $\alpha = \{t(1), \cdots, t(n)\}$. Let $\Gamma \subset \mathbb{F}_\Lambda$ be the copy of $\mathbb{F}_n$ generated by $\{g_{t(j)} \mid 1 \le j \le n\}$. We claim that T_0 extends to an operator $\tilde{T} : C^*(G) \to M_\Gamma$ such that $\|\tilde{T}\|_{\mathrm{dec}} \le 2\mathrm{ex}(X)(1+\varepsilon)$.

By a result due to Thorbjørnsen and Haagerup [30] (see [47, page 331]) recently refined in [14] we have (here we denote by (g_j) the free generators of $\mathbb{F}_n$):

For any n and N there is an n-tuple of $N \times N$-unitary matrices $(u_j^{(N)})_{1\le j\le n}$ such that for any exact operator space X and any $x_j \in X$ we have

$$\lim_{N\to\infty} \left\| \sum u_j^{(N)} \otimes x_j \right\|_{M_N(X)} \le \mathrm{ex}(X) \left\| \sum \lambda_{\mathbb{F}_n}(g_j) \otimes x_j \right\|_{\min}, \tag{5.4}$$

and

$$\left\{u_j^{(N)} \mid 1 \le j \le n\right\} \textit{ converges in moments to } \left\{\lambda_{\mathbb{F}_n}(g_j) \mid 1 \le j \le n\right\}. \tag{5.5}$$

Let $X = C_\Lambda^*$. This gives us by (5.3)

$$\lim_{N\to\infty} \left\| \sum u_j^{(N)} \otimes \varphi_{t(j)} \right\|_{M_N(X)} \le 2\mathrm{ex}(X).$$

For some n_0 we have

$$\sup_{N\ge n_0} \left\| \sum u_j^{(N)} \otimes \varphi_{t(j)} \right\|_{M_N(X)} \le 2\mathrm{ex}(X)(1+\varepsilon).$$

This gives us a map $T_1 : C_\Lambda \to (\oplus \sum_{N\ge n_0} M_N)_\infty$ with $\|T_1\|_{\mathrm{cb}} \le 2\mathrm{ex}(X)(1+\varepsilon)$, such that $T_1(U_G(t(j))) = \oplus_{N\ge n_0} u_j^{(N)}$. Let ω be a nontrivial ultrafilter on $\mathbb{N}$. By (5.5), we have an isometric embedding $M_\Gamma \subset (\oplus \sum_{N\ge 1} M_N)_\infty/\omega$ and a surjective unital c.p. map $Q_\omega : (\oplus \sum_{N\ge 1} M_N)_\infty \to M_\Gamma$, such that

$$Q\left(\oplus_{N\ge n_0} u_j^{(N)}\right) = \lambda_\Gamma(g_j).$$

Since $(\oplus \sum_{N\ge 1} M_N)_\infty$ is injective there is an extension of T_1 denoted $\tilde{T}_1 : C^*(G) \to (\oplus \sum_{N\ge 1} M_N)_\infty$ such that $\|\tilde{T}_1\|_{\mathrm{dec}} = \|\tilde{T}_1\|_{\mathrm{cb}} \le \|T_1\|_{\mathrm{cb}} \le 2\mathrm{ex}(X)(1+\varepsilon)$, and hence setting $\tilde{T} = Q\tilde{T}_1$, we obtain the claim.

Then we conclude by Corollary 2.9. □

Corollary 5.2. *Let $\Lambda \subset G$. The operator space $C_\Lambda \subset C^*(G)$ is completely isomorphic to $\ell_1(\Lambda)$ (with its maximal o.s. structure) iff Λ is completely Sidon.*

Remark 5.3. By the same argument, we can replace the exactness assumption of Theorem 5.1 by the subexponentiality (or tameness) in the sense of [49].

Remark 5.4. By the same argument, the following can be proved. Let $\{x_j\} \subset A$ be a bounded sequence in a C^*-algebra A. Assume that for some constant c, for any N and any sequence (a_j) in M_N with only finitely many nonzero terms we have

$$c\left\|\sum a_j \otimes x_j\right\| \geq \max\left\{\left\|\sum a_j^* a_j\right\|^{1/2}, \left\|\sum a_j a_j^*\right\|^{1/2}\right\}.$$

Let E be the closed span of $\{x_j\}$. If E^* is exact then $\{x_j \otimes x_j\}$ is completely Sidon in $A \otimes_{\max} A$ (with constant $4c^2\mathrm{ex}(E^*)^2$). See [52] for more on that theme.

Remark 5.5.

(i) Let us first observe that the Varopoulos result mentioned above remains valid for a non-commutative group G. We will show that if C_Λ is isomorphic to $\ell_1(\Lambda)$, then the usual linear mapping taking the canonical basis of $\ell_1(\Lambda)$, namely $(\delta_t)_{t\in\Lambda}$, to $(U_G(t))_{t\in\Lambda}$ is an isomorphism. Actually it suffices to assume that $C_\Lambda^* \simeq \ell_\infty(\Lambda)$ as a Banach space or that, say, C_Λ^* is a $\mathcal{L}_\infty$-space, or that $(C_\Lambda^*, C_\Lambda^*)$ is a GT-pair in the sense of [48, Def. 6.1], to which we refer for all unexplained terminology in the sequel.
With the preceding notation, let $W_x : C_\Lambda^* \to C_\Lambda$ be the linear operator associated to the tensor $x = \sum_{t\in\alpha} x(t)U_G(t) \otimes U_G(t) \in C_\Lambda \otimes C_\Lambda$. Let $\|\ \|_\vee$ be the norm in the injective tensor product (in the usual Banach space sense) of $C^*(G)$ with itself. Note

$$\|W_x\| = \|x\|_\vee \leq \|x\|_{\min} = \left\|\sum\nolimits_{t\in\alpha} x(t)U_G(t)\right\|_{C^*(G)}.$$

Let $(z(t)) \in \mathbb{T}^\Lambda$. Let $T_z : C_\Lambda \to C_\Lambda^*$ be the linear operator associated to the tensor $\sum_{t\in\alpha} z(t)\varphi_t \otimes \varphi_t \in C_\Lambda^* \otimes C_\Lambda^*$. A simple verification shows that, denoting by $\gamma_2(T_z)$ the norm of factorization through Hilbert space of T_z, we have $\gamma_2(T_z) \leq 1$.
Then Grothendieck's Theorem, or our Banach space assumption (see [48, §6]), implies that for any finite rank map $w : C_\Lambda^* \to C_\Lambda$ we have $|\mathrm{tr}(wT_z)| \leq K\gamma_2(T_z)\|w\|_\vee \leq K\|w\|$, where K is a constant independent of w, z. Therefore, we have

$$\left|\sum\nolimits_{t\in\alpha} x(t)z(t)\right| = |\mathrm{tr}(W_x T_z)| \leq K\|x\|_{C^*(G)},$$

and hence taking the sup over all z's and α's

$$\sum\nolimits_{t\in\Lambda} |x(t)| \leq K\|x\|_{C^*(G)}.$$

Thus we conclude that C_Λ is isomorphic to $\ell_1(\Lambda)$ by the usual (basis to basis) isomorphism. Such sets are called weak Sidon in [44], where the term Sidon is reserved for the sets that span $\ell_1(\Lambda)$ in the reduced C^*-algebra $C_\lambda^*(G)$.

(ii) Let C_Λ^λ be the closed span of Λ in $C_\lambda^*(G)$, *i.e.*, $C_\Lambda^\lambda = \overline{\operatorname{span}}\{\lambda_G(t) \mid t \in \Lambda\}$. The preceding argument applies equally well to C_Λ^λ, and shows that if C_Λ^λ is isomorphic to $\ell_1(\Lambda)$ (by an arbitrary isomorphism) then it actually is so by the usual isomorphism, and Λ is Sidon in the sense of [44].

(iii) Lastly, we apply the same idea to slightly generalize Theorem 5.1. Fix $N \geq 1$. Let $z = (z(t)) \in U(N)^\Lambda$ and $x = (x(t)) \in M_N^\Lambda$. Consider the tensors

$$\mathbf{T_z} = \sum\nolimits_{t\in\alpha} \varphi_t \otimes [z(t) \otimes \varphi_t] \in C_\Lambda^* \otimes M_N(C_\Lambda^*),$$

and

$$\mathbf{W_x} = \sum\nolimits_{t\in\alpha} x(t) \otimes U_G(t) \otimes U_G(t) \in M_N(C_\Lambda \otimes C_\Lambda).$$

Then it can be checked on the one hand that

$$\max\left\{\|\mathbf{T_z}\|_{C_\Lambda^* \otimes_h M_N(C_\Lambda^*)}, \|{}^{\mathbf{t}}\mathbf{T_z}\|_{M_N(C_\Lambda^*)\otimes_h C_\Lambda^*}\right\} \leq 1.$$

Thus if the pair $(C_\Lambda^*, M_N(C_\Lambda^*))$ satisfies (uniformly over N) the o.s. version of Grothendieck's theorem described in [48, Proposition 18.2] we find for some constant K (independent of N)

$$\|\mathbf{T_z}\|_{C_\Lambda^* \otimes_\wedge M_N(C_\Lambda^*)} \leq K.$$

Here $\otimes_\wedge$ is the projective tensor product in the operator space sense. A fortiori, this implies

$$\|\mathbf{T_z}\|_{M_N(C_\Lambda^* \otimes_\wedge C_\Lambda^*)} \leq K.$$

On the other hand, we have obviously

$$\|\mathbf{W_x}\|_{M_N(C_\Lambda \otimes_{\min} C_\Lambda)} \leq \left\|\sum\nolimits_{t\in\alpha} x(t) \otimes U_G(t)\right\|_{M_N(C_\Lambda)}.$$

Thus we obtain

$$\begin{aligned}\left\|\sum z(t) \otimes x(t)\right\| &\leq \|\mathbf{T_z}\|_{M_N(C_\Lambda^* \otimes_\wedge C_\Lambda^*)} \|\mathbf{W_x}\|_{M_N(C_\Lambda \otimes_{\min} C_\Lambda)} \\ &\leq K \left\|\sum\nolimits_{t\in\alpha} x(t) \otimes U_G(t)\right\|_{M_N(C_\Lambda)}.\end{aligned}$$

The latter implies that Λ is completely Sidon.

6. Remarks and open questions

6.1. Free sets

We start by the characterization of the case $C = 1$ announced in Proposition 3.

Proposition 6.1. *The following properties of a subset $\Lambda \subset G$ are equivalent:*

(i) *Λ is completely Sidon with a constant $C = 1$;*

(ii) *For any finite subset* $S \subset \Lambda$ *we have* $\| \sum_{s\in S} \lambda_G(s)\| = 2\sqrt{|S|-1}$;
(iii) Λ *is a (left say) translate of a free set enlarged by including the unit;*
(iv) *For every* m *and every* $2m$*-tuple* $t_1, t_2, t_3, \cdots, t_{2m-1}, t_{2m}$ *in* Λ *with* $t_1 \neq t_2 \neq \cdots t_{2m-1} \neq t_{2m}$ *we have* $t_1^{-1}t_2t_3^{-1}t_4 \cdots t_{2m-1}^{-1}t_{2m} \neq 1$.

Proof. We start by (iii) $\Rightarrow$ (i). Assume (iii). Since translation has no significant effect, it suffices to prove (i) for $\Lambda = S \cup \{1\}$ with S free. We may assume that S generates G. Let $z : \Lambda \to U(A)$ such that $z(1) = 1$. By the freeness of S there is a unitary representation $\pi : G \to A$ extending z. By Remark 1 Λ is completely Sidon set with $C = 1$.

Conversely, let us show (i) $\Rightarrow$ (iii).
Assume (i). Pick and fix an element $s \in \Lambda$. We may assume after (left say) translation by s^{-1} that $1 \in \Lambda$. Then the correspondence $t \mapsto g_t$ $(t \neq s)$ extends to a unital completely contractive map from the span of Λ in $C^*(G)$ to that of $\{1\} \cup \{g_t \mid t \in \Lambda \setminus \{s\}\}$ in $C^*(\mathbb{F}_\Lambda)$. By [46, Prop. 6] the latter mapping is the restriction of a unital $*$-homomorphism from $C^*(G)$ to $C^*(\mathbb{F}_\Lambda)$, which (by the maximality of $C^*(\mathbb{F}_\Lambda)$) must be a $*$-isomorphism. Translating back by s yields (iii).
(iii) $\Leftrightarrow$ (iv) is due to Akemann-Ostrand [1, Def. III.B and Th. III.D], as well as (iii) $\Rightarrow$ (ii) and the converse is due to Lehner [36]. □

Since free sets (or their left or right translates) are the fundamental completely Sidon examples, and the latter are stable by finite unions it is natural to ask: Is any completely Sidon set a finite union of translates of free sets? In other words (see Proposition 6.1): is every completely Sidon set with constant $C < \infty$ a finite union of sets with $C = 1$? Of course this would imply that any group G that contains an infinite completely Sidon set contains a copy of $\mathbb{F}_\infty$ as a subgroup, but we do not even know whether this is true, although non-amenability is known (see Remark 4).

Remark 6.2. In [45] we asked whether an L-set (see the definition below) is a finite union of left translates of free sets, but Fendler gave a simple counterexample in Coxeter groups in [20].

6.2. *L*-sets

In [45] (following [28]) we study a class of subsets of discrete groups that we call L-sets. By definition, L-sets are the sets satisfying (6.1) below. These sets are the same as those called strong 2-Leinert sets in [7]. L-sets seem to be somehow the *reduced* C^*-algebraic analogue of our completely Sidon sets. Indeed, $\Lambda \subset G$ is an L-set iff the linear map taking $\lambda_{\mathbb{F}_\Lambda}(g_t)$ to $t \in \Lambda$ extends to a complete isomorphism v from the span of $\tilde{\Lambda}$ in $C^*_\lambda(\mathbb{F}_\Lambda)$ to that of Λ in $C^*_\lambda(G)$. If (6.1) holds we have $\|v\|_{\mathrm{cb}} \leq C'$ and $\|v^{-1}\|_{\mathrm{cb}} \leq 1$ always holds. The connection between completely Sidon sets and L-sets is unclear. However our Proposition 6.3 below suggests that completely Sidon sets are probably L-sets.

Proposition 6.3. *Assume that* $C^*_\lambda(G)$ *is an exact* C^**-algebra (*G *is then called an "exact group"). Let* $\Lambda \subset G$ *be a completely Sidon set. There is a constant* C' *such*

that for any k and any finitely supported function $a : \Lambda \to M_k$ we have

$$\left\|\sum\nolimits_{t\in\Lambda} a(t)\otimes\lambda_G(t)\right\| \le C' \max\left\{\left\|\sum a(t)^* a(t)\right\|^{1/2}, \left\|\sum a(t)a(t)^*\right\|^{1/2}\right\}. \quad (6.1)$$

In other words Λ is an L-set in the sense of [45].

Proof. Fix k. Let $(U_t)_{t\in\Lambda}$ be an i.i.d. family of random matrices uniformly distributed in the unitary group $U(k)$. Let $z(t) = \overline{U_t}$. By (ii) in Proposition 1.1 we have $\|u_z\|_{\mathrm{cb}} \le C$ and hence

$$\left\|\sum\nolimits_{t\in\Lambda}[a(t)\otimes\lambda_G(t)\otimes U_t]\otimes\overline{U_t}\right\| \le C\left\|\sum\nolimits_{t\in\Lambda}[a(t)\otimes\lambda_G(t)\otimes U_t]\otimes U_G(t)\right\|. \quad (6.2)$$

Since $U_G\otimes\lambda_G$ is equivalent to λ_G (by Fell's absorption principle, see *e.g.* [47, page 149]) and we may permute the factors

$$\left\|\sum\nolimits_{t\in\Lambda}[a(t)\otimes\lambda_G(t)\otimes U_t]\otimes U_G(t)\right\| = \left\|\sum\nolimits_{t\in\Lambda} a(t)\otimes\lambda_G(t)\otimes U_t\right\|,$$

and since the operators $U_t\otimes\overline{U_t}$ have a common eigenvector

$$\left\|\sum\nolimits_{t\in\Lambda} a(t)\otimes\lambda_G(t)\right\| \le \left\|\sum\nolimits_{t\in\Lambda}[a(t)\otimes\lambda_G(t)\otimes U_t]\otimes\overline{U_t}\right\|.$$

Therefore (6.2) implies

$$\left\|\sum\nolimits_{t\in\Lambda} a(t)\otimes\lambda_G(t)\right\| \le C\left\|\sum\nolimits_{t\in\Lambda} a(t)\otimes\lambda_G(t)\otimes U_t\right\|.$$

We now recall that the matrices U_t are random $k\times k$ unitaries and we let $k\to\infty$. By [14] (actually [29, Theorem B] suffices for our needs) the announced inequality follows with $C' = 2C$. □

Remark 6.4. In Proposition 6.3 it clearly suffices to assume that $C^*_\lambda(G)$ is "completely tight" or "subexponential" in the sense of [49].

Remark 6.5. We refer to [47, §9.7] for all the terms used here. By Remark 6.6 below applied with $p = 1$, if $\Lambda\subset G$ (assumed infinite for simplicity) is completely Sidon, then the span of Λ in $L_1(\tau_G) = M_{G*}$ is completely isomorphic to the operator space $R + C$. But we see no reason why it should be completely complemented in $L_1(\tau_G)$, so we do not see how to deduce from this that the span of Λ in M_G or in $C^*_\lambda(G)$ is completely isomorphic to the operator space $R\cap C = (R+C)^*$.

Note that the question whether $C^*_\lambda(G)$ is an exact C^*-algebra for all groups G remained open for a long time, until Ozawa [42] proved that a group constructed by Gromov in [25] (the so-called "Gromov monster") is a counterexample. See also [2] and also [40,41] for more recent examples. This shows that the assumption that G is exact in Proposition 6.3 is a serious restriction, although it holds in many examples.

In the converse direction we do not have any example at hand of an L-set that is not completely Sidon.

6.3. $\Lambda(p)$-sets

In [5,6] Bożejko considered the analogue of Rudin's $\Lambda(p)$-sets in a non-Abelian discrete group G. He proved that any sequence in G contains a subsequence forming a $\Lambda(p)$-set with $\Lambda(p)$-constant growing like $\sqrt{p}$ (we call such sets "subgaussian" in [51]). In this direction, a natural question arises: which sequences in G contain a completely Sidon subsequence? similarly, which contain a subsequence forming an L-set? Obviously this is not true for any infinite sequence. It seems interesting to understand the underlying combinatorial (or operator theoretic) property that allows the extraction. In this context, we recall Rosenthal's famous dichotomy [54] for a sequence in a Banach space: it contains either a weak Cauchy subsequence or a ℓ_1-sequence (*i.e.*, the analogue of a Sidon sequence). Is there an operator space analogue of Rosenthal's theorem?

6.4. $\Lambda(p)_{\mathrm{cb}}$-sets

L-sets are also $\Lambda(p)_{\mathrm{cb}}$-sets in the sense of Harcharras [31] for any $2 < p < \infty$. In fact L-sets are just $\Lambda(p)_{\mathrm{cb}}$-sets with uniformly bounded $\Lambda(p)_{\mathrm{cb}}$-constant when $p \to \infty$. We refer to [31] for more information on these operator space analogues of Rudin's $\Lambda(p)$-sets.

Remark 6.6. If $\Lambda \subset G$ is completely Sidon, then a fortiori it is "weak Sidon" in the sense of [44]. This means that any bounded scalar valued function on Λ is the restriction of a multiplier in $B(G)$. Since the latter are c.b. multipliers on $L_p(\tau_G)$ simultaneously for all $1 \le p < \infty$ (by Proposition 2.4 and complex interpolation) we can use the Lust-Piquard-Khintchine inequalities (see [47, page 193]) to show that for any $1 \le p < \infty$ the span of Λ in $L_p(\tau_G)$ is isomorphic to that of $\tilde{\Lambda}$ in $L_p(\tau_{\mathbb{F}_\Lambda})$. Therefore, Λ is $\Lambda(p)_{\mathrm{cb}}$ for any $2 < p < \infty$ and the corresponding constant is $O(\sqrt{p})$ when $p \to \infty$. Such sets could be called "completely subgaussian". Whether conversely the $\Lambda(p)_{\mathrm{cb}}$-constant being $O(\sqrt{p})$ implies weak Sidon probably fails but we do not have any counterexample. It is natural to ask whether this "completely subgaussian" property implies that the set defines an unconditional basic sequence in the *reduced* C^*-algebra of G. In this form this is correct for commutative groups by our result from 1978 (see [51]), but what about amenable groups?

In [11] it is proved that the generators in any Coxeter group satisfy the weak Sidon property and the preceding remark is explicitly applied to that case.

6.5. Exactness

It is a long standing problem raised by Kirchberg whether the exactness of the full C^*-algebra $C^*(G)$ of a discrete group G implies the amenability of G. We feel that the preceding results may shed some light on this.

Let $\Lambda \subset A$ be a subset of a C^*-algebra A. Let $\mathbb{F}_\Lambda$ be the free group with generators (g_t) indexed by Λ. Following [52] we say that $\Lambda \subset A$ is completely

Sidon with constant C if the linear map taking $t \in \Lambda$ to $U_{\mathbb{F}_\Lambda}(g_t)$ is c.b. with c.b-norm $\leq C$.

For any $n \geq 1$, let Λ_n be linearly independent *finite* sets in the unit ball of A with $|\Lambda_n| \to \infty$. Let $C(\Lambda_n)$ be the completely Sidon constant. By [47, Theorem 21.5, page 336] if $C(\Lambda_n) = o(\sqrt{|\Lambda_n|})$ then A cannot be exact. In particular, if this holds for $A = C^*(G)$ then G is not amenable. A fortiori, if $A = C^*(G)$ contains an infinite completely Sidon set then G is not amenable.

Thus one approach to the preceding Kirchberg problem could be to show conversely that if G is non-amenable then there is a sequence (Λ_n) of such sets in $A = C^*(G)$ or even in G.

The analogous fact for the reduced C^*-algebra was proved by Andreas Thom [55].

6.6. Interpolation sets

Sidon sets are examples of "interpolation sets". Given an abstract set G given with a space $X \subset \ell_\infty(G)$ of functions on G, a subset $\Lambda \subset G$ is called an interpolation set for X if any bounded function on Λ is the restriction of a function in X.

It is known (see [45]) that $\Lambda \subset G$ is an L-set iff any (real or complex) function bounded on Λ and vanishing outside it is a c.b. (*i.e.*, "Herz-Schur") multiplier on the von Neumann algebra of G. In other words Λ is an interpolation set for the class of such multipliers, *with an additional property:* that the indicator function of Λ is also a c.b. (Herz-Schur) multiplier.

In [44] Picardello introduces the term "weak Sidon set" for a subset $\Lambda \subset G$ such that any bounded function on Λ is the restriction of one in $B(G) = C^*(G)^*$. In other words, Λ is an interpolation set for $B(G)$. By Hahn-Banach this is the same as saying that the closed span of Λ in the full C^*-algebra $C^*(G)$ is isomorphic as a Banach space to $\ell_1(\Lambda)$ by the natural correspondence.

In [44] the term Sidon (respectively strong Sidon) is then (unfortunately in view of our present work) reserved for the interpolation sets for $B_\lambda(G) = C^*_\lambda(G)^*$ (respectively for the sets such that any function in $c_0(\Lambda)$ extends to one in $A(G)$). Simeng Wang observed recently in [57] that Sidon and strong Sidon in Picardello's sense are equivalent.

Remark 6.7 (operator valued interpolation). A subset $\Lambda \subset G$ is completely Sidon iff it is an interpolation set for operator valued functions more precisely iff any bounded $B(H)$-valued function on Λ is the restriction of one in $CB(G, B(H))$. Indeed, this is Proposition 1.1. Moreover, if this holds then by Theorem 4.1 for any unital C^*-algebra A any bounded A-valued function on Λ is the restriction of one in $D(G, A)$.

Remark 6.8 (final remark). In [52] we prove a version of the union theorem for subsets of a general C^*-algebra A. We can recover the group case when $A = C^*(G)$.

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