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Tate-Shafarevich kernel, weak Brauer and R-equivalence on connected reductive groups over local and global fields

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Abstract. We introduce a new equivalence relation on k-points of connected reductive groups over an arbitrary field, which coincides with the usual Brauer equivalence when the characteristic is 0, and study its relation with R-equivalence relation and other basic arithmetic-geometric invariants of the given group over local and global fields of any characteristic via some local-global exact sequences.

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1. Introduction

In [8], Colliot-Thélène and Sansuc studied the arithmetic and geometry of tori over fields via R-equivalence, Brauer equivalence relations and weak approximation in connection with smooth compactifications of such tori. Especially, some arithmetic local-global relations between the group of Brauer equivalence classes and R-equivalence classes, the defect of weak approximation, the Tate-Shafarevich kernel etc. have been discovered via some interesting exact sequences. Some of these results have been extended further by Colliot-Thélène and Sansuc, first in [36] and then in [5] to the case of connected linear algebraic groups defined over number fields.

An interesting feature of these exact sequences is that they connect nicely several arithmetic, geometric (birational) and cohomological invariants (or obstructions) of an algebraic group and that of its smooth compactification thus they reveal some beautiful (and also mysterious) connections between the objects of very different nature.

Our aim in the paper is to extend some of these results to the case of any connected reductive group defined over an arbitrary global field. We have a complete

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analog for several exact sequences especially in the case when the global field k has no real places (*e.g.* global function field). Among some main tools we use, the flasque resolutions of connected reductive groups constructed in [5] and the interconnection between the newly introduced weak Brauer equivalence relation and the R-equivalence relation play an essential role.

Many of the results presented here are already known in the case of local and global fields of characteristic 0 and the new feature is the treatment of local and global fields uniformly regardless of their characteristic. That is why we present most of our results in its general form, though it will be clearly stated which is known in the case of characteristic 0.

We should remark that it is a difficult problem to extend these exact sequences to the case char.$k > 0$ in their full generality since the Brauer equivalence defined by Colliot-Thélène and Sansuc [8, Section 7], makes use of the cohomological Brauer group of smooth compactifications and in general, the latter may not exist over an arbitrary global field of characteristic > 0. (For example, if $k = \mathbb{F}_p(t)$, the one-dimensional unipotent k-group $G \subset \mathbb{G}_a^2$ given by the equation $y^p = y + tx^p$ provides an example of a smooth k-group, which has no smooth compactifications.)[1]

Another problem is also due to non-standard behavior of unipotent groups in char.$k = p > 0$. (Recall that the unipotent radical of a connected linear algebraic group is not necessarily defined over the ground field and for any $p > 0$ and there are smooth connected unipotent groups G of dimension 1 defined over the global function field $k := \mathbb{F}_p(t)$, such that $G(k) = 1$!) Therefore it is reasonable for us to restrict to connected reductive groups only.

The main tools in the present paper are the following. First, we use the methods employed in [42–44], combined with the approach given in [46], where the main tool is the flasque (or co-flasque) resolution. Another point is that in order to extend the exact sequences mentioned above to the case of connected reductive groups over global function fields, it is necessary to extend some other auxiliary results in the theory developed by Colliot-Thélène and Sansuc in [8, 36] and [5] to the local and global function field case, most importantly those related to Brauer and R-equivalences.

Especially, we revisit the Brauer equivalence relations defined on connected reductive groups and their connections with the R-equivalence relation. The usual definition given by Colliot-Thélène and Sansuc [8, Section 7], uses the (cohomological) Brauer group of a smooth compactification of a given variety. To avoid the problem of the existence of a smooth compactification of a given k-group G in positive characteristic case, we introduce the so-called *weak Brauer equivalence relation* on connected reductive k-groups defined over an arbitrary field k, where some fragments of its construction were already suggested in [8, Section 7]. This equivalence relation is shown to coincide with the usual Brauer equivalence (in-

[1] I thank M. Brion for indicating this simple example to me. I also thank M. Brion and the referee for suggesting that instead of smooth compactification, a *regular compactification* would be sufficient for our purpose, still under the assumption that the resolution of singularities for algebraic varieties holds.

troduced and considered earlier by Colliot-Thélène and Sansuc and Manin) in the case the ground field has characteristic 0. Moreover, over local and global function fields (where the existence of a smooth compactification of the given group G is not known a priori), this equivalence relation enjoys many properties that the usual Brauer equivalence was shown to have in characteristic 0. This weak Brauer equivalence relation allows us to explore further connections between some arithmetic, geometric and cohomological invariants of connected reductive groups over local and global function fields. Therefore, it can be considered as a substitution for Brauer equivalence relation on connected reductive groups in the case char. $k > 0$.

The plan of the paper is as follows. In Section 2, we recall some preliminary notions and concepts which will be used in the paper.

In Section 3, after recalling the definitions of various Brauer equivalence relations, we introduce a new equivalence relation, called weak Brauer equivalence on the group of rational points of a connected reductive algebraic group G over an arbitrary field and prove some basic properties (see Propositions 3.2, 3.4, Theorem 3.5). It can be considered as a substitution for the original Brauer equivalence introduced in [8] which was introduced in the presence of smooth compactifications (which is known to exist in characteristic 0 case) thus it is particularly of interest when the base field has characteristic > 0.

We show that in the characteristic 0 case, this new equivalence relation coincides with the Brauer equivalence relation introduced in [8], that in the case of local or global fields (of characteristic 0), it also coincides with the Brauer equivalence relation considered by Borovoi and Kunyavskii in [2] and that over finitely generated fields of characteristic 0, the corresponding groups of equivalence classes are finite Abelian groups. (see Theorem 3.6).

In Section 4, we study interconnections between R-equivalence relation, various Brauer equivalence relations and the weak Brauer equivalence relation, especially over local and global fields and show, for example, that over fields of characteristic 0, the group of Brauer equivalence classes of a connected linear algebraic group is a stably birational invariant. Moreover, over any field, which is finitely generated over $\mathbb{Q}$, this group is a *finite Abelian group*, and over any global field the group of those elements which are equivalent to the identity element has the approximation property. The main results of this section are Theorems 4.3 and 4.7.

Finally, in Section 5, by using the newly introduced weak Brauer equivalence, we extend some of the exact sequences established for tori by Colliot-Thélène and Sansuc relating the group of R-equivalence classes, Brauer equivalence and the obstruction to weak approximation to the case of connected reductive groups over local and global fields of any characteristic. (See Theorems 5.3, 5.6, 5.11, 5.14 and 5.16.)

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2. Preliminaries

For general notions and results of algebraic groups and algebraic varieties, we refer to [11,31,34] and [40].

If k is a field, we denote by k_s the separable closure of k in an algebraic closure $\bar{k}$ of k. For a ring A, A^* denotes the group of invertible elements of A. If k is a global field (that is, a finite extension of either $\mathbb{Q}$, or the rational function field $\mathbb{F}_q(t)$ over the finite field $\mathbb{F}_q$) we denote by ∞ the set of all archimedean places of k, by $\mathbb{A}_k$ the adèle ring of k and set $\mathbb{A}_s := \mathbb{A}_k \otimes_k k_s$. A local field always means a locally compact field (thus a completion of a global field k with respect to a valuation of k).

If X is a variety (that is an irreducible geometrically integral, reduced and separated scheme of finite type) defined over a field k, K/k an extension field, $X \times K$ denotes the base change from k to K, $K(X)$ denotes the function field of X over K and we use $\mathscr{X}$ to denote a k-compactification of X (to avoid the notation $\mathcal{X}$, which usually denotes an integral model of X).

The symbol $\mathbb{G}_a$ (respectively $\mathbb{G}_m$) denotes the additive (respectively multiplicative) group and $\mathbb{A}^n$ (respectively $\mathbb{P}^n$) denotes the n-dimensional affine (respectively projective) space. For a connected reductive group G defined over a field k, G^{ss} denotes the derived subgroup of G which is the semisimple part of G. If H is another k-group, which is isomorphic to G over some finite extension L/k, then H is called a *k-form* of G. We refer to [34, page 66], [31, page 80], [40, pages 219, 285], for the definition of inner forms of connected reductive groups, groups of inner type and related properties.

Let k be a global field and let $\Gamma := \mathrm{Gal}(k_s/k)$ be the absolute Galois group of k. Denote by V the set of all places of k and let k_v be the completion of k at $v \in V$. For an affine algebraic k-group scheme G, we denote by $\mathrm{H}^i_{\mathrm{fppf}}(k, G) := \mathrm{H}^i_{\mathrm{fppf}}(\bar{k}/k, G(\bar{k}))$ the flat cohomology in degree i (≤ 1 if G is non-commutative) of G (which is isomorphic to Galois cohomology $\mathrm{H}^i(k, G) := \mathrm{H}^i(\Gamma, G(k_s))$ in degree i if G is smooth) and let

$$Ш^1(G) := \mathrm{Ker}\left(\mathrm{H}^1_{\mathrm{fppf}}(k, G) \to \prod_{v \in V} \mathrm{H}^1_{\mathrm{fppf}}(k_v, G)\right)$$

be the *Tate-Shafarevich kernel* of G. We set also

$$Ш^i(G) := \mathrm{Ker}\left(\mathrm{H}^i_{\mathrm{fppf}}(k, G) \to \prod_{v \in V} \mathrm{H}^i_{\mathrm{fppf}}(k_v, G)\right), i \geq 0,$$

and

$$Ч^i(G) := \mathrm{Coker}\,(\mathrm{H}^i_{\mathrm{fppf}}(k, G) \to \prod_{v \in V} \mathrm{H}^i_{\mathrm{fppf}}(k_v, G)), i \geq 0,$$

whenever it makes sense. Denote by $\hat{T} := \mathrm{Hom}_{k-gr}(T, \mathbb{G}_m)$ the character group of the multiplicative group T, $P^* := \mathrm{Hom}(P, \mathbb{Z})$ the $\mathbb{Z}$-dual of the Abelian group P and $Y^D = \mathrm{Hom}(Y, \mathbb{Q}/\mathbb{Z})$, the Pontrjagin dual of an Abelian group Y.

One denotes by $\mathrm{A}(G) := \prod_{v \in V} G(k_v)/\overline{G(k)}$ (respectively $\mathrm{A}(S, G) := \prod_{v \in S} G(k_v)/\overline{G(k)}$) the *defect* (or *obstruction*) *to the weak approximation property of G over k* (respectively *obstruction to weak approximation at S*), where $\overline{G(k)}$ denotes the closure of $G(k)$ in the product of $G(k_v)$ (respectively in $\prod_{v \in S} G(k_v)$). We say that G has the weak approximation property with respect to a finite subset $S \subset V$ if $G(k)$ is dense in the product $\prod_{v \in S} G(k_v)$ via the diagonal embedding and that G has the weak approximation property over k if the above holds for any finite set $S \subset V$, which is equivalent to $\mathrm{A}(G) = 1$.

Let M be a Γ-module, which is a free $\mathbb{Z}$-module of finite type. Denote by M^* the dual module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. M is called a *permutation* Γ-module if there is a $\mathbb{Z}$-basis of M which is permuted by Γ. M is called a *flasque* (respectively *co-flasque*) Γ-module, if for every open subgroup $\Theta \subset \Gamma$, we have $\mathrm{H}^1(\Theta, M^*) = 0$ (respectively $\mathrm{H}^1(\Theta, M) = 0$).

For a k-torus T, we denote its character module by $\hat{T} := X^*(T)$ and its co-character module by $X_*(T)$. If X is a proper, smooth geometrically integral k-variety which is k_s-rational, then it is known that the Γ-module $\mathrm{Pic}(X_s)$ is a torsion free Abelian group of finite type. Then a k-torus S is called a *Néron-Severi torus* for such X if $\hat{S} \simeq \mathrm{Pic}(X_s)$.

A k-torus T is called *induced* (respectively *flasque*, *co-flasque*), if $\hat{T}$ is a permutation (respectively flasque, co-flasque) Γ-module. Two k-tori S, T are in the same similarity class if there are induced k-tori P_1, P_2 such that we have $S \times P_1 \simeq T \times P_2$ and we refer to [5, Section 2], [8, Section 1, Section 2] for more information regarding further notions and results related with tori.

If we are given a pairing $A \times B \xrightarrow{\varphi} C$ between two groups A, B with values in a group C, then for $a \in A$ (respectively $b \in B$) we write $a \perp B$ (respectively $A \perp b$), if we have $\varphi(a, x) = 0$ (respectively $\varphi(y, b) = 0$) for any element $x \in B$ (respectively, $y \in A$).

For a connected reductive group G defined over a field k, a *z-extension* of G is a connected reductive k-group H such that the semisimple part of H (the derived subgroup of H) is simply connected and H is an extension (in the sense of algebraic groups) of G by an induced k-torus Z. Then for any connected reductive group G defined over a field k, there exists a z-extension for G (*cf.* [32, Proposition 3.1] in the case characteristic 0 and [45, Lemma 2.3.1], in the general case).

Let H be a connected linear algebraic k-group. Then H is called *quasi-trivial* (after Colliot-Thélène [5, Section 2], if $k_s[H]^*/k_s^*$ is a permutation Γ-module and the Picard group $\mathrm{Pic}(H_{k_s}) = 0$, where $k_s[H]$ stands for the affine algebra of H. Then if H^{tor} denotes the maximal torus quotient of H, $P := H^{\mathrm{tor}}$ is an induced k-torus. According to [5, Proposition 3.1, 4.1], for any connected linear algebraic k-group G, supposed to be reductive if char.$k > 0$, there exist a flasque k-torus F, a quasi-trivial connected linear algebraic k-group H (which is also reductive, if G is), with the following exact sequence

$$1 \to F \to H \to G \to 1,$$

called a *flasque resolution* of G, and also there exist an induced k-torus P, an extension H of a co-flasque k-torus Q by a simply connected k-group (which is semisimple, if G is), with the following exact sequence

$$1 \to P \to H \to G \to 1$$

called a *co-flasque resolution* of G, which, in the case G is reductive, is also a z-extension of G. The flasque torus F plays an important role in the arithmetic and geometry of G, so it will be called a *flasque kernel of* G in the sequel. The above exact sequence induces the following long exact sequence of Galois cohomology

$$1 \to F(k) \to H(k) \to G(k) \overset{\delta_{F,H}}{\to} \mathrm{H}^1(k, F),$$

where the coboundary map is denoted by $\delta_{F,H}$ to indicate that it depends on the choice of F, H.

3. Brauer equivalence relations over local and global fields

Let X be a smooth algebraic variety over a field k and assume that $X(k) \neq \emptyset$. In this section we consider various equivalence relations over $X(k)$, related with the Brauer group $\mathrm{H}^2_{et}(X, \mathbb{G}_m)$ (or some of its subgroups) and we also introduce a new notion of *weak Brauer equivalence* relation on a connected reductive group G and discuss some of its main properties. This new equivalence relation seems to be more manageable than the original equivalence relation and is proven to coincide with previously known ones in the case of characteristic 0.

3.1. Brauer equivalence relations

For the following we refer the readers to the basic sources such as [8,9,27,28] (and also [2,29,39]) and references there for more details.

3.1.1.

Let X be a smooth variety over k, such that $X(k) \neq \emptyset$ and consider a smooth compactification $\mathscr{X}$ of X *i.e.*, a smooth complete k-variety containing X as an open dense subset.

Let $Br(X) := \mathrm{H}^2_{et}(X, \mathbb{G}_m)$ denote the cohomological Brauer group of a k-variety X. If $K = k_s$, then we set $X_s = X \times k_s$ for short. Then we have natural homomorphisms $Br(k) \to Br(X) \to Br(X_s)$, where the image of the former lies in the kernel of the latter. We set

$$Br_1(X) := \mathrm{Ker}\ (Br(X) \to Br(X_s)),$$
$$Br_0(X) := \mathrm{Im}\ (Br(k) \to Br(X)),$$

and $Br_a(X) := Br_1(X)/Br_0(X)$.

We consider the following natural pairings

$$X(k) \times Br(X) \to Br(k), \quad (x, b) \mapsto b(x), \tag{3.1}$$

$$X(k) \times Br_1(X) \to Br(k), \quad (x, b) \mapsto b(x), \tag{3.2}$$

$$X(k) \times Br(\mathscr{X}) \to Br(k), \quad (x, b) \mapsto b(x), \tag{3.3}$$

$$X(k) \times Br_1(\mathscr{X}) \to Br(k), \quad (x, b) \mapsto b(x), \tag{3.4}$$

where $b(x)$ denotes the equivalence class of central simple algebras over k, which is considered as an element of $Br(k)$.

Following Colliot-Thélène and Sansuc [8, Section 7, page 212], one defines Brauer equivalence relations as follows.

Definition. We say that $x, y \in X(k)$ are *$Br(X)$-equivalent* (respectively *$Br_1(X)$-equivalent*, *$Br(\mathscr{X})$-equivalent*, *$Br_1(\mathscr{X})$-equivalent*) if for all $b \in Br(X)$ (respectively $Br_1(X)$, $Br(\mathscr{X})$, $Br_1(\mathscr{X})$) we have $b(x) = b(y)$. If this is the case, then we write $x \sim_{Br(X)} y$ (respectively $x \sim_{Br_1(X)} y, x \sim_{Br(\mathscr{X})} y, x \sim_{Br_1(\mathscr{X})} y$) to make it clear which groups we use in order to define the equivalence relation.[2]

The just defined relations are clearly equivalence relations on $X(k)$ and we denote by $X(k)/Br$ (respectively $X(k)/Br_1$, and for a fixed compactification $\mathscr{X}$, $X(k)/\mathscr{B}r$, $X(k)/\mathscr{B}r_1$) the set of corresponding Brauer equivalence classes of $X(k)$. (The last two sets may depend a priori on the choice of $\mathscr{X}$.)

Since $X(k) \neq \emptyset$, by [8, Lemma 15], there is the following exact sequence

$$0 \to Br(k) \to Br_1(\mathscr{X}) \xrightarrow{s} \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \to 0$$

which is split: any k-point $* \in X(k)$ defines a section

$$t_* : \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \to Br_1(\mathscr{X})$$

[2] As the referee suggested, one may also define the pairing with the prime-to-p part of the unramified Brauer group $Br_{nr}(X)$ of X (*i.e.*, the prime-to-p part of the Brauer group $Br(\mathscr{X})$ for a smooth compactification $\mathscr{X}$ of X). We hope to pursue this approach in the future.

of s. Then by means of t_*, we may define the Picard pairing

$$X(k) \times \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \to Br(k), (x, p) \mapsto t_*(p)(x). \tag{3.5}$$

Definition. We say that that the pair $x, y \in X(k)$ are *Picard equivalent* (Pic-equivalent) (with respect to $\mathscr{X}$) if $t_*(p)(x) = t_*(p)(y)$ for all $p \in \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s))$. We write then $x \sim_{\mathrm{Pic}(\mathscr{X})} y$ and denote the corresponding set of equivalence classes by $X(k)/\mathrm{Pic}$.

Notice that if $*'$ is another k-point of X, then it is easy to see that the result of the pairing (3.5) by using $*'$ will differ from the original one (by using $*$) by a constant, thus *the set* $X(k)/\mathrm{Pic}$ *does not depend on the choice of the k-point*.[3]

3.1.2.

By [36, Lemma 6.1], we have the following natural inclusions (where the first inclusion is by mean a k-point $*$ of X) $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \subseteq Br_1(\mathscr{X}) \subseteq Br_1(X) \subseteq Br(X)$, thus we have

$$x \sim_{Br(X)} y \Rightarrow x \sim_{Br_1(X)} y \Rightarrow x \sim_{Br_1(\mathscr{X})} y \Rightarrow x \sim_{\mathrm{Pic}(\mathscr{X})} y.$$

From this it is easy to obtain the following well-defined surjective maps

$$X(k)/Br \twoheadrightarrow X(k)/Br_1 \twoheadrightarrow X(k)/\mathscr{B}r_1 \twoheadrightarrow X(k)/\mathrm{Pic}. \tag{3.6}$$

By using the definition (3.2), Borovoi and Kunyavskii computed the set $X(k)/Br_1$ of Brauer equivalence classes of homogeneous spaces X under semisimple simply connected groups with connected stabilizers over number fields (*cf.* [2]).

The following statement establishes some elementary (but very basic) properties of the Brauer equivalence. Though the proof should be known to experts, we give the proof here for the convenience of the readers and also for the following reasons: The properties (1), (3) below have been stated (without proof) and used in [43, Proposition 1.3]; the proof of (1) and (3) is not quite obvious, while the proof of (2) given in [8, page 212], is too sketchy.

Proposition 3.1. *Let G be a connected linear algebraic group defined over a field k, e the identity element of G. Then*

(1) $BG(k) := \{g \in G(k) \mid g \sim_{Br(G)} e\}$, $B_1G(k) := \{x \in G(k) \mid x \sim_{Br_1(G)} e\}$ *are normal subgroups of* $G(k)$ *and we have canonically*

$$G(k)/Br = G(k)/BG(k), G(k)/Br_1 = G(k)/B_1G(k).$$

(2) [8, page 212] *If* $\mathrm{char}.k = 0$*, then for any k-variety X, the equivalence class for the Brauer equivalence* $\sim_{\mathscr{B}r(\mathscr{X})}$ *(respectively* $\sim_{\mathscr{B}r_1(\mathscr{X})}$, $\sim_{\mathrm{Pic}(\mathscr{X})}$*) defined by* (3.3) *(respectively* (3.5)*) does not depend on the choice of the smooth compactification* $\mathscr{X}$.

[3] I thank the referee for this remark.

(3) *If char.$k = 0$ and $\mathcal{G}$ is a smooth k-compactification of G, then*

$$\mathcal{B}G(k) := \{x \in G(k) \mid x \sim_{Br(\mathcal{G})} e\},$$
$$\mathcal{B}_1 G(k) := \{x \in G(k) \mid x \sim_{Br_1(\mathcal{G})} e\}$$

and $\operatorname{Pic} G(k) := \{x \in G(k) \mid x \sim_{\operatorname{Pic}(\mathcal{G})} e\}$ *are normal subgroups of* $G(k)$. *We have canonically* $G(k)/\mathcal{B}r = G(k)/\mathcal{B}G(k)$, $G(k)/\mathcal{B}r_1 = G(k)/\mathcal{B}r_1 G(k)$, $G(k)/\operatorname{Pic} = G(k)/\operatorname{Pic} G(k)$, *which do not depend on the choice of* $\mathcal{G}$.

Proof. (1). We denote by B one of the following groups $Br(G)$ or $Br_1(G)$. We set $B_x := \{y \in G(k) \mid y \sim_B x\}$.

From the very definition, it implies that for any two elements $x, y \in G(k)$, we have $x \sim_B y$ if and only if $B_x = B_y$. We need to show that

$$\begin{aligned} &\text{(i) If } x, y, z \in G(k) \text{ such that } x \sim_B y \text{ then we have } xz \sim_B yz \\ &\quad\ \text{and } zx \sim_B zy; \\ &\text{(ii) If } x \sim_B e \text{ then } x^{-1} \sim_B e; \\ &\text{(iii) If } x \in B_e,\ y \in G(k) \text{ then } yxy^{-1} \in B_e. \end{aligned} \tag{3.7}$$

(i) Let $f_z : G \to G$, $g \mapsto gz$ be the right translation by z, which is a k-automorphism of the k-variety G. Then by functoriality, f_z defines isomorphisms of Brauer groups, $f_z^* : Br(G) \to Br(G)$ and $f_z^* : Br_1(G) \to Br_1(G)$, respectively. Then we have the following commutative diagrams

$$\begin{array}{ccccc} G(k) & \times & B & \to & Br(k) \\ f_z \downarrow & & \uparrow f_z^* & & \updownarrow = \\ G(k) & \times & B & \to & Br(k). \end{array} \tag{3.8}$$

For any $b \in B$, we have $f_z^*(b)(x) = b(f_z(x))$. Since $x \sim_B y$ we have $b(xz) = b(f_z(x)) = f_z^*(b)(x) = f_z^*(b)(y) = b(yz)$, hence $xz \sim_B yz$. Similarly, we have $zx \sim_B zy$.

(ii) From (i), if $x \sim_B e$ then we have $e = x.x^{-1} \sim_B e.x^{-1} = x^{-1}$. It implies that if $x \sim_B e$, $y \sim_B e$, then $xy^{-1} \sim_B e$. Therefore B_e is a subgroup of $G(k)$.

(iii) If x is an element from one of the subgroups (denoted by B) as above, and $y \in G(k)$, then we have $x \sim_B e$, so $yx \sim_B ye = y$ and $yxy^{-1} \sim_B yy^{-1} = e$. Hence B is a normal subgroup in $G(k)$ and one may define a group structure on $G(k)/B$.

(2) Let char.$k = 0$, $i : X \to \mathcal{X}$, $j : X \to \mathcal{X}'$ be two open embeddings of X into smooth complete k-varieties $\mathcal{X}$, $\mathcal{X}'$. For $x \in X(k)$, we denote $[x]_B$ the equivalence class of x with respect to B-equivalence, where $B = Br(\mathcal{X})$, $Br_1(\mathcal{X})$ or $\mathrm{H}^1(k, \operatorname{Pic}(\mathcal{X}_s))$. Let $x, y \in X(k)$ be such that with respect to the embedding

i, we have $x \sim_{Br(\mathcal{X})} y$, *i.e.*, for all $b \in Br(\mathcal{X})$, we have $b(i(x)) = b(i(y))$. We have to show that $x \sim_{Br(\mathcal{X}')} y$, *i.e.*, for all $b' \in Br(\mathcal{X}')$, then $b'(j(x)) = b'(j(y))$. Consider the following commutative diagram

$$\begin{array}{ccc} i(X(k)) & \times & Br(\mathcal{X}) \\ \downarrow f & & \wr \uparrow f_1^* \\ j(X(k)) & \times & Br(\mathcal{X}'). \end{array} \tag{3.9}$$

Here $f : i(X) \to j(X)$ is a k-isomorphism given by $i(x) \mapsto j(x)$, so it defines a birational k-rational map $f_1 : \mathcal{X} \dashrightarrow \mathcal{X}'$. For any $b \in Br(\mathcal{X})$, we have $(b(i(x)) = b(i(y))$. By [18, Groupes de Brauer III, Theorem 6.1 and Theorem 7.4], f_1 induces an isomorphism of Brauer groups $f_1^* : Br(\mathcal{X}') \simeq Br(\mathcal{X})$, so for each $b \in Br(\mathcal{X})$ there exists a unique $b' \in Br(\mathcal{X}')$ such that $b = f_1^*(b')$. Then we have $b(i(x)) = f_1^*(b')(i(x)) = b'(f(i(x))) = b'(j(x))$. Thus $b'(j(x)) = b(i(x)) = b(i(y)) = b'(j(y))$. Since f_1^* is an isomorphism, the last equality holds for all $b' \in Br(\mathcal{X}')$, which means that $x \sim_{Br(\mathcal{X}')} y$ in term of the open embedding j. Hence we have $[x]_{Br(\mathcal{X})} = [x]_{Br(\mathcal{X}')}$ for any $x \in X(k)$.

Next we consider the case of Br_1-equivalence. It follows from [18, Groupes de Brauer III, Corollary 7.3], that for any smooth complete k-varieties $\mathcal{X}$, $\mathcal{Y}$ and a birational k-morphism $f : \mathcal{X} \to \mathcal{Y}$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & Br_1(\mathcal{Y}) & \to & Br(\mathcal{Y}) & \to & Br(\mathcal{Y} \times k_s) \\ & \downarrow \alpha & & \simeq\downarrow \beta & & \simeq\downarrow \gamma \\ 0 \to & Br_1(\mathcal{X}) & \to & Br(\mathcal{X}) & \to & Br(\mathcal{X} \times k_s) \end{array}$$

where α, β, γ are naturally induced from f. It follows from [Gr, Groupes de Brauer III, Corol. 7.3] that α is also an isomorphism. Hence the assertion that $x \sim_{Br_1(\mathcal{X})} y \Leftrightarrow x \sim_{Br_1(\mathcal{X}')} y$, follows from the diagram similar to (3.8). Thus $[x]_{Br_1(\mathcal{X})} = [x]_{Br_1(\mathcal{X}')}$ for any $x \in X(k)$.

Regarding the Pic-equivalence, according to [9, Proposition 2A1, page 461], the birational map $f_1 : \mathcal{X} \dashrightarrow \mathcal{X}'$ defines an isomorphism $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{X}_s)) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{X}'_s))$. Therefore, after we identify $\mathcal{B}r_1(\mathcal{X})$ and $\mathcal{B}r_1(\mathcal{X}')$, then by (3.5), the Picard equivalence defined by means of $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{X}_s))$ and $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{X}'_s))$ may differ only by a constant. Therefore, by passing to equivalence classes, we obtain $[x]_{\mathrm{H}^1(k,\mathrm{Pic}(\mathcal{X}_s))} = [x]_{\mathrm{H}^1(k,\mathrm{Pic}(\mathcal{X}'_s))}$ for any $x \in X(k)$.

(3) Assume that char.$k = 0$ and $\mathcal{G}$ is a fixed smooth k-compactification of G. Then according to (2) above, or [8, page 212], the Brauer equivalence relations does not depend on the choice of a particular smooth compactification $\mathcal{G}$, which means that if $\mathcal{G}_1, \mathcal{G}_2$ are two smooth k-compactifications of G then $x \sim_{Br(\mathcal{G}_1)} y \Leftrightarrow x \sim_{Br(\mathcal{G}_2)} y$, $x \sim_{Br_1(\mathcal{G}_1)} y \Leftrightarrow x \sim_{Br_1(\mathcal{G}_2)} y$. Then in particular, we have $[x]_{B_1} = [x]_{B_2}$, *i.e.*,

the equivalence class does not depend on the choice of a smooth compactification, where $B_i = Br(\mathscr{G}_i)$, $Br_1(\mathscr{G}_i)$ or $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{G}_{i,s}))$, $i = 1, 2$.

First we prove the assertion for $\mathscr{B}r$-equivalence relation. For simplicity, we set $B := Br(\mathscr{G})$. We show that the three assertions of (3.7) hold for $B = Br(\mathscr{G})$ and as the above proof of (3.7) shows, it suffices to establish (i) for $Br(\mathscr{G})$. Thus, to show that $xz \sim_B yz$, it suffices to show that $xz \sim_{Br(\mathscr{G})} yz$ for some particular smooth k-compactification $\mathscr{G}$. Take an embedding $G \hookrightarrow \mathbb{A}^n \hookrightarrow \mathbb{P}^n$ for some n such that the k-isomorphism (of varieties) $f_z : G \to G$, $g \mapsto zg$ is given by a k-polynomial map on $\mathbb{A}^n$. If we denote by $\mathscr{G}_1$ the Zariski closure of G in $\mathbb{P}^n$ then it is clear that f_z also induces a k-morphism $\mathscr{G}_1 \to \mathscr{G}_1$, denoted by f'_z, which gives rise to the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & \mathscr{G}_1 \\ f_z \downarrow & & \downarrow f'_z \\ G & \xrightarrow{i} & \mathscr{G}_1. \end{array}$$

Notice that since G is open and dense in $\mathscr{G}$, f'_z is a birational k-morphism. The complete k-variety $\mathscr{G}_1$ may not be smooth, so let Z be its singular locus and set $Z' := f_z'^{-1}(Z)$. Now in the above diagram, we blow up the top copy of $\mathscr{G}_1$ with center in Z' (to get $\mathscr{G}_2$) and the bottom copy of $\mathscr{G}_1$ with center in Z (to get $\mathscr{G}_2$) and by using [22, Chapter 2, Corollary 7.15] to get the following commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{i} & \mathscr{G}_1 & \xleftarrow{\pi} & \mathscr{G}_2 \\ f_z \downarrow & & \downarrow f'_z & & \downarrow f_z^{(2)} \\ G & \xrightarrow{i} & \mathscr{G}_1 & \xleftarrow{\pi'} & \mathscr{G}_2 \end{array} \tag{3.10}$$

where $\mathscr{G}_2, \mathscr{G}_2$ are complete k-varieties, k-birationally equivalent to $\mathscr{G}_1$, thus also to G. After a finitely many number of blow-ups, by Hironaka's Theorem [23], we may assume from the beginning that in the commutative diagram (3.10), $\mathscr{G}_2$ and $\mathscr{G}_2$ are already smooth. There are open subsets $V \subset \mathscr{G}_2$, $V' \subset \mathscr{G}_2$ such that $\pi|_V : V \to \pi(V)$ $\pi|_{V'} : V' \to \pi'(V')$ are k-isomorphisms. Since G is outside the singular locus, we may arrange that $G \subset \mathrm{Im}\ (\pi)$, $G \subset \mathrm{Im}\ (\pi')$. Therefore there are open embeddings $j : G \hookrightarrow \mathscr{G}_2$, $j' : G \hookrightarrow \mathscr{G}_2$ making commutative the following diagram

$$\begin{array}{ccc} G & \xrightarrow{j} & \mathscr{G}_2 \\ f_z \downarrow & & \downarrow f_z^{(2)} \\ G & \xrightarrow{j'} & \mathscr{G}_2 \end{array}$$

thus also

$$\begin{array}{ccccccc}
G(k) & \xrightarrow{j} & \mathcal{G}_2(k) & \times & B_2 & \to & Br(k) \\
f_z \downarrow & & f_z^{(2)} \downarrow & & \uparrow f_z^{(2)*} & & \updownarrow = \\
G(k) & \xrightarrow{j'} & \mathcal{G}_2'(k) & \times & B_2' & \to & Br(k)
\end{array}$$

where $B_2 = Br(\mathcal{G}_2)$ (respectively $B_2' = Br(\mathcal{G}_2'))$. As mentioned above, the equivalence $x \sim_{Br(\mathcal{G})} y$ does not depend on the choice of $\mathcal{G}$, so we also have $x \sim_{Br(\mathcal{G}_2)} y$, *i.e.*, $x \sim_{B_2} y$, and from the last diagram, for any $b \in B_2'$, we have $b(f_z(x)) = f_z^{(2)*}(b)(x)$. By [18, Groupes de Brauer III, Corolary 7.3], the birational k-morphism $f_z^{(2)}$ induces a canonical isomorphism of Brauer groups $f_z^{(2)*} : Br(\mathcal{G}_2) \simeq Br(\mathcal{G}_2')$. Since $f_z^{(2)*}$ is an isomorphism, when b runs over B_2', $f_z^{(2)*}(b)$ runs over all B_2. So $f_z^{(2)*}(b)(x) = f_z^{(2)*}(b)(y) = b(f_z(x)) = b(f_z(y))$, hence $xz \sim_{B_2'} yz$.

If $x \sim_B e$, then $x^{-1} \sim_B e$ by (3.7), so we have $e = x.x^{-1} \sim_B e.x^{-1} = x^{-1}$. The rest can be proved as in the proof of (1)–(2) as above.

In the case of $\mathcal{B}r_1$-equivalence relation, the proof is similar as above, since the birational k-morphism $f_z^{(2)}$ induces canonical isomorphism of Brauer groups $Br_1(\mathcal{G}_2) \simeq Br_1(\mathcal{G}_2')$.

Finally, in the case of Pic($\mathcal{G}$)-equivalence, we consider the following commutative diagram, which is similar to (3.8)

$$\begin{array}{ccccc}
G(k) & \times & \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) & \to & Br(k) \\
f_z \downarrow & & \uparrow f_z^* & & \updownarrow = \\
G(k) & \times & \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) & \to & Br(k).
\end{array} \tag{3.11}$$

Then we may finish the proof as above. □

Below we propose another definition of Brauer equivalence for connected reductive groups, which is equivalent to the one given by (3.1) in the case of characteristic 0. This definition was inspired by [8, Proposition 17] in the case of tori and also by [9, Proposition 2.7.10].

3.1.3.

Let k be a field, G a connected reductive k-group. Let $1 \to F \to H \to G \to 1$ be a flasque resolution of G. By [36, Lemma 6.1 and Proposition 6.10], we have the following exact sequence

$$\mathrm{Pic}(G) \to \mathrm{Pic}(H) \to \mathrm{Pic}(F) \stackrel{\theta_G}{\to} Br_1(G) \to Br_1(H).$$

Let $Br_f := \theta_G(\mathrm{Pic}(F)) \subseteq Br_1(G)$. Then by using the pairing

$$G(k) \times Br_1(G) \to Br(k)$$

we may define a pairing

$$G(k) \times Br_f \to Br(k), (x, b) \mapsto b(x). \tag{3.12}$$

Definition. We say that $x, y \in G(k)$ are *weakly Brauer equivalent* (Br_f-equivalent) if for all $b \in Br_f$, we have $b(x) = b(y)$ and write $x \sim_{B_f} y$.

It is clear that this gives rise to an equivalence relation on $G(k)$ and we denote by

$$B_f G(k) := \{x \in G(k) \mid x \sim_{B_f} e\} \tag{3.13}$$

the subset of $G(k)$ consisting of all elements which are B_f-equivalent to e. Since $Br_f \subseteq Br_1(G)$, the Br_f-equivalence is coarser than the Br_1-equivalence. (Note that a priori the definition of the weak Brauer equivalence is depending on the choice of a flasque resolution, but we will show latter on that it does not depend on the resolution. In some cases, we show that weak Brauer equivalence coincides with the usual Brauer equivalence, for example if char.$k = 0$ (see Theorem 3.6 below).)

3.1.4.

We need some constructions related with a flasque resolution

$$1 \to F \to H_1 \to G \to 1$$

of G (*cf.* [5, Section 3.1, 3.2]). Take a co-flasque resolution

$$1 \to F \to P_2 \to Q \to 1$$

of F, then consider the embedding of F into the direct product $H_1 \times P_2$ via

$$\varphi : F \to H_1 \times P_2,\ \ f \mapsto (f, f^{-1})$$

and consider the quotient group $H := (H_1 \times P_2)/\varphi(F)$. By [5, Proof of 4.1] we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & F & \to & H_1 & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & P_2 & \to & H & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & Q & = & Q & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{3.14}$$

hence also the following commutative diagram with exact rows and columns, where we denote $P_1 = H_1^{\mathrm{tor}}$, $T := H^{\mathrm{tor}}$

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_1^{ss} & = & H^{ss} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & H_1 & \to & H & \to & Q & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
1 & \to & P_1 & \to & T & \to & Q & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1. & & & &
\end{array} \tag{3.15}$$

From the exact sequences $1 \to F \to P_2 \to Q \to 1$ and $1 \to P_1 \to T \to Q \to 1$ in diagrams (3.14) and (3.15), we derive the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & P_1 & \to & T \times_Q P_2 & \to & P_2 & \to & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
1 & \to & P_1 & \to & T & \to & Q & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{3.16}$$

where $T \times_Q P_2$ is the fiber product of T and P_2 over Q. Since P_1 and P_2 are induced tori, the above middle row is split, *i.e.*, $T \times_Q P_2$ is an induced k-torus. This implies that in (3.16), the first column $1 \to F \to T \times_Q P_2 \to T \to 1$ is a flasque resolution of the torus T. From above we derive the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P_2 & = & P_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & F & \xrightarrow{\varphi} & H_1 \times P_2 & \to & H & \to & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow \pi & & \\
1 & \to & F & \to & H_1 & \to & G & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1. & &
\end{array} \tag{3.17}$$

By our construction, $H_1 \times P_2$ is a quasi-trivial k-group, hence

$$1 \to F \xrightarrow{\varphi} H_1 \times P_2 \to H \to 1$$

is a flasque resolution of H. Recall that the semisimple part H^{ss} of H is simply connected, thus also isomorphic to that of $H_1 \times P_2$. Therefore we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (H_1 \times P_2)^{ss} & \simeq & H^{ss} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & F & \xrightarrow{\varphi} & H_1 \times P_2 & \to & H & \to & 1 \\
 & & \downarrow= & & \downarrow & & \downarrow \pi & & \\
1 & \to & F & \to & (H_1 \times P_2)^{\mathrm{tor}} & \to & H^{\mathrm{tor}} & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1. & &
\end{array}$$

Since H_1 is a quasi-trivial reductive group, so is $H_1 \times P_2$ and $(H_1 \times P_2)^{\mathrm{tor}}$ is an induced k-torus. Thus the last row of the above diagram is a flasque resolution of the torus $T = H^{\mathrm{tor}}$. The long exact sequence of Galois cohomology associated with $1 \to F \to H_1 \to G \to 1$ gives us a coboundary map $\delta_{F,H_1} : G(k) \to \mathrm{H}^1(k, F)$. By [36, Corollary 6.11] and the constructions given in the beginning of Subsection 3.1.5, we have the following exact sequence

$$\begin{aligned}
&1 \to \hat{G}(k) \to \hat{H}_1(k) \to \hat{F}(k) \to \mathrm{Pic}(G) \to \mathrm{Pic}(H_1) \to \mathrm{Pic}(F) \\
&\xrightarrow{\theta_G} Br_1(G) \to Br_1(H_1).
\end{aligned}$$

By [36, Lemma 6.9] we have $\mathrm{Pic}(F) \simeq \mathrm{H}^1(k, \hat{F})$. It is known (see [46, Theorem 3.8(i)]) that $\mathrm{Pic}(H_1) = 0$, thus we have an injective homomorphism θ_G : $\mathrm{H}^1(k, \hat{F}) \hookrightarrow Br_1(G)$. Let H, T be constructed as above. We have the following exact sequences $1 \to P_2 \to H \xrightarrow{\pi} G \to 1$, $1 \to F \to H_1 \times P_2 \to H \to 1$ and

$$1 \to H^{ss} \to H \xrightarrow{p} T \to 1. \qquad (3.18)$$

We state the following observations, which follow from [5, Proof of Propositions 3.1, 3.2, 4.1], the diagrams (3.14) - (3.16) and above discussion for the later use.

With notation as in Subsection 3.1.4, *H is a co-flasque z-extension of G.* (3.19)

The flasque kernel F of G is also a flasque kernel of H and of the torus $T = H^{\mathrm{tor}}$. (3.20)

T is a co-flasque k-torus. (3.21)

We have:

Proposition 3.2. *Let k be a field, G a connected reductive k-group and let*

$$1 \to F \to H_1 \to G \to 1$$

be a flasque resolution of G, $\delta_{F,H_1} : G(k) \to \mathrm{H}^1(k, F)$ the corresponding coboundary map in Galois cohomology.

(a) *With notation as above, the following diagram is anti-commutative, where* $\delta_1 := \delta_{F,H_1}$

$$\begin{array}{ccccc} G(k) & \times & Br_1(G) & \to & Br(k) \\ \downarrow \delta_1 & & \uparrow \theta_G & & \updownarrow = \\ \mathrm{H}^1(k,F) & \times & \mathrm{H}^1(k,\hat{F}) & \to & Br(k). \end{array}$$

(b) *For* $x, y \in G(k)$, $Br_f := \theta_G(\mathrm{Pic}(F))$, *we have* $x \sim_{B_f} y$ *if and only if* $(\delta_1(x) - \delta_1(y)) \perp \mathrm{H}^1(k,\hat{F})$. *In particular,* $x \in B_f G(k)$ *(*i.e., $x \sim_{B_f} e$*) if and only if* $\delta_1(x) \perp \mathrm{H}^1(k,\hat{F})$. *Also, the group* $G(k)/B_f$ *is an Abelian group, which is a subquotient of* $\mathrm{H}^1(k,F)$ *and any quasi-trivial k-group has trivial weak Brauer equivalence relation.*

(c) *If* $\mathrm{H}^1(k,H_1) = 1$, *then* $G(k)/B_f$ *is isomorphic to the image of* $\mathrm{H}^1(k,F)$ *via the natural homomorphism*

$$\mathrm{H}^1(k,F) \xrightarrow{\omega} \mathrm{Hom}(\mathrm{H}^1(k,\hat{F}), Br(k)).$$

Proof. (a) We use the same notation as in Subsection 3.1.5. As above, we have the following commutative diagrams

$$\begin{array}{ccccc} H(k) & \times & Br_1(H) & \to & Br(k) \\ \downarrow p & & \uparrow p' & & \updownarrow = \\ T(k) & \times & Br_1(T) & \to & Br(k) \end{array}$$

and

$$\begin{array}{ccccc} H(k) & \times & Br_1(H) & \to & Br(k) \\ \downarrow \pi & & \uparrow \pi' & & \updownarrow = \\ G(k) & \times & Br_1(G) & \to & Br(k). \end{array}$$

As is clear in the diagram (3.16), the exact sequence $1 \to F \to T \times_Q P_2 \to T \to 1$ is a flasque resolution of T. From this we derive an injection $\alpha : \mathrm{H}^1(k,\hat{F}) \hookrightarrow \mathrm{H}^2(k,\hat{T})$. By [36, Lemma 6.9] and its proof,

$$\mathrm{H}^2(k,\hat{T}) \simeq Br_a(T) \simeq Br_e(T) \hookrightarrow Br_1(T),$$

where $Br_e(T) := \mathrm{Ker}\ (sp : Br(T) \to Br(k))$, and $sp : b \mapsto b(e)$ is the specialization homomorphism. By [CTS1, Proof of Proposition 17], the following diagram is anti-commutative, where $\delta_2 := \delta_{F,T\times_Q P_2}$ coming from the above exact sequence

$$\begin{array}{ccccc} T(k) & \times & \mathrm{H}^2(k,\hat{T}) & \xrightarrow{\cup} & Br(k) \\ \downarrow \delta_2 & & \uparrow \alpha & & \updownarrow = \\ \mathrm{H}^1(k,F) & \times & \mathrm{H}^1(k,\hat{F}) & \xrightarrow{\cup} & Br(k) \end{array}$$

i.e., according to the isomorphism $\mathrm{H}^2(k,\hat{T}) \simeq Br_a(T)$, by using the fact that α is injective and by identifying $\mathrm{H}^2(k,\hat{F})$ with a subgroup of $Br_1(T)$, the same argument as in the proof of [36, diagram (8.11.2)] shows that the following diagram is anti-commutative

$$\begin{array}{ccccc} T(k) & \times & Br_1(T) & \stackrel{ev}{\to} & Br(k) \\ \downarrow \delta_2 & & \uparrow \alpha & & \updownarrow = \\ \mathrm{H}^1(k, F) & \times & \mathrm{H}^1(k, \hat{F}) & \stackrel{\cup}{\to} & Br(k). \end{array}$$

On the one hand, we have the following commutative diagrams, where $\delta_3 := \delta_{F, H_1 \times P_2}$ coming from the exact sequence (3.18)

$$\begin{array}{ccc} H(k) & \stackrel{p}{\to} & T(k) \\ \downarrow \delta_3 & & \downarrow \delta_2 \\ \mathrm{H}^1(k, F) & \stackrel{=}{\to} & \mathrm{H}^1(k, F) \end{array}$$

and

$$\begin{array}{ccc} H(k) & \stackrel{\pi}{\to} & G(k) \\ \downarrow \delta_3 & & \downarrow \delta_1 \\ \mathrm{H}^1(k, F) & \stackrel{=}{\to} & \mathrm{H}^1(k, F). \end{array}$$

On the other hand, on the level of Brauer groups, we have the following commutative diagrams

$$\begin{array}{ccc} Br_1(H) & \stackrel{p'}{\longleftarrow} & Br_1(T) \\ \uparrow \theta_H & & \uparrow \theta_T \\ \mathrm{H}^1(k, \hat{F}) & \stackrel{=}{\to} & \mathrm{H}^1(k, \hat{F}) \end{array}$$

and

$$\begin{array}{ccc} Br_1(H) & \stackrel{\pi'}{\longleftarrow} & Br_1(G) \\ \uparrow \theta_H & & \uparrow \theta_G \\ \mathrm{H}^1(k, \hat{F}) & \stackrel{=}{\to} & \mathrm{H}^1(k, \hat{F}). \end{array}$$

From these diagrams, we derive the following diagram of boxes Fon

$$\begin{tikzcd}
G(k) \arrow[dd, "\delta_1"'] \arrow[dr, dash, "\times"] & & H(k) \arrow[ll, "\pi"'] \arrow[rr, "p"] \arrow[dd, dotted, "\delta_3"] \arrow[dr, dash, "\times"] & & T(k) \arrow[dd, dotted, "\delta_2"] \arrow[dr, dash, "\times"] & \\
& \mathscr{B}r_1(G) \arrow[rr, "\pi'"'] & & \mathscr{B}r_1(H) & & \mathscr{B}r_1(T) \arrow[ll, "p'"'] \\
\mathrm{H}^1(k,F) \arrow[dr, dash, "\times"] \arrow[rr, phantom, "="] & & \mathrm{H}^1(k,F) \arrow[dr, dash, "\times"] \arrow[rr, phantom, "="] & & \mathrm{H}^1(k,F) \arrow[dr, dash, "\times"] & \\
& \mathrm{H}^1(k,\hat{F}) \arrow[uu, "\theta_G"] \arrow[rr, phantom, "="] & & \mathrm{H}^1(k,\hat{F}) \arrow[uu, "\theta_H"] \arrow[rr, phantom, "="] & & \mathrm{H}^1(k,\hat{F}). \arrow[uu, "\theta_T"]
\end{tikzcd}$$

In this diagram, the sign $\times$ means the pairing with values in $Br(k)$, all the horizontal squares are commutative and the vertical square on the right (relating T

with F) is anti-commutative. We show that the middle vertical square is also anti-commutative. Indeed, let $h \in H(k)$, $\hat{f} \in \mathrm{H}^1(k, \hat{F})$. Then, by using the anti-commutativity of the right vertical diagram, we have

$$\begin{aligned}(\delta_3(h), \hat{f}) &= (\delta_2(p(h)), \hat{f}) = -(p(h), \theta_T(\hat{f}))\\ &= -(h, p'(\theta_T(\hat{f}))) = -(h, \theta_H(\hat{f})).\end{aligned}$$

Using this, we show that the vertical square on the left is also anti-commutative as well. Let $g \in G(k)$, $\hat{f} \in \mathrm{H}^1(k, \hat{F})$. Since $\pi : H(k) \to G(k)$ is surjective, there is $h \in H(k)$ such that $g = \pi(h)$. Then we have

$$\begin{aligned}(\delta_1(g), \hat{f}) &= (\delta_1(\pi(h))), \hat{f}) = (\delta_3(h), \hat{f})\\ &= -(h, \theta_H(\hat{f})) = -(h, \pi'(\theta_G(\hat{f}))\\ &= -(\pi(h), \theta_G(\hat{f})) = -(g, \theta_G(\hat{f})).\end{aligned}$$

and the assertion of the proposition follows:

(b) This is obvious and follows immediately from the above proof of (a).

(c) Follows from (b), since in this case δ_1 is surjective. □

Next we show that the definition of the weak Brauer equivalence does not depend on the choice of a particular flasque resolution. First we need the following:

Lemma 3.3. (On lifting flasque resolutions) *Let $\varphi : G_1 \to G_2$ be a k-morphism of connected reductive k-groups. Then φ can be lifted to a morphism of a flasque resolution of G_1 to that of G_2,* i.e.*, we have a commutative diagram with exact rows being flasque resolutions*

$$\begin{array}{ccccccccc} 1 & \to & F_1 & \to & H_1 & \to & G_1 & \to & 1\\ & & \downarrow & & \downarrow & & \downarrow\varphi & & \\ 1 & \to & F_2 & \to & H_2 & \to & G_2 & \to & 1.\end{array}$$

Proof. Let $1 \to F_2 \to H_2 \to G_2 \to 1$ be a flasque resolution of G_2. Let H be the fiber product of H_2 and G_1 over G_2. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & F_2' & \to & H & \xrightarrow{p} & G_1 & \to & 1\\ & & \downarrow = & & \downarrow h & & \downarrow\varphi & & \\ 1 & \to & F_2 & \to & H_2 & \to & G_2 & \to & 1.\end{array}$$

Further, we let $1 \to J \to H_1 \xrightarrow{g} H \to 1$ be a flasque resolution of H. From above commutative diagram we derive the following commutative diagram, where

$l = p \circ g$ anf $F_1 := \mathrm{Ker}\,(l)$

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & J & = & J & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & F_1 & \to & H_1 & \xrightarrow{l} & G_1 & \to & 1 \\ & & \downarrow f & & \downarrow g & \searrow l & \downarrow = & & \\ 1 & \to & F_2 & \to & H & \xrightarrow{p} & G_1 & \to & 1 \\ & & \downarrow = & & \downarrow h & & \downarrow \varphi & & \\ 1 & \to & F_2 & \to & H_2 & \to & G_2 & \to & 1 \end{array}$$

where $l = p \circ g$. Since g is surjective, it follows that so is f and the sequence $1 \to J \to F_1 \to F_2 \to 1$ is exact. Since J and F_2 are flasque tori, so is F_1. Therefore $1 \to F_1 \to H_1 \to G_1 \to 1$ is a flasque resolution of G_1. We derive from above the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & F_1 & \to & H_1 & \xrightarrow{l} & G_1 & \to & 1 \\ & & \downarrow f & & \downarrow h \circ g & & \downarrow \varphi & & \\ 1 & \to & F_2 & \to & H_2 & \to & G_2 & \to & 1. \end{array}$$

It is the diagram drawn in the lemma and the lemma follows. □

Definition (*cf.* [6, page 312]). A field k_0 with absolute Galois group Γ has *finite cohomology* if for any finite discrete Γ-module M, $\mathrm{H}^n(\Gamma, M)$ is finite for any n.

For examples, $\mathfrak{p}$-adic fields, finite fields, or the field of Laurent series over such fields, are fields with finite cohomology. From the above we derive the following

Proposition 3.4.

(a) *Let k be a field and let G be a connected reductive k-group. With above notation, the definition of weak Brauer equivalence does not depend on the choice of a flasque resolution of G.*

(b) *The correspondence $G \mapsto G(k)/B_f$ gives rise to an additive functor from the category of connected reductive k-groups to the category of Abelian groups.*

(c) *If k is a field, such that for any flasque k-torus F, $\mathrm{H}^1(k, F)$ is finite then for any connected reductive k-group G, the group of weak Brauer equivalence classes $G(k)/B_f$ is a finite Abelian group. In particular, $G(k)/B_f$ is a finite Abelian group in the following cases:*

(i) *either $k = k_0$ or $k = k_0((t))$, the field of Laurent power series in a variable t and k_0 is a field finitely generated over the prime ground field;*

(ii) *k is the quotient field of an excellent Henselian local domain with residue field k_0 and k_0 is a field of characteristic 0 with finite cohomology.*

(iii) *k is a local field.*

Proof. (a) We recall that, by [5, Proposition 3.2(ii), (iii)], if $1 \to F_i \to H_i \to G \to 1$, $i = 1, 2$ are two flasque resolutions of G, we have the following commutative diagram

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \uparrow & & \uparrow & & \\ 1 \to & F_2 & \stackrel{\lambda'}{\to} & H_2 & \stackrel{\gamma'}{\to} & G & \to 1 \\ & || & & \uparrow \alpha & & \uparrow \gamma & \\ 1 \to & F_2 & \stackrel{\beta'}{\to} & E & \stackrel{\alpha'}{\to} & H_1 & \to 1 \\ & & & \uparrow \beta & & \uparrow \lambda & \\ & & & F_1 & = & F_1 & \\ & & & \uparrow & & \uparrow & \\ & & & 1 & & 1 & \end{array}$$

where E is the fiber product of H_1 and H_2 over G. Then by [5, Proposition 3.2(ii), (iii)], there are isomorphisms of k-groups $\varphi : E \simeq F_1 \times H_2$, $\psi : E \simeq F_2 \times H_1$, and an isomorphism of Γ-modules $f : \hat{F}_2 \oplus \hat{P}_1 \simeq \hat{F}_1 \oplus \hat{P}_2$, where $P_i := H_i^{\mathrm{tor}}$. We choose a Γ-homomorphism $\hat{\alpha} : \hat{F}_2 \to \hat{F}_1$ as the composition

$$\hat{\alpha} : \hat{F}_2 \hookrightarrow \hat{F}_2 \oplus \hat{P}_1 \stackrel{f}{\to} \hat{F}_1 \oplus \hat{P}_2 \stackrel{pr_1}{\to} \hat{F}_1.$$

By taking the dual, this induces another Γ-homomorphism $\alpha : F_1 \to F_2$, and also natural isomorphisms

$$\hat{\alpha}' : \mathrm{H}^1(k, \hat{F}_2) \simeq \mathrm{H}^1(k, \hat{F}_1), \alpha' : \mathrm{H}^1(k, F_1) \simeq \mathrm{H}^1(k, F_2)$$

which are compatible in the sense that the following diagram is commutative

$$\begin{array}{ccccc} \mathrm{H}^1(k, F_1) & \times & \mathrm{H}^1(k, \hat{F}_1)) & \stackrel{\cup}{\to} & Br(k) \\ \wr \downarrow \alpha' & & \wr \uparrow \hat{\alpha}' & & \updownarrow = \\ \mathrm{H}^1(k, F_2) & \times & \mathrm{H}^1(k, \hat{F}_2) & \stackrel{\cup}{\to} & Br(k). \end{array}$$

From the two flasque resolutions of G

$$1 \to F_1 \to H_1 \to G \to 1, \ 1 \to F_2 \to H_2 \to G \to 1,$$

we derive the following exact sequences of Galois cohomology

$$1 \to F_1(k) \to H_1(k) \to G(k) \stackrel{\delta_{F_1,H_1}}{\to} \mathrm{H}^1(k, F_1),$$

$$1 \to F_2(k) \to H_2(k) \to G(k) \stackrel{\delta_{F_2,H_2}}{\to} \mathrm{H}^1(k, F_2).$$

We obtain also the following exact sequences

$$1 \to F_1 \times F_2 \to H_1 \times F_2 \to G \to 1, \tag{3.22}$$

$$1 \to F_2 \times F_1 \to H_2 \times F_1 \to G \to 1, \tag{3.23}$$

and since $F_1 \times H_2 \simeq E \simeq F_2 \times H_1$, one checks that we have the following commutative diagram

$$\begin{array}{ccccccc}
E & \xrightarrow{\varphi} & F_1 \times H_2 & \xrightarrow{pr_2} & H_2 & \xrightarrow{\gamma'} & G \\
\| & & \uparrow \chi & & & & \| \\
E & \xrightarrow{\psi} & F_2 \times H_1 & \xrightarrow{pr_2} & H_2 & \xrightarrow{\gamma} & G
\end{array}$$

where $\chi := \varphi \circ \psi^{-1}$. All of the diagrams above give us an isomorphism of the above exact sequences (3.22) and (3.23)

$$\begin{array}{ccccccccc}
1 & \to & F_2 \times F_1 & \to & F_2 \times H_1 & \to & G & \to & 1 \\
 & & \zeta \downarrow\simeq & & \chi \downarrow\simeq & & \downarrow= & & \\
1 & \to & F_1 \times F_2 & \to & F_1 \times H_2 & \to & G & \to & 1.
\end{array}$$

This implies that we have also the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & F_1 & \to & H_1 & \to & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow= & & \\
1 & \to & F_2 \times F_1 & \to & F_2 \times H_1 & \to & G & \to & 1 \\
 & & \zeta \downarrow\simeq & & \chi \downarrow\simeq & & \downarrow= & & \\
1 & \to & F_1 \times F_2 & \to & F_1 \times H_2 & \to & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow= & & \\
1 & \to & F_2 & \to & H_2 & \to & G & \to & 1
\end{array}$$

and the following commutative diagram

$$\begin{array}{ccccccc}
G(k) & \xrightarrow{\delta_{F_1,H_1}} & \mathrm{H}^1(k, F_1) & \times & \mathrm{H}^1(k, \hat{F}_1) & \xrightarrow{\cup} & Br(k) \\
\downarrow= & & \beta \downarrow & & \uparrow \hat{\beta} & & \updownarrow= \\
G(k) & \xrightarrow{\delta'} & \mathrm{H}^1(k, F_2 \times F_1) & \times & \mathrm{H}^1(k, \hat{F}_1 \times \hat{F}_2) & \xrightarrow{\cup} & Br(k) \\
\downarrow= & & \beta' \downarrow\simeq & & \uparrow \hat{\beta}' & & \updownarrow= \\
G(k) & \xrightarrow{\delta''} & \mathrm{H}^1(k, F_1 \times F_2) & \times & \mathrm{H}^1(k, \hat{F}_2 \times \hat{F}_1) & \xrightarrow{\cup} & Br(k) \\
\downarrow= & & \beta'' \downarrow & & \uparrow \hat{\beta}'' & & \updownarrow= \\
G(k) & \xrightarrow{\delta_{F_2,H_2}} & \mathrm{H}^1(k, F_2) & \times & \mathrm{H}^1(k, \hat{F}_2) & \xrightarrow{\cup} & Br(k).
\end{array}$$

Let $g \in G(k)$ be such that $\delta_{F_1,H_1}(g) \perp \mathrm{H}^1(k, \hat{F}_1)$. Since $\hat{\beta}$ is surjective, so for any $x \in \mathrm{H}^1(k, \hat{F}_1 \times \hat{F}_2)$, we have

$$0 = (\delta_{F_1,H_1}(g), \hat{\beta}(x)) = (\beta(\delta_{F_1,H_1}(g)), x) = (\delta'(g), x).$$

Since β' and $\hat{\beta}'$ are isomorphisms, we have $\beta'(\delta'(g)) \perp \mathrm{H}^1(k, \hat{F}_2 \times \hat{F}_1)$, *i.e.*, $\delta''(g) \perp \mathrm{H}^1(k, \hat{F}_2 \times \hat{F}_1)$. In particular, this implies that

$$\delta''(g) \perp \mathrm{Im}\ (\hat{\beta}'') \subset \mathrm{H}^1(k, \hat{F}_2 \times \hat{F}_1),$$

i.e., for any element $y \in \mathrm{H}^1(k, \hat{F}_2)$ we have

$$0 = (\delta''(g), \hat{\beta}''(y)) = (\delta_{F_2,H_2}(g), \hat{\beta}''(x)),$$

i.e., $\delta_{F_2,H_2}(g) \perp \mathrm{H}^1(k, \hat{F}_2)$. Changing the role of F_1 and F_2, we obtain the converse statement. This means that the weak Brauer equivalence does not depend on the choice of a particular flasque resolution.

(b) Now for a given morphism $\varphi : G_1 \to G_2$, we assume, according to Lemma 3.3, that there is given a commutative diagram as in the lemma. The long exact sequence of Galois cohomology gives us the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & F_1(k) & \to & H_1(k) & \to & G_1(k) & \overset{\delta_{F_1,H_1}}{\to} & \mathrm{H}^1(k, F_1) \\
 & & \downarrow g & & \downarrow & & \downarrow \varphi & & \downarrow \psi \\
1 & \to & F_2(k) & \to & H_2(k) & \to & G_2(k) & \overset{\delta_{F_2,H_2}}{\to} & \mathrm{H}^1(k, F_2)
\end{array}$$

From these diagrams, we derive the following box of diagrams

$$\begin{tikzcd}
G_1(k) \arrow[rr, "\varphi"] \arrow[dr, dash, "\times"] \arrow[dd, "\delta_{F_1,H_1}"'] & & G_2(k) \arrow[dd, dotted, "\delta_{F_2,H_2}"] \arrow[dr, dash, "\times"] & \\
 & \theta_{G_1}(\mathrm{H}^1(k, \hat{F}_1)) & & \theta_{G_2}(\mathrm{H}^1(k, F_2)) \arrow[ll, "\hat{\varphi}"'] \\
\mathrm{H}^1(k, F_1) \arrow[rr, dotted, "\psi"] \arrow[dr, dash, "\times"] & & \mathrm{H}^1(k, F_2) \arrow[dr, dash, "\times"] & \\
 & \mathrm{H}^1(k, \hat{F}_1) \arrow[uu, "\theta_{G_1}"] & & \mathrm{H}^1(k, \hat{F}_2) \arrow[ll, "\hat{\psi}"] \arrow[uu, "\theta_{G_2}"']
\end{tikzcd}$$

where all the diagrams, except possibly for the top one, are commutative (or commutative up to sign) and $\times$ means taking the pairing. Now given an element $x \in G_1(k)$, such that $x \perp \theta_{G_1}(\mathrm{H}^1(k, \hat{F}_1))$, we need to show that $\varphi(x) \perp \theta_{G_2}(\mathrm{H}^1(k, \hat{F}_2))$. From the above diagram-box, for any $\hat{f}_2 \in \mathrm{H}^1(k, \hat{F}_2)$, we have

$$\begin{aligned}
(\varphi(x), \theta_{G_2}(\hat{f}_2)) &= \pm(\delta_{F_2,H_2}(\varphi(x)), \hat{f}_2) = \pm(\psi(\delta_{F_1,H_1}(x)), \hat{f}_2) \\
&= \pm(\delta_{G_1}(x), \hat{\psi}(\hat{f}_2)) = \pm(x, \theta_{G_1}(\hat{\psi}(\hat{f}_2))) = 0
\end{aligned}$$

(by definition of x). Therefore $\varphi(B_f G_1(k)) \subseteq B_f G_2(k)$, and φ induces a homomorphism $G_1(k)/B_f \to G_2(k)/B_f$. The rest of checking that $G \mapsto G(k)/B_f$ is a functor is trivial so we omit the details. To show that this functor is additive, *i.e.*, for any two connected reductive k-groups G_1, G_2, one has an isomorphism

$$(G_1 \times G_2)(k)/B_f \simeq G_1(k)/B_f \times G_2(k)/B_f,$$

we consider a flasque resolution $1 \to F_i \to H_i \to G_i \to 1$ of G_i, where $i = 1, 2$. Let e_i be the identity element of G_i, $i = 1, 2$. Then

$$1 \to F_1 \times F_2 \to H_1 \times H_2 \to G_1 \times G_2 \to 1$$

is a flasque resolution of $G_1 \times G_2$. By [36, Remark 6.11.3], the image of $\theta_{G_i} : \mathrm{Pic}(F_i) \to Br_1(G_i)$ lies in the subgroup

$$Br_{e_i}(G_i) := \{b \in Br_1(G_i) \mid b(e_i) = 0\}.$$

Thus the pairing $G_i(k) \times \theta_{G_i}(\mathrm{Pic}(F_i)) \to Br(k)$ (see (3.12)) can be considered as the restriction of the pairing $G_i(k) \times Br_{e_i}(G_i) \to Br(k)$. Since $\mathrm{Pic}(G)$ and $Br_e(G)$ are additive in G (see [36, Lemma 6.6] and [36, Proof of Lemma 6.9]), we have $\mathrm{Pic}(F_1 \times F_2) \simeq \mathrm{Pic}(F_1) \times \mathrm{Pic}(F_2)$ and

$$Br_{(e_1,e_2)}(G_1 \times G_2) \simeq Br_{e_1}(G_1) \times Br_{e_2}(G_2).$$

This implies that

$$(G_1 \times G_2)(k)/B_f \simeq G_1(k)/B_f \times G_2(k)/B_f.$$

(c) The first assertion follows from (a). The exact sequence

$$1 \to F(k) \to H(k) \to G(k) \overset{\delta_{F,H}}{\to} \mathrm{H}^1(k, F)$$

gives us on the one hand $\mathrm{Ker}\,(\delta_{F,H}) \subseteq B_f G(k) \subseteq G(k)$, thus a surjective homomorphism $f : G(k)/\mathrm{Ker}\,(\delta_{F,H}) \to G(k)/B_f G(k)$. On the other hand,

$$G(k)/\mathrm{Ker}\,(\delta_{F,H}) \subseteq \mathrm{H}^1(k, F).$$

By assumption, the last Abelian group is finite since F is a flasque k-torus.

(i) If $k = k_0$ is finitely generated over the prime field, then according to [8, Theorrem 1], $\mathrm{H}^1(k, F)$ is finite, so $G(k)/B_f$ is also finite. (Similarly, if $k = k_0((t))$, then by [6, Theorem 3.2], $\mathrm{H}^1(k, F)$ is finite hence so is $G(k)/B_f$.)
(ii) Follows from [6, Theorem 3.4] and the arguments as above.
(iii) Follows from the finiteness of Galois cohomology of connected reductive groups (*cf.* [37, Chapter III]). □

The following result shows how the weak Brauer equivalence relation behaves via co-flasque resolutions, which are particular z-extensions.

Theorem 3.5. *Let k be a field, G a connected reductive k-group. For a z-extension $1 \to Z \to H \to G \to 1$, which is a co-flasque resolution of G as in Subsection* 3.1.4, *let $T := H^{\mathrm{tor}} := H/H^{ss}$.*

(a) *p induces a natural isomorphism of Abelian groups $H(k)/B_f \simeq G(k)/B_f$.*

(b) *If $\mathrm{H}^1(k, H^{ss}) = 1$ then the projections $p : H \to G, q : H \to T$ induce isomorphisms of Abelian groups*

$$G(k)/B_f \simeq H(k)/B_f \simeq T(k)/B_f \simeq \mathrm{H}^1(k, F)/\mathrm{Im}\ (\delta_F),$$

where $1 \to F \to H_1 \to G \to 1$ is a flasque resolution of G and $\delta_F : T(k) \to \mathrm{H}^1(k, F)$ is a certain coboundary map. In particular, if k is a field such that semisimple simply connected groups have trivial 1-degree Galois cohomology (e.g. *k is a non-archimedean local field*[4]*), then the group of weak Brauer equivalence classes of any connected reductive k-group G can be computed (via the same group of a co-flasque torus T) as quotient of* $\mathrm{H}^1(k, F)$.

(c) *Moreover, if k is a local field, then $B_f G(k)$ is an open subgroup of $G(k)$ and for any flasque resolution $1 \to F \to H_1 \xrightarrow{\pi} G \to 1$ of G, we have $B_f G(k) = \pi(H_1(k))$.*

Proof. (a) By applying the exact sequence [36, Proposition 6.10, (6.10.3)] (which is functorial in the corresponding variables) to two rows of the diagram (3.17) we derive the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(F) & \xrightarrow{\theta_H} & Br_1(H) \\ \uparrow = & & \uparrow p' \\ \mathrm{Pic}(F) & \xrightarrow{\theta_G} & Br_1(G). \end{array}$$

By functoriality, we have the following commutative diagram

$$\begin{array}{ccccc} H(k) & \times & Br_1(H) & \to & Br(k) \\ \downarrow p & & \uparrow p' & & \updownarrow = \\ G(k) & \times & Br_1(G) & \to & Br(k). \end{array}$$

Since H is a z-extension of G, $\pi : H(k) \to G(k)$ is surjective, hence so is $p'' : H(k)/B_f \to G(k)/B_f$. To show that it is injective, let $h \in H(k)$ such that its image in $G(k)/B_f$ is trivial. This means that $p(h) \perp \theta_G(\mathrm{Pic}(F))$, *i.e.*, $(p(h), \theta_G(f)) = 0$ for all $f \in \mathrm{Pic}(F)$. Since $(p(h), \theta_G(f)) = (h, p'(\theta_G(f))) = (h, \theta_H(f)) = 0$ for any $f \in \mathrm{Pic}(F)$, so by definition, this implies that $h \in B_f H(k)$ as required. Thus we have a natural isomorphism $H(k)/B_f \simeq G(k)/B_f$.

(b) As mentioned above (*cf.* (3.21)), in the diagram (3.16), the exact sequence

$$1 \to F \to T \times_Q P_2 \to T \to 1$$

is a flasque resolution of T (recall that, according to [5, Proposition 4.1], T is co-flasque), from the above we derive the following commutative diagram with exact

[4] Or more generally, totally imaginary number fields, or all fields k with $cd(k) \leq 2$, assuming that Serre's conjecture II holds, [16].

rows and columns

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & H^{ss} & = & H^{ss} & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & F & \xrightarrow{\varphi} & H_1 \times P_2 & \to & H & \to & 1 \\ & & \downarrow= & & \downarrow & & \downarrow q & & \\ 1 & \to & F & \to & T \times_Q P_2 & \to & T & \to & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1. & & \end{array} \tag{3.24}$$

As above, from the two middle rows of (3.24), we derive the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(F) & \xrightarrow{\theta_H} & Br_1(H) \\ \uparrow= & & \uparrow q' \\ \mathrm{Pic}(F) & \xrightarrow{\theta_T} & Br_1(T). \end{array}$$

By the functoriality again we have the following commutative diagram

$$\begin{array}{ccccc} H(k) & \times & Br_1(H) & \to & Br(k) \\ \downarrow q & & \uparrow q' & & \updownarrow= \\ T(k) & \times & Br_1(T) & \to & Br(k). \end{array}$$

By using the triviality of $\mathrm{H}^1(k, H^{ss})$, thus the surjectivity of $H(k)/B_f \to T(k)/B_f$, arguing as above, we have $H(k)/B_f \simeq T(k)/B_f$.

By a similar argument as in the proof of Proposition 3.4, by using [5, Proposition 4.2], one checks that the above isomorphisms do not depend on the choice of the particular flasque and co-flasque resolutions. Since

$$1 \to F \to T \times_Q P_2 \to T \to 1$$

is a flasque resolution of T, we have a surjective homomorphism (the corresponding co-boundary map) $\delta_F : T(k) \to \mathrm{H}^1(k, F)$ and

$$T(k)/B_f \simeq \mathrm{H}^1(k, F)/(\mathrm{H}^1(k, \hat{F}))^{\perp} = \mathrm{H}^1(k, F)/\mathrm{Im}\ (\delta_F).$$

(c) For a flasque resolution $1 \to F \to H_1 \xrightarrow{\pi} G \to 1$ of G, we have by definition $B_f G(k) = \{g \in G(k) \mid \delta_{F,H_1}(g) \perp \mathrm{H}^1(k, \hat{F})\}$, which contains $\pi(H_1(k))$. Since π is a smooth morphism of algebraic groups, it is well-known that its differential $d\pi$ is surjective, hence π (when restricted to $H_1(k)$) defines an open mapping according to the Implicit Function Theorem (*cf.* [38, Part II, Chapter 3, Theorem 2]). In particular, $\pi(H_1(k))$ is an open subgroup of $G(k)$ and so is $B_f G(k)$.

For the last assertion, assume that k is non-archimedean, since the case k is archimedean case is trivial. Then we have $\mathrm{H}^1(k, H_1) = 1$. It follows that the map

$\delta_{F,H_1} : G(k) \to \mathrm{H}^1(k, F)$ is surjective. Thus by definition we have $B_f G(k) = \{g \in G(k) \mid \delta_{F,H_1}(g) \perp \mathrm{H}^1(k, \hat{F})\}$. Since $\mathrm{H}^1(k, F)$ and $\mathrm{H}^1(k, \hat{F})$ are in a perfect duality according to Tate-Nakayama, it implies that

$$B_f G(k) = \{g \in G(k) \mid \delta_{F,H_1}(g) = 1\},$$

i.e., $B_f G(k) = \pi(H_1(k))$. □

3.2. A comparison between Brauer equivalences

We wish to compare the Brauer equivalences introduced above. To do so, we need the following results (*cf.* [9, Proposition 2.7.10] and [5, Theorem 5.6]). Let X be a smooth geometrically integral variety defined over a field k, such that $k_s[X]^* = k_s^*$, that is, the only invertible regular k_s-functions on X are constants. Let $\Gamma = \mathrm{Gal}(k_s/k)$. Then it is known [9, (2.0.2), page 408], that for any k-group S of multiplicative type, there exists the following exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^1(k, S) \xrightarrow{i_1} \mathrm{H}^1_{\mathrm{fppf}}(X, S) \xrightarrow{\chi} \mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}(X_s)) \xrightarrow{\delta} \mathrm{H}^2(k, S) \\ \to \mathrm{H}^2_{\mathrm{fppf}}(X, S). \end{aligned}$$

Definition (*cf.* [9, page 408]). For an element $\lambda \in \mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}(X_s))$, we say that a X-torsor $\mathscr{T}$ under S (with its class $[\mathscr{T}]$ in $\mathrm{H}^1(X, S)$) *is of type* λ if we have $\chi([\mathscr{T}]) = \lambda$. Further, if the natural Γ-embedding $k_s^* \hookrightarrow k_s(X)^*$ admits a Γ-retraction $\sigma : k_s(X)^* \to k_s^*$, then the embedding $i_1 : \mathrm{H}^1_{\mathrm{fppf}}(k, S) \to \mathrm{H}^1_{\mathrm{fppf}}(X, S)$ admits a section $\sigma' : \mathrm{H}^1_{\mathrm{fppf}}(X, S) \to \mathrm{H}^1_{\mathrm{fppf}}(k, S)$ and we say that a X-torsor $\mathscr{T}$ under S is *trivial at* σ if $\sigma'([\mathscr{T}]) = 0$.

Now we show that the weak Brauer equivalence relation on $G(k)$ coincides with the usual Brauer equivalence relation with respect to a smooth k-compactification $\mathscr{G}$ of G, in the case char.$k = 0$.

Theorem 3.6. *Let k be a field of characteristic* 0, $\Gamma := \mathrm{Gal}(k_s/k)$*, G a connected linear algebraic k-group, and let $\mathscr{G}$ be a smooth k-compactification of G. Then the definitions of Brauer equivalence given in Subsection* 3.1.1 *by pairing with either of the groups $Br(\mathscr{G})$, $Br_1(\mathscr{G})$, or* $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{G}_s))$ *and that of weak Brauer equivalence B_f given in Subsection* 3.1.3 *(by pairing with the group $Br_f = \theta_G(\mathrm{Pic}(F))$) are equivalent on $G(k)$. These relations are coarser than the ones given in Subsection* 3.1.1 *by pairing with either of the groups $Br(G)$, $Br_1(G)$.*

Proof. Let $1 \to F \to H \to G \to 1$ be a flasque resolution of G (*cf.* [5, Proposition 3.1]). The proof is easily reduced to the case of connected reductive groups, so for simplicity, we assume that G is a reductive group. From the long exact sequence of Galois cohomology we obtain the boundary map $\delta_{F,H} : G(k) \to \mathrm{H}^1(k, F)$. By assumption, G has a smooth compactification $\mathscr{G}$ and since G is k_s-rational, so is $\mathscr{G}$. Since $\mathscr{G}(k) \neq \emptyset$, we may choose $\sigma : k_s(\mathscr{G})^* \to k_s^*$ to be a Γ-retraction for

$k_s^* \to k_s(\mathcal{G})^*$ corresponding to the identity element e of G as in [9, Proposition 2.7.10].

First, as was mentioned in [36, Remarks 6.1.13], we have $\theta_G(\mathrm{Pic}(F)) \subseteq Br_e(G) := \mathrm{Ker}\,(Br_1(G) \overset{sp}{\to} Br(k))$, where sp denotes the specialization (evaluation) map defined by e. By [8, Lemma 16], the Brauer equivalence on $\mathcal{G}(k)$ is the same by pairing with either $Br_1(\mathcal{G})$ or $Br(\mathcal{G})$ and also coincides with the one given by the following pairing $\mathcal{G}(k) \times \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \to Br(k)$. More precisely, this pairing can be defined by any embedding $t_\sigma : \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \hookrightarrow Br_1(\mathcal{G})$, which is a section to the natural map σ, where σ is defined by a k-rational element belonging to $\mathcal{G}(k)$. In our case, we choose the k-rational element to be the identity element $e \in G(k)$. Thus we have the embeddings $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \hookrightarrow Br_1(\mathcal{G}) \hookleftarrow \theta_G(\mathrm{Pic}(F))$. In particular, we identify $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s))$ with a subgroup of $Br_1(\mathcal{G})$ via t_σ. We show that

*For an arbitrary element $P \in G(k)$ (considered as an element of $\mathcal{G}(k)$) we have if and only if $P \perp \theta_G(\mathrm{H}^1(k, \hat{F}))$ (*i.e.* P is weakly Brauer trivial in the sense of Subsection* 3.1.3*).* (3.25)

On the one hand, by Proposition 3.4, $P \perp \theta_G(\mathrm{H}^1(k, \hat{F}))$ if and only if $\delta_{F,H}(P) \perp \mathrm{H}^1(k, \hat{F})$ (where $\delta_{F,H} : G(k) \to \mathrm{H}^1(k, F)$ is the coboundary map resulting from the flasque resolution $1 \to F \to H \to G \to 1$). On the other hand, we apply [9, Proposition 2.7.10] to our situation. Let F_0 be the Néron-Severi torus of $\mathcal{G}$, *i.e.* $\hat{F}_0 := \mathrm{Pic}(\mathcal{G}_s)$ and let $\mathcal{T} \to \mathcal{G}$ be a universal torsor with trivial fiber at e and let $\mathcal{T}_G$ be its restriction to G, as constructed in [5, Proposition 5.2]. There it was shown that there is an isomorphism of k-varieties $\varphi : F \times_k \mathcal{T}_G \simeq F_0 \times_k H$. Let U be the functor on the category of k-varieties with values in that of Abelian groups given by: $X \mapsto U(X) := k_s[X]^*/k_s^*$. Then φ induces a Γ-isomorphism (denoted also by φ)

$$\varphi : U(F \times \mathcal{T}_G) = U(F) \oplus U(\mathcal{T}_G) \simeq U(F_0 \times H) = U(F_0) \oplus U(H).$$

Let $i_1 : U(F) \hookrightarrow U(F) \oplus U(\mathcal{T}_G)$, $p_2 : U(F_0) \oplus U(H) \to U(F_0)$ be the canonical embedding and projection, respectively. Then we have a homomorphism $\lambda = p_2 \circ \varphi \circ i_1 : U(F) \to U(F_0)$. By Rosenlicht Lemma (*cf.* [36, Lemma 6.5]), $U(F_0) = \hat{F}_0$, $U(F) = \hat{F}$, thus we have a homomorphism (denoted also by λ) $\lambda : \hat{F} \to \hat{F}_0$. By the very definition of H and by [5, Proposition 5.1], $U(H)$ and $U(\mathcal{T}_G)$ are Γ-permutation modules, so φ induces an isomorphism

$$\varphi' : \mathrm{H}^1(k, \hat{F}) \simeq \mathrm{H}^1(k, \hat{F}_0) = \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)),$$

which also coincides with the map (induced by λ) $\lambda' : \mathrm{H}^1(k, \hat{F}) \overset{\sim}{\to} \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s))$.

Then by [9, Proposition 2.7.10] applied to the case $X = \mathcal{G}$ and $S = F$, we have the following diagram, where $\mathcal{T}^{(\sigma)}$ denotes a torsor under $\mathcal{G}$ which is trivial at

σ and the big rectangle on the right of which is commutative up to a sign

$$\begin{array}{ccccccc} G(k) & \subset & \mathscr{G}(k) & \times & \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{G}_s)) & \stackrel{\cup}{\to} & Br(k) \\ & \delta_{F,H} \searrow & \downarrow \mathscr{T}^{(\sigma)} & & \simeq\uparrow \lambda & & \updownarrow = \\ & & \mathrm{H}^1(k, F) & \times & \mathrm{H}^1(k, \hat{F}) & \stackrel{\cup}{\to} & Br(k) \end{array} \tag{3.26}$$

except possibly for the triangle on the left. By [5, Theorem 5.4] applied to G, where $X = \mathscr{G}$, and $\mathscr{T} = \mathscr{T}^{(\sigma)}$, we may endow on $\mathscr{T}_G^{(\sigma)}$ with a structure of linear algebraic k-group such that we have an exact sequence of k-groups

$$1 \to \mathrm{Ker}\ (p) \to \mathscr{T}_G^{(\sigma)} \stackrel{p}{\to} G \to 1, \tag{3.27}$$

which is a flasque resolution of G, with Ker (p) being k-isomorphic to F_0. By Proposition 3.4, the weak Brauer equivalence does not depend on the choice of a particular flasque resolution of G and we have just seen that (3.27) is exactly such a resolution with flasque kernel isomorphic to F_0. Therefore, we may assume from the very beginning that $F = F_0$. We show that with the assumption just made, the triangle on the left of the diagram (3.26) is commutative. Indeed, naturally, for the coboundary map $\delta_{F_0, \mathscr{T}_G}$ coming from the exact sequence of Galois cohomology associated with (3.27), for each $P \in G(k)$, we have $[\delta_{F_0, \mathscr{T}_G}(P)] = [p^{-1}(P)] = [\mathscr{T}_G^{(\sigma)}(P)]$, *i.e.*, the above triangle is commutative.

This implies that the Brauer equivalence on $\mathscr{G}(k)$ (by pairing with $Br(\mathscr{G})$, $Br_1(\mathscr{G})$ or $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{G}_s))$ when restricted to $G(k)$ coincides with the weak Brauer equivalence as desired. It is also clear that these equivalence relations on $G(k)$ are coarser than the one obtained by pairing with the group $Br_1(G)$ or $Br(G)$, simply because of the injections $Br_1(\mathscr{G}) \hookrightarrow Br_1(G) \hookrightarrow Br(G)$ by [36, Lemma 6.1]). ☐

Remark 3.7. Assume that char.$k = 0$. If G is a quasi-trivial k-group, then $\mathscr{B}r(\mathscr{G})$ is trivial, so $G(k)/\mathscr{B}r = G(k)/\mathscr{B}r_1 = G(k)/\mathrm{Pic} = 1$.[5]

The following corollary extends a result established for tori by Colliot-Thélène and Sansuc to the case of arbitrary connected linear algebraic group.

Corollary 3.8. **(*cf.* [8, Corollary 1(i), page 217] for tori)** *Let char.$k = 0$ and let k be as in Proposition* 3.4(c) *(*e.g. *finitely generated over the prime field $\mathbb{Q}$). Then for any connected linear algebraic k-group G, $G(k)/\mathscr{B}r$ is a finite Abelian group.*

Proof. Follows from Proposition 3.4 and Theorem 3.6 ☐

Remark 3.9. As an application, Theorems 3.5 and 3.6 give us another proof for [43, Proposition 3.6(1)] (which was also corrected in [44, Remark, page 314]) (that if $1 \to Z \to H \to G \to 1$ is a z-extension of G, then we have $H(k)/\mathscr{B}r \simeq G(k)/\mathscr{B}r$). They give also a another proof for [43, Proposition 3.6.3] in the case k is a local $\mathfrak{p}$-adic field and G is a linear connected algebraic k-group. (The proof given in [43] uses of [36, Theorem 9.5], thus cannot be extended to the case of global function fields.)

5 I thank the referee for indicating this short argument to me.

4. *R*-equivalence relation and its connection with Brauer equivalence relations

In this section we establish some relations among the R-equivalence and the (weak) Brauer equivalence relations introduced in Section 3.1, especially when k is any local field. When char.$k = 0$, we recover (and correct) some results announced and proved in [43,44].

4.1. *R*-equivalence relation

One defines the R-equivalence following Manin (*cf.* [8,9,27–29]) as follows. Let X be a smooth algebraic variety over a field k.

Definition. We say that $x, y \in X(k)$ are *R-equivalent* if there is a sequence of points $z_i \in X(k), x = z_1, y = z_n$, such that for each pair z_i, z_{i+1} there is a k-rational map $f : \mathbb{P}^1 \to X$, regular at 0 and 1, with $f(0) = z_i$, $f(1) = z_{i+1}$, $1 \leq i \leq n-1$. X is called *rationally connected over* k, if any two points $x, y \in X(k)$ are R-equivalent.

We then write $x \sim_R y$ and denote by $X(k)/R$ the set of R-equivalent classes of $X(k)$. Then X is rationally connected over k, if $X(k)/R = (1)$. It is known (*cf.* [8, Section 4, Proposition 10]), that if char.$k = 0$, then $X(k)/R$ is a *birational invariant* of smooth complete algebraic varieties X defined over k. (However, it is not clear if it is so over any field.) If G is a smooth affine k-group, then $G(k)/R$ has a natural group structure, which is compatible with the group structure on $G(k)$, *i.e.*, the projection $G(k) \to G(k)/R$ is a group homomorphism. Moreover,

$$RG(k) := \{g \in G(k) \mid g \sim_R 1\}$$

is a normal subgroup of $G(k)$ and we have canonically $G(k)/R \simeq G(k)/RG(k)$ (*cf.* [12, Lemma II.1.1(a)], [13, page 292]).

It is natural to compare the R-equivalence and the Brauer equivalences. It has been proved in [8, Proposition 16 and its proof] that for a smooth k-variety X, the Brauer equivalence relations Br introduced in Section 3.1 are coarser than the R-equivalence relation, that is, if $x \in X(k)$, $y \in X(k)$, then

$$x \sim_R y \Longrightarrow x \sim_{Br} y. \tag{4.1a}$$

Without loss of generality, we may assume that x, y are in the image of $\mathbb{P}^1(k)$ via a regular k-map $\mathbb{P}^1 \to X$. Let $[x]_R$ (respectively $[x]_{Br}$) be the R-equivalence (respectively Br-equivalence) class of x. Then the main ingredients of the proof are the facts that $Br_1(\mathbb{P}^1) = 0$ (which follows from [8, Lemma 15(i)]) and that $Br(\mathbb{P}^1 \times k_s) = 0$ (which follows from [18, Groupes de Brauer III, Corollary 5.8]). It implies that the correspondence

$$[x]_R \mapsto [x]_{Br} \tag{4.1b}$$

defines a well-defined map $X(k)/R \to X(k)/Br$. Indeed, if $y \in [x]_R$, then $[x]_R = [y]_R$ and (4.1a) shows that $[x]_{Br} = [y]_{Br}$. It also implies that the natural induced map $X(k)/R \to X(k)/Br$ is surjective. Therefore, combined with (3.6), we have the following natural well-defined surjective maps

$$X(k)/R \twoheadrightarrow X(k)/Br \twoheadrightarrow X(k)/Br_1 \twoheadrightarrow X(k)/\mathscr{B}r_1 \twoheadrightarrow X(k)/\operatorname{Pic}, \qquad (4.1c)$$

where for the last two arrows, one assumes char.$k = 0$. Also, if $X = G$ is a connected reductive k-group, then by Proposition 3.1d, we have the natural well-defined surjective homomorphisms

$$G(k)/R \twoheadrightarrow G(k)/Br \twoheadrightarrow G(k)/Br_1 \twoheadrightarrow G(k)/B_f \qquad (4.1d)$$

and if char.$k = 0$, we have also the following surjective homomorphism, followed by isomorphisms

$$G(k)/R \twoheadrightarrow G(k)/\mathscr{B}r \simeq G(k)/\mathscr{B}r_1 \simeq G(k)/\operatorname{Pic} \simeq G(k)/B_f. \qquad (4.1e)$$

In particular, if k is a global field and G is a connected reductive k-group, then from the corresponding result for R-equivalence [8, Corollary page 205]) it follows immediately that

For almost all places v of k, $G(k_v)/B$ is trivial, where B stands for B_f, Br_1, Br (or $\mathscr{B}r$, $\mathscr{B}r_1$, Pic *if char.$k = 0$).* (4.1f)

Also, from respective results for R-equivalence (see [12, Lemma II.1.1], [14, Corollary 0.3] we derive the following analog for Br-equivalence relation.

Proposition 4.1. *Let k be an infinite field and let B stand for B_f, Br_1, Br (or $\mathscr{B}r$, $\mathscr{B}r_1$,* Pic *if char.$k = 0$). Let $k(t)$ (respectively $k((t))$) be the field of rational functions (respectively Laurent series) in the variable t over k. Then for any connected reductive k-group G, we have a canonical isomorphism $G(k)/B \simeq G(k(t))/B$, and if char.$k \neq 2$, then also isomorphism $G(k)/B \simeq G(k((t)))/B$.*

Proof. Step 1. First we show that if X is a smooth geometrically irreducible k-variety, then the natural maps $h_1 : X(k)/B \to X(k(t))/B$, $h_2 : X(k)/B \to X(k((t)))/B$ are injective, where B stands for Br, Br_1, $\mathscr{B}r$, $\mathscr{B}r_1$, Pic.

Notice that we have natural injections

$$Br(k) \hookrightarrow Br(k(t)),\ \ Br(k) \hookrightarrow Br(k((t))). \qquad (4.2)$$

Indeed, let F be the perfect closure of k in $\bar{k}$. Then by Fadeev's Theorem [17, Corollary 6.4.6, page 156], we have the following commutative diagram with exact second line

$$\begin{array}{ccccc} & & Br(k) & \stackrel{\alpha}{\to} & Br(k(t)) \\ & & \downarrow f & & \downarrow g \\ 0 & \to & Br(F) & \stackrel{\alpha'}{\to} & Br(F(t)) \end{array} \qquad (4.3)$$

Since F/k is purely inseparable, it implies that Ker $(f) = 0$, so Ker $(\alpha) = 0$, too. Thus $Br(k) \hookrightarrow Br(k(t))$. The injection $Br(k) \hookrightarrow Br(k((t)))$ follows similarly, by using Witt's Theorem [17, Corollary 6.3.7, page 149]. Hence (4.2) holds true.

It implies that for any smooth k-variety X with a smooth k-compactification $\mathscr{X}$, we have the following injections

$$\begin{aligned} Br(X) &\hookrightarrow Br(X \times k(t)), \\ Br_1(X) &\hookrightarrow Br_1(X \times k(t)), \\ \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) &\hookrightarrow \mathrm{H}^1(k(t), \mathrm{Pic}(\mathscr{X}_s)) \text{ (if } \mathscr{X}(k) \neq \emptyset), \end{aligned} \tag{4.4}$$

and the similar injections, where $k(t)$ is replaced by $k((t))$

$$\begin{aligned} Br(X) &\hookrightarrow Br(X \times k((t))), \\ Br_1(X) &\hookrightarrow Br_1(X \times k((t))), \\ \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) &\hookrightarrow \mathrm{H}^1(k((t)), \mathrm{Pic}(\mathscr{X}_s)) \text{ (if } \mathscr{X}(k) \neq \emptyset). \end{aligned}$$

Indeed, for a fixed k-point $x \in \mathscr{X}(k)$, we have (*cf.* [8, Lemma 15(i)]) the splitting exact sequence

$$0 \to Br(k) \to Br_1(\mathscr{X}) \to \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \to 0,$$

by mean of which we may write

$$Br_1(\mathscr{X}) = Br(k) \oplus \mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)),$$

$$Br_1(\mathscr{X} \times k(t)) = Br(k(t)) \oplus \mathrm{H}^1(k(t), \mathrm{Pic}(\mathscr{X}_s)).$$

The projection $p_x : Br_1(\mathscr{X}) \to Br(k)$ (with respect to x) is compatible with its base change $p_{x,k(t)} : Br_1(\mathscr{X} \times k(t)) \to Br(k(t))$, thus so is the embedding of $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \to Br_1(\mathscr{X})$ and $\mathrm{H}^1(k(t), \mathrm{Pic}(\mathscr{X}_s)) \to Br_1(\mathscr{X} \times k(t))$. Therefore the injectivity of the homomorphism $\mathrm{H}^1(k, \mathrm{Pic}(\mathscr{X}_s)) \hookrightarrow \mathrm{H}^1(k(t), \mathrm{Pic}(\mathscr{X}_s))$ follows from the injectivity of $Br_1(\mathscr{X}) \to Br_1(\mathscr{X} \times k(t))$. The latter in turn follows from the corresponding statement for Br. Thus we will focus on Br only. We have the following commutative diagram with exact second line, where $k(X)$ denotes the function field of X and β, β' are homomorphisms of base change

$$\begin{array}{ccccc} & & Br(X) & \xrightarrow{\beta} & Br(X \times k(t)) \\ & & \downarrow f' & & \downarrow g' \\ 0 & \to & Br(k(X)) & \xrightarrow{\beta'} & Br(k(X)(t)). \end{array} \tag{4.5}$$

Since β' is injective (see above) and f' is also injective (*cf.* [18, Groupes de Brauer II, Proposition 1.7]), it implies that so is β. Let $x \in X(k)$, $y \in X(k)$ such that they have the same B-equivalence classes $[x]_{B(t)} = [y]_{B(t)}$ as B-equivalence classes of

x, y in $X(k(t))$. For simplicity, we assume that $B = Br$, since the other cases are similar. From the commutative diagram

$$\begin{array}{ccccc} X(k) & \times & Br(X) & \to & Br(k) \\ \downarrow \pi & & \downarrow \pi^* & & \downarrow \\ X(k(t)) & \times & Br(X \times k(t)) & \to & Br(k(t)) \end{array}$$

and the injectivity of π^*, it implies that $[x]_B = [y]_B$ (as the classes in $X(k)$). Hence $X(k)/B \to X(k(t))/B$ is injective. In a similar way the same assertion holds if $B = Br_1(X)$, $Br(\mathscr{X})$, $Br_1(\mathscr{X})$, and also $B_f G$, if $X = G$ is a connected reductive k-group. The case $k((t))$ is similar.

Step 2. Now we show that the natural homomorphism $h : G(k)/B \to G(k(t))/B$ is surjective. We have the following commutative diagram

$$\begin{array}{ccc} G(k)/R & \xrightarrow{r} & G(k(t))/R \\ \downarrow f & & \downarrow g \\ G(k)/B & \xrightarrow{h} & G(k(t))/B \end{array}$$

By [12, Lemma II.1.1], we know that there is an isomorphism $r : G(k)/R \simeq G(k(t))/R$. Let $x \in G(k(t))/B$. Then by (1), there is $y \in G(k(t))/R$ such that $g(y) = x$. Hence for some $z \in G(k)$, we have $x = g(y) = g(r(z)) = h(f(z))$, so h is surjective. We have similar injections, where $k(t)$ is replaced by $k((t))$.

If char.$k \neq 2$, then by [14, Corollary 03], we have $G(k)/R \simeq G(k((t)))/R$, so we may apply the same argument as above to get the isomorphism

$$G(k)/B \simeq G(k((t)))/B. \qquad \square$$

Next we consider some relations between the set of the R-equivalence classes (respectively Brauer equivalence classes) for algebraic groups and that of their smooth compactifications defined over a field of characteristic 0 and derive some consequence. From some important results established in [8, Section 4, Section 7] for tori) one derives the following observation which generalizes such results to connected linear algebraic groups. For smooth complete varieties $\mathscr{X}$ over a field k of characteristic 0, it was established in [8, Proposition 16] that the set of Brauer equivalence classes $\mathscr{X}(k)/\mathscr{B}r$ is a birational invariant of $\mathscr{X}$. As shown below, this is also a *stable birational invariant* of $\mathscr{X}$.

Theorem 4.2. *Assume that char.$k = 0$, G is a connected linear algebraic k-group and $\mathscr{G}$ is any smooth k-compactification of G.*

(1) (*cf.* [14, Proposition 1.8], [8, Propositions 13, 14 for the case of tori and quasi-split groups]) *The identity map defines a bijection $G(k)/R \simeq \mathscr{G}(k)/R$.*[6]
(2) *If $\mathscr{X}, \mathscr{Y}$ are stably birationally equivalent smooth complete k-varieties and $\mathscr{X}(k) \neq \emptyset$, $\mathscr{Y}(k) \neq \emptyset$, then there is a bijection $\mathscr{X}(k)/Br \simeq \mathscr{Y}(k)/Br$.*

[6] I am indebted to the referee for indicating [14] to me.

(3) (*cf.* [8, Proposition 17(iii), (iv)] for the case of tori) *We have the following bijection* $G(k)/B \simeq \mathcal{G}(k)/B$, *where* B *stands for either* $\mathcal{B}r$, $\mathcal{B}r_1$ *or* Pic *and the* B*-equivalence relation on* $G(k)$ *is induced from that on* $\mathcal{G}(k)$. *As a consequence, if* G *and* H *are stably birationally equivalent connected reductive* k*-groups then there is a bijection* $G(k)/B \simeq H(k)/B$, *where* B *stands for* $\mathcal{B}r$, $\mathcal{B}r_1$, Pic *or* B_f.

Proof. (2) Let $X := \mathcal{X} \times \mathbb{P}^n \simeq \mathcal{Y} \times \mathbb{P}^m =: Y$ be a k-isomorphism of varieties. By [8, Proposition 10] we have $X(k)/Br \simeq Y(k)/Br$. Since $\mathcal{X}$ and X are stably birationally equivalent, by [8, Remark after Proposition 10], we have a bijection $f : X(k)/R \simeq \mathcal{X}(k)/R$. Similarly, $\mathcal{Y}(k)/R \simeq Y(k)/R$. Also, by assumption, we have $X(k)/R \simeq Y(k)/R$ [8, Proposition 10]. We show that the natural map $g : X(k)/Br \to \mathcal{X}(k)/Br$ induced by the projection $\mathcal{X} \times \mathbb{P}^n \to \mathcal{X}$ is bijective. To prove that it is surjective, let $\alpha : \mathcal{X}(k)/R \to \mathcal{X}(k)/Br$, $\beta : X(k)/R \to X(k)/Br$ be the natural projections [8, Proposition 10]. From the commutative diagram

$$\begin{array}{ccc} X(k)/R & \overset{f}{\simeq} & \mathcal{X}(k)/R \\ \alpha \downarrow & & \downarrow \beta \\ X(k)/Br & \overset{g}{\to} & \mathcal{X}(k)/Br \end{array}$$

where α, β are surjective (see (4.1a)-(4.1d), or [8, Proof of Proposition 13], we infer that g is also surjective. In particular, if Card denotes the cardinality, then it implies that $\mathrm{Card}(X(k)/Br) \geq \mathrm{Card}(\mathcal{X}(k)/Br)$ and we have a similar statement for the pair $Y, \mathcal{Y}$.

Now let $i : \mathcal{X} \to \mathcal{X} \times \mathbb{P}^n$ be the embedding $x \mapsto (x, e)$, where $e \in \mathbb{P}^n(k)$ is a fixed k-point. Then i induces a natural map $h : \mathcal{X}(k)/Br \to (\mathcal{X} \times \mathbb{P}^n)(k)/Br$. From the commutative diagram

$$\begin{array}{ccc} \mathcal{X}(k)/R & \overset{f'}{\simeq} & X(k)/R \\ \beta \downarrow & & \downarrow \alpha \\ \mathcal{X}(k)/Br & \overset{g'}{\to} & X(k)/Br \end{array}$$

where we know that α, β are surjective it implies that g' is also surjective, so $\mathrm{Card}(\mathcal{X}(k)/Br) \geq \mathrm{Card}(X(k)/Br)$. We have a similar statement for the pair $\mathcal{Y}, Y$. From above it follows that

$$\mathrm{Card}(\mathcal{X}(k)/Br) = \mathrm{Card}(X(k)/Br), \mathrm{Card}(\mathcal{Y}(k)/Br) = \mathrm{Card}(Y(k)/Br),$$

thus $\mathrm{Card}(\mathcal{X}(k)/Br) = \mathrm{Card}(\mathcal{Y}(k)/Br)$ and we have a bijection $\mathcal{X}(k)/Br \simeq \mathcal{Y}(k)/Br$.

(3) Since the Brauer equivalence relation does not depend on the choice of the smooth compactification, so by the very definition, we have an injective map $f : G(k)/B \hookrightarrow \mathcal{G}(k)/B$, where we use $\mathcal{B}r(\mathcal{G})$, $\mathcal{B}r_1(\mathcal{G})$, or $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s))$ in order

to get the group $G(k)/B$. We need to show that f is surjective. By (1), for any $x \in \mathcal{G}(k)$, there is $g \in G(k)$ such that $g \sim_R x$. Since Br-equivalence or Br_1-equivalence are coarser than R-equivalence, this implies that $g \sim_B x$, too, *i.e.*, f is surjective.

Finally, assume that G, H are stably birationally equivalent connected reductive k-groups. Let $\mathcal{G}$ and $\mathcal{H}$ be smooth k-compactifications of G, H, respectively. Since G and H are stably birationally equivalent, so are $\mathcal{G}$ and $\mathcal{H}$. Then by (2), we have a bijection $\mathcal{G}(k)/\mathcal{B}r \simeq \mathcal{H}(k)/\mathcal{B}r$. From above we know that $\mathcal{G}(k)/\mathcal{B}r \simeq G(k)/\mathcal{B}r$ and $\mathcal{H}(k)/\mathcal{B}r \simeq H(k)/\mathcal{B}r$, so by Theorem 3.6, we have

$$G(k)/B_f \simeq G(k)/\mathcal{B}r \simeq \mathcal{G}(k)/\mathcal{B}r \simeq \mathcal{H}(k)/\mathcal{B}r \simeq H(k)/\mathcal{B}r \simeq H(k)/B_f$$

hence also $G(k)/B \simeq H(k)/B$ where B stands for either $\mathcal{B}r$, $\mathcal{B}r_1$, Pic or B_f as required. □

Next we consider some basic relations between the groups (or sets) of R-equivalence classes via an exact sequence provided by a central isogeny over a local non-archimedean field or a global function field.

Definition (*cf.* [36, page 14]). Let G be a connected reductive group defined over a field k. An exact sequence $1 \to \mu \to G_1 \xrightarrow{\pi} G \to 1$ defines a *special covering* of G, if π is a central k-isogeny, $G_1 = G_1^{ss} \times P$, where G_1^{ss} is a semisimple simply connected k-group and P is an induced k-torus.

We have the following important theorem, which is the correct statement of [43, Lemma 4.20]. This lemma, being designed to prove [43, Theorem 4.12], was stated there in too general terms and remains unproven, since its proof given in [43, page 282] is not correct. (Though, we do not have any counter-example to the statement yet.) The problem is that there were given in [12] several exact sequences computing the group of R-equivalence but we did not use the right ones and interpreted the others wrongly. In fact, we need the lemma here *only in the case of local and global fields* and the correct formulation of [43, Lemma 4.20] should be as in Theorem 4.3 below. (For another proof of the theorem in characteristic 0 case, see [3, Theorem 1, page 333].)

Theorem 4.3. **(*cf.* [12, Theorem III.4.3(b)], [5, Theorem 9.3], [3, Theorems 4.8, 8.4, Theorem 1, page 333] for other cases not treated here).** *Let k be a local or global field, G a connected reductive k-group, H a co-flasque z-extension of G, $T = H/H^{ss}$, Then we have the following exact sequence*

$$H^{ss}(k)/R \to G(k)/R \xrightarrow{\chi_R} T(k)/R \to 1, \tag{4.6}$$

with $T(k)/R \simeq \mathrm{H}^1(k, F)$, where F is any flasque kernel of G. The sequence (4.6) *is functorial with respect to the class of groups G having simply connected semisimple parts.*

Proof. Since $H(k)/R \simeq G(k)/R$ (see [3, Corollary 4.16]) and since a flasque kernel for G is also a flasque kernel for H by (3.20), we may assume from the beginning that $\tilde{G} := G^{ss}$ is simply connected. There exist a number m, induced k-tori P and Q, such that $G^m \times P$ has a special covering

$$1 \to \mu \to (\tilde{G})^m \times P \to G^m \times Q \to 1$$

(see [36, Lemma 1.10]). Then we have the following induced sequence

$$[(\tilde{G})^m \times P](k)/R \to [G^m \times Q](k)/R \xrightarrow{\sim} [H^m \times Q](k)/R \to [T^m](k)/R \to 1,$$

or equivalently

$$\tilde{G}^m(k)/R \to G^m(k)/R \to T^m(k)/R \to 1, \tag{4.7}$$

since P, Q are k-rational. It is easy to see that to prove the exactness of the sequence (4.6) it is sufficient to prove the same thing for the sequence (4.7). We may therefore assume from the very beginning that G has a special covering, thus we have the following exact sequence $1 \to \mu \to \tilde{G} \times T' \xrightarrow{\pi} G \to 1$, where π is a central k-isogeny and T' is an induced k-torus. Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu \cap \tilde{G} & \to & \mu & \xrightarrow{\tau} & \mu' & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \tilde{G} & \to & \tilde{G} \times T' & \xrightarrow{\lambda} & T' & \to & 1 \\
 & & \downarrow \psi & & \downarrow \pi & & \downarrow \varphi & & \\
1 & \to & \tilde{G} & \to & G & \xrightarrow{\chi} & T & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $\mu \cap \tilde{G}$ denotes the schematic-intersection. Since ψ is a k-central isogeny and $\tilde{G}$ is semisimple, simply connected, ψ is an isomorphism by [1, Proposition 2.24]. It implies that $\mu \cap \tilde{G} = 1$, so $\tau : \mu \to \mu'$ is also a k-isomorphism and we may identify μ and μ' via the isomorphism τ. Let

$$1 \to \mu \xrightarrow{\zeta} F \to E \to 1 \tag{4.8}$$

be a flasque resolution of μ. Here F is a flasque k-torus and E is an induced k-torus. Since $\mu \simeq \mu'$ (see the proof above), we may consider the embedding of μ into the direct product $T' \times F$ via $\varphi : f \mapsto (f, f^{-1})$ and consider the quotient group $H := (T' \times F)/\varphi(\mu)$. From the last column of the diagram and (4.8) we derive the following commutative diagram with exact rows and columns (similar to that of (3.14))

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & \mu & \to & F & \to & E & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & T' & \to & H & \to & E & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & T & = & T & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1. & & & & \end{array} \tag{4.9}$$

Since T' and E are induced k-tori, the second row of (4.9) implies that so is H. Therefore the second column is a flasque resolution of T and F is a flasque kernel of T. In particular, by [8, Theorem 2] we have

$$T(k)/R \simeq \mathrm{H}^1(k, F). \tag{4.10}$$

(a) Let k be a local field. If $k \simeq \mathbb{R}$ (or $\mathbb{C}$), then it is well-known that $H(k)/R = 1$ for any connected linear algebraic k-group so there is nothing to prove.

If k is a local non-archimedean field, then by Kneser's Theorem in characteristic 0 (*cf.* [24,25]) and by Bruhat-Tits' Theorem in any characteristic (*cf.* [4, Section 4.7, Theorem (ii)]), we have $\mathrm{H}^1(k, \tilde{G}) = 1$, so the homomorphism $\chi : G(k) \to T(k)$ is surjective. In particular, the induced homomorphism $\chi_R : G(k)/R \to T(k)/R$ is also surjective. Then we may assume that $\tilde{G}$ is absolutely almost simple k-group. If $\tilde{G}$ is k-isotropic, then it is well-known by the affirmative solution of the Kneser–Tits conjecture for local fields by Platonov (see [34, Chapter 7, Proposition 7.7], or [35, Main Theorem]) that $\tilde{G}(k)$ has no proper non-central subgroups. Since $RG(k)$ is a normal subgroup of $G(k)$, so we have $\tilde{G}(k)/R = 1$. If $\tilde{G}$ is k-anisotropic, then by [25, Satz 3] and [4, Section 4.5, 4.6, page 696] $\tilde{G}$ is of type ${}^1\mathrm{A}_n$. We may assume that $\tilde{G}(k) = \mathrm{SL}_1(D)$, the group of norm-1-elements of D^*, where D is a central simple division algebra over k. It is known (see [15, Theorems 7.1, 7.2]) that for any $n \geq 2$, $A := \mathrm{M}_n(D)$ and for the k-group G_n defined by $G_n(k) = \mathrm{SL}_1(A)$, we have $G_n(k)/R = \tilde{G}(k)/R = \mathrm{SL}_1(D)/[D^*, D^*]$. From above it follows that for G_n, we have $G_n(k)/R = 1$, thus in this case $\tilde{G}(k)/R = 1$. Also $T'(k)/R = 1$ since T' is induced k-torus. From (4.10) and [12, Proposition III.2.7] (*cf.* also Remarks 4.12 for another short proof), it follows that

$$T(k)/R \simeq \mathrm{H}^1(k, F), \quad G(k)/R \simeq \mathrm{H}^1(k, F)$$

thus $G(k)/R \simeq T(k)/R$ and the assertion (a) holds.
(b) Let k be a global field. Then by [12, Theorem III.3.1], we have the following exact sequences

$$(\tilde{G} \times T')(k)/R \xrightarrow{\pi_R} G(k)/R \xrightarrow{\delta_1} \mathrm{H}^1(k, F) \to 1. \tag{4.11}$$

By (4.10) we have $T(k)/R \simeq \mathrm{H}^1(k, F)$ and the exact sequence (4.11) becomes

$$\tilde{G}(k)/R \overset{\pi_R}{\to} G(k)/R \overset{\delta_1}{\to} T(k)/R \to 1,$$

which is what to be proved. From this the functoriality of the sequence (4.6) also follows. □

Consequently, we have the following statement, which was proved in [3, Lemma 4.12] for fields of characteristic 0 of geometric type (gl), (sl), (ll) [7] and in [3, Lemma 8.3] for number fields, where G has no anisotropic factors of type E_6.

Corollary 4.4. *Let G be a connected reductive group defined over k, where k is either a local field or a global field k having no real places. If $\tilde{G} := G^{ss}$ is simply connected, then the natural homomorphism $G \to G^{\mathrm{tor}} := G/G^{ss}$ induces an isomorphism of groups $G(k)/R \simeq G^{\mathrm{tor}}(k)/R$. In particular, if G is quasi-trivial, then $G(k)/R = 1$.*

Proof. First we show that under the assumption, $\tilde{G}(k)/R$ is trivial. We may assume that $\tilde{G}$ is absolutely almost simple k-group. If k is a local field, then this follows from the proof of the theorem. If k is a global field and $\tilde{G}$ is k-isotropic, this follows from [15, Theorem 8.1]. If k has no real places and G is k-anisotropic, then by [34, Theorem 6.25], $\tilde{G}$ is of type A_n, so again by [15, Theorem 8.3], $\tilde{G}(k)/R = 1$. Now it follows from the theorem that we have $G(k)/R \simeq G^{\mathrm{tor}}(k)/R$. If G is quasi-trivial, then G^{tor} is an induced torus, so we are done. □

From Theorem 4.3 [12,Theorem III.3.1] and from recent results on the Kneser-Tits conjecture [15, Theorem 8.3] one deduces immediately the following assertion, which will be used in the sequel. For another proof in the case of a group G having no anisotropic factors of type E_6 defined over number fields, we refer to [3, Propositions 8.1, 8.2, Theorem 8.4, Corollary 8.5]. The case of geometric fields (gl), (ll), (sl) where G has no E_8 factors if k is (gl), was treated in [3, Theorem 4.5]. The following is also a global analog of [12, Proposition III.2.7] for fields with no real places.

Corollary 4.5. (*cf.* [5, Theorem 9.1(i)] for $\mathfrak{p}$-adic fields, [12, Proposition III.2.7] for local non-archimedean fields). *Let G be a connected reductive group defined over a global field k having no real places. Let $1 \to \mu \to G_1 \to G \to 1$ be a special covering of G and let $1 \to \mu \to F \to E \to 1$ be a flasque resolution of μ,* i.e., *F is a flasque k-torus and E is a k-induced torus. Then we have $G(k)/R \simeq \mathrm{H}^1(k, F)$.*

Proof. Let $G_1 = \tilde{G} \times P$, where P is an induced k-torus and $\tilde{G}$ is a semisimple simply connected k-group. Since $P(k)/R = 1$, we need only show that $\tilde{G}(k)/R = 1$ if k has no real places or $\tilde{G}$ has no anisotropic factors of type E_6. We may assume

[7] We refer to [6] for the definition of such fields.

that $\tilde{G}$ is absolutely almost simple over k. If $\tilde{G}$ is k-isotropic, then by [15, Theorem 8.1], $\tilde{G}(k)$ is projectively simple, so $\tilde{G}(k)/R = 1$. If $\tilde{G}$ is k-anisotropic then in the first case, by [34, Theorem 6.25, Chapter 6], $\tilde{G}$ is of type A_n. Then by [15, Theorem 8.3], $\tilde{G}(k)$ is projectively simple and again $\tilde{G}(k)/R = 1$. For the second case, the assertion also follows from [15, Theorem 8.3].

According to the proof of Theorem 4.3 and Corollary 4.4, now we have $G(k)/R \simeq G^{\mathrm{tor}}(k)/R$. From the proof of Theorem 4.3, it also follows that F is also a flasque kernel of G^{tor}. Thus by [8, Theorem 2], we have $\mathrm{H}^1(k, F) \simeq G^{\mathrm{tor}}(k)/R$. $\square$

Remarks 4.6. (a) By Proposition 3.4 (respectively Corollary 3.8), $G(k)/B_f$ (respectively $G(k)/\mathscr{B}r$) is finite over such fields k, that $\mathrm{H}^1(k, F)$ is finite for any flasque k-torus F (respectively finitely generated over the prime field $\mathbb{Q}$).

With above notation, from Theorem 4.3 it also implies that (*cf.* [12, Theorem B]) $G(k)/R$ is a finite group. For, by a result by Margulis [30, Corollary 2.4.9] and Prasad [35, Theorem C, page 569], we know that $\tilde{G}(k)/R$ is finite, that (by [8, Corollary 2, page 200]) $T(k)/R$ is finite. Hence the exact sequence of Theorem 4.3 implies that $G(k)/R$ is also finite.

The harder problem on the finiteness of the group of R-equivalence classes $G(k)/R$ over fields, which are finitely generated over the prime field is still open even in characteristic 0, and we refer to [5, Theorem 8.1 (iii)] for some related results regarding the finiteness of certain *quotient group* of $G(k)/R$.

(b) The equality $G_1^{ss}(k)/R = 1$ holds if a general conjecture of Margulis [30, 2.4.8] (*cf.* also [34, Conjecture 9.2, page 511]) for anisotropic almost simple simply connected groups of type E_6 defined over a real number field is true, see [15, Théorème 8.3].

(c) Some related results for connected reductive k-groups over fields k with $cd_p(k_s) \leq 2$ was given in [13, Theorem 6].

4.2. Some density relations between the weak Brauer and R-equivalences

In this section we consider some relations between the following subgroups of $G(k)$ (see 3.1.1 and 4.1 for notation): $RG(k)$, $B_fG(k)$, $BG(k)$, $B_1G(k)$ (respectively $\mathscr{B}G(k)$, $\mathscr{B}_1G(k)$ if char.$k = 0$) over a global field k and and their analogs over the completions k_v. Denote $G_S := \prod_{v\in S} G(k_v)$, $RG_S := \prod_{v\in S} RG(k_v)$, $BG_S = \prod_{v\in S} BG(k_v)$, $B_1G_S = \prod_{v\in S} B_1G(k_v)$, $B_fG_S := \prod_{v\in S} B_fG(k_v)$, and if char.$k = 0$, $\mathscr{B}G_S = \prod_{v\in S} \mathscr{B}G(k_v)$, $\mathscr{B}_1G_S = \prod_{v\in S} \mathscr{B}_1G(k_v)$.

We have the following relations among these groups over local and global fields. Notice that by Theorem 3.6, the Brauer and weak Brauer equivalence relations are the same if char.$k = 0$, so the following (and other subsequent ones) extends the results related to Brauer equivalence to the case char.$k > 0$. The following was stated in [43, Theorem 4.22] for Brauer and R-equivalence relations in characteristic 0. We give a unified proof, which not only corrects the one given in [43], but also is valid over any global field.

Theorem 4.7. **(*cf.* [43, Theorem 4.22] for $\mathscr{B}r$ and R-equivalence over number fields).** *Let k be a global field, S a finite set of places of k and let G be a connected reductive k-group. Then we have the following equalities*

$$\overline{RG(k)} = RG_S = \overline{BG(k)} = BG_S = \overline{B_1G(k)} = B_1G_S = \overline{B_fG(k)} = B_fG_S \subseteq \overline{G(k)}.$$

Especially, if char.k $= 0$*, these subgroups are also equal to* $\overline{\mathscr{B}G(k)} = \mathscr{B}G_S = \overline{\mathscr{B}_1G(k)} = \mathscr{B}_1G_S$.

We need the following approximation result for quasi-trivial groups (see [5, Proposition 9.2] (for number fields) and [46, Proof of Proposition 2.2] (for any global field)).

Proposition 4.8 ([5, Proposition 9.2], [45, Proof of Proposition 2.2]). *If H is a quasi-trivial reductive group defined over a global field k, then H has the weak approximation property over k.*

The following lemma plays a crucial role in the sequel.

Lemma 4.9 (*cf.* [44, Theorem 2.1] for the characteristic 0 case). *If k is a local field, F a flasque kernel of a connected reductive k-group G, then there are canonical isomorphisms*

(a) $G(k)/B \simeq \mathrm{H}^1(k, F)$, *where B stands for B_f, Br, Br_1 (and $\mathscr{B}r$, $\mathscr{B}r_1$,* Pic *if char.k* $= 0$*),*
(b) (*cf.* [5, Theorem 9.1(i)] for the characteristic 0 case) $G(k)/R \simeq \mathrm{H}^1(k, F)$.

Proof. By surjective homomorphisms given in 4.1, it is clear that once we can prove the assertions regarding the relation between R-equivalence and B_f-equivalence, then the other cases also follow. It is clear that we may assume k is a non-archimedean local field.

(a) $G(k)/B_f \simeq \mathrm{H}^1(k, F)$. Let $1 \to F \to H_1 \to G \to 1$ be a flasque resolution G. Since k is non-archimedean, and H_1 is quasi-trivial, we have $\mathrm{H}^1(k, H_1) = 1$ (by [CT, Proposition 9.2] for the case char.$k = 0$, but the same proof holds for char.$k > 0$, by using Bruhat-Tits' Theorem [4, Section 4.7, Theorem (ii)]). Hence δ is surjective, where δ is the coboundary homomorphism $\delta : G(k) \to \mathrm{H}^1(k, F)$, related to the flasque resolution $1 \to F \to H_1 \to G \to 1$ of G. By Corollary 4.4, we have $H_1(k)/R = 1$, so δ induces a surjective homomorphism $\delta'' : G(k)/R \to \mathrm{H}^1(k, F)$ and we have the following diagram, which is obviously commutative
Here $\delta' : \mathrm{H}^1(k, F) \to G(k)/B_f$ is derived from the projection $G(k) \to G(k)/B_f$ and $\delta : G(k) \to \mathrm{H}^1(k, F)$, $\varphi = \delta' \circ \delta''$, ω is an isomorphism by Tate-Nakayama duality and $\omega = \gamma_G \circ \delta'$. Since γ_G and ω are injective, so is δ'. Since δ' is surjective, it follows that δ' is an isomorphism.

(b) $G(k)/R \simeq \mathrm{H}^1(k, F)$. For a co-flasque z-extension H of G as constructed in Subsection 3.1.4 and $T = H^{\mathrm{tor}}$, we have $T(k)/B_f \simeq H(k)/B_f \simeq G(k)/B_f$ by

$$\begin{tikzcd} G(k)/R \arrow[r, "\varphi"] \arrow[d, "\delta''"'] & G(k)/B_f \arrow[d, "\gamma_G"] \\ \mathrm{H}^1(k,F) \arrow[r, "\omega"] \arrow[ur, "\delta'"] & \mathrm{H}^1(k,\hat{F})^D. \end{tikzcd}$$

Theorem 3.5. By (3.20), F is also a flasque kernel for T, so by [8, Theorem 2], $T(k)/R \simeq \mathrm{H}^1(k, F)$. By ($a$) we also have an isomorphism $T(k)/B_f \simeq \mathrm{H}^1(k, F)$. Hence $T(k)/R \simeq T(k)/B_f \simeq \mathrm{H}^1(k, F)$. By Theorem 4.3 we have $H(k)/R \simeq T(k)/R$ and by [3, Corollary 4.16] we have $H(k)/R \simeq G(k)/R$, thus

$$\mathrm{H}^1(k, F) \simeq T(k)/R \simeq G(k)/R.$$

So finally we have $G(k)/R \simeq G(k)/B_f \simeq \mathrm{H}^1(k, F)$. □

Proof of Theorem 4.7. First we show that

(a) $RG_S = B_f G_S = BG_S = B_1 G_S$ $(= \mathscr{B}G_S = \mathscr{B}_1 G_S$ *if char.k* $= 0)$ *and they are open subgroups of* G_S.

The equalities follow from Lemma 4.9. Since G is connected and reductive, G is unirational over k, so by [42, Lemma 2.5], RG_S is an open subgroup of G_S. Hence (a).

(b) *We have the following equalities*

$$\overline{RG(k)} = RG_S = \overline{BG(k)} = BG_S = \overline{B_1G(k)} = B_1G_S = \overline{B_fG(k)} = B_fG_S \subseteq \overline{G(k)}$$

and if char.k $= 0$*, these subgroups are also equal to* $\overline{\mathscr{B}G(k)} = \mathscr{B}G_S = \overline{\mathscr{B}_1G(k)} = \mathscr{B}_1G_S$).
To prove these equalities, notice that we have the following obvious inclusions

$$\overline{RG(k)} \subseteq \overline{BG(k)} \subseteq \overline{B_1G(k)} \subseteq \overline{B_fG(k)} \subseteq \overline{G(k)}.$$

Therefore, according to (a), it suffices to prove that $RG_S = \overline{RG(k)}$. (The following argument corrects the wrong argument of the proof in the case of number fields (*cf.* [43, Proof of Theorem 4.22, page 287])). Remark that if $\tilde{G}$ is a semisimple simply connected k-group then we have $\overline{R\tilde{G}(k)} = \tilde{G}_S$.[8] Indeed, it is known that $R\tilde{G}(k)$ is a normal subgroup of $\tilde{G}(k)$ ([Gi1, Lemma II.1.1]), hence so is $\overline{R\tilde{G}(k)}$ in $\overline{\tilde{G}(k)} = \tilde{G}_S$, since the weak approximation holds for simply connected semisimple groups according to Kneser, Harder and Platonov [34, Chapter 7, Proposition 7.9] (*cf.* also Proposition 4.8). Therefore, by the Kneser–Tits conjecture for local fields,

[8] Conjecturally, we have a stronger identity: $R\tilde{G}(k) = \tilde{G}(k)$ for any simply connected semisimple k-group $\tilde{G}$, see Remarks 4.6(b).

we have $\overline{R\tilde{G}(k)} = \tilde{G}_S$. For any quasi-trivial reductive k-group H, with (simply connected) semisimple part H^{ss} and (induced) torus quotient $T = H/H^{ss}$, by Theorem 4.3 we have an exact sequence $H^{ss}(k)/R \to H(k)/R \to T(k)/R \to 1$. Since $T(k)/R = 1$, it implies that the including map $H^{ss}(k) \hookrightarrow H(k)$ induces a surjective homomorphism $H^{ss}(k)/R \to H(k)/R$, thus we have $H(k) = H^{ss}(k).RH(k)$. By Proposition 4.8, $H(k) = H^{ss}(k).RH(k)$ is dense in H_S. From above remark we have

$$\begin{aligned} H_S = \overline{H(k)} = \overline{H^{ss}(k).RH(k)} &\subseteq \overline{\overline{H^{ss}(k)}.RH(k)} \\ = \overline{\overline{RH^{ss}(k)}.RH(k)} &\subseteq \overline{\overline{RH(k)}.\overline{RH(k)}} = \overline{RH(k)}, \end{aligned}$$

so finally we have $H_S = \overline{RH(k)}$. Therefore, for a flasque resolution

$$1 \to F \to H \xrightarrow{\pi} G \to 1,$$

from above Lemma 4.9 and (a) we have

$$RG_S = B_f G_S = \pi(H_S) = \pi(\overline{RH(k)}) \subseteq \overline{\pi(RH(k))} \subseteq \overline{RG(k)}.$$

Thus $\overline{RG(k)} = RG_S$ and we are done. □

As applications of Lemma 4.9 and its proof, we derive the following corollary. The statement regarding the triviality of $G(k)/R$ was proved in [6, Corollary 4.11] for semisimple groups over fields k of characteristic 0, with $cd(k) \leq 2$, over which the index and exponent of 2-primary and 3-primary central simple k-algebras are equal (*cf.* also [33]).

Corollary 4.10. *Keep the notation as in Lemma 4.9. Let G_1, G_2 be connected reductive groups over a local field k, which are inner forms of each other. Then we have $G_1/R \simeq G_1(k)/B \simeq G_2(k)/B \simeq G_2/R$. In particular, for an inner form G of a connected reductive k-group, which is split over a metacyclic extension of k, we have $G(k)/R = G(k)/B_f = 1$.*

Proof. The first assertion follows from the following simple lemma, whose proof is omitted. □

Lemma 4.11. *Let k be a field, H_1 a connected reductive k-group, P a central k-subtorus of H_1 and*

$$1 \to P_1 \to H_1 \xrightarrow{p} G_1 \to 1 \tag{4.12}$$

an exact sequence. Assume that G_2 is an inner k-form of G_1. Then there are k-groups P_2, H_2 and an exact sequence

$$1 \to P_2 \to H_2 \to G_2 \to 1 \tag{4.13}$$

such that (4.13) *is obtained by* (4.12) *by an inner twisting. In particular, if* (4.12) *is a flasque resolution (respectively z-extension) for G_1, then so is* (4.13) *for G_2.*

To prove the last assertion, we may assume that k is a non-archimedean local field and that G is split over a metacyclic extension M/k. We may choose a maximal k-torus T_G of G such that T_G is split over M and that T_G contains a maximal k-split torus S of G. Consider the centralizer $Z_G(S)$ of S in G. Then $Z_G(S) = S_1 H_1$ (almost direct product), where S_1 is the connected center of $Z_G(S)$ and H_1 the semisimple part of $Z_G(S)$. It is known that H_1 is k-anisotropic, so we may assume, by [4, Section 4.6, page 696], that H_1 is the almost direct product of absolutely almost simple k-groups of type ^{1}A. According to Bruhat decomposition, G and $Z_G(S)$ are birationally k-equivalent, so by [8, Corolllary page 197], we have a bijection $G(k)/R \simeq Z_G(S)(k)/R$. Therefore we may assume that the semisimple part G^{ss} of G is k-anisotropic. Let $\tilde{G}$ be the simply connected k-covering of G^{ss}, $G = G^{ss}P$, where P is a k-torus. We have a central k-isogeny $1 \to F \xrightarrow{j} \tilde{G} \times P \to G \to 1$. Since $F \subset Z(\tilde{G})$ (the center of $\tilde{G}$) and $\tilde{G}$ is a direct product of absolutely almost simple k-groups of type ^{1}A, we may choose a k-embedding $i : F \hookrightarrow S_0$, where S_0 is a k-split torus and let $\varphi : F \hookrightarrow S_0 \times (\tilde{G} \times P)$, $f \mapsto (i(f), j(f^{-1}))$. We consider the k-group $H := (S_0 \times (\tilde{G} \times P))/\varphi(F)$. Then we have exact sequences $1 \to \tilde{G} \to H \to T \to 1$, $1 \to S_0 \to H \xrightarrow{\pi} G \to 1$, where the last one is a z-extension. Take a maximal k-torus T_H of H, which projects onto T_G via π and take a maximal k-torus $\tilde{T}$ of $\tilde{G}$ such that $T_H/\tilde{T} = T$. Since T_G is split over M, it is clear that so are T_H and $\tilde{T}$, thus also $T = T_H/\tilde{T}$. By [8, Corollary 3, page 200], Lemma 4.9 and Theorem 3.5 we have $1 = T(k)/R = T(k)/B_f = H(k)/B_f = G(k)/B_f = G(k)/R$. (Another argument goes as follows. Let G^{qs} the quasi-split k-inner form of G. Since G is split over M, the group G^{qs}, being the quasi-split inner form of the split F-group G over M, is itself M-split. We may choose a maximal k-torus T of G^{qs} such that T contains a maximal k-split torus of G^{qs} and T is also M-split. Since M is a metacyclic Galois extension of k, as above we have $T(k)/R = 1$. By the Bruhat decomposition, we know that (*cf.* [8, Proposition 14]) $G^{qs}(k)/R \simeq T(k)/R$ and from the previous part we know that $G(k)/R \simeq G^{qs}(k)/R$, hence $G(k)/R \simeq T(k)/R = 1$. By Lemma 4.9 we have $G(k)/B_f = G(k)/R = 1$.) $\square$

Remarks 4.12. (1) The proof of Subsection 4.2.4 gives another proof of [43, Proposition 4.4].

(2) We derive from the proof of Lemma 4.9 also a short proof of [12, Proposition III.2.7] (*i.e.*, Corollary 4.4 in the case of local fields). Let k be a local non-archimedean field, G a connected reductive k-group with special covering $1 \to \mu \xrightarrow{\alpha} G_1 \to G \to 1$, where $G_1 = \tilde{G} \times P$, $\tilde{G}$ is semisimple simply connected k-group and P is an induced k-torus. Consider a flasque resolution $1 \to \mu \xrightarrow{\beta} F \to E \to 1$ for μ. We need to show that $G(k)/R \simeq \mathrm{H}^1(k, F)$. We consider the embedding of μ into the direct product $G_1 \times F$ via $\varphi : m \mapsto (\alpha(m), \beta(m)^{-1})$ and let $H := (G_1 \times F)/\varphi(\mu)$. From these two exact sequences involving μ as above, we derive a commutative diagram with exact rows and columns, similar to (3.14). One can check that H is a quasi-trivial reductive k-group, hence

$1 \to F \to H \to G \to 1$ is a flasque resolution of G. Therefore by Lemma 4.9 we have $G(k)/R \simeq G(k)/B \simeq \mathrm{H}^1(k, F)$ as desired.

5. Some exact sequences relating arithmetic, geometric and cohomological invariants of connected groups

In this section we establish some exact sequences that connect the arithmetic invariant $\mathrm{A}(G)$, cohomological invariant $Ш^1(G)$, geometric invariants $Br_a(G)$, $G(k)/R$, $G(k)/B$ (where B stands for various Brauer equivalence relations considered in Subsection 3.1.1) and Subsection 3.1.3) of connected reductive groups G defined over an arbitrary global field k and their variants over the completions of k, which were established earlier by Colliot-Thélèneand Sansuc for algebraic tori. The main tool here is the use of weak Brauer equivalence relation in a comparison with R-equivalence relation and other Brauer equivalence relations.

5.1. An exact sequence relating weak Brauer equivalence and weak approximation

We recall the following local-global principle for Br-equivalences basically due to Manin and Tsfasman. Notice that in [29, page 53 and page 69], k is assumed perfect and global, thus k is a number field and in [2, Theorem 3.4], k is also a number field. The assertion was originally stated there for the Brauer Br-equivalence, but the proof (stated in [BK]) also holds for Br_1-equivalence (and $\mathscr{B}r$-, or $\mathscr{B}r_1$-equivalence if a smooth compactification exists) over any global field.

Theorem 5.1 (*cf.* [2,29])). *Let k be a global field. Then for any smooth k-variety X, there is the following injection $X(k)/B \hookrightarrow \prod_v X(k_v)/B$, where B stands for either Br, Br_1 (or $\mathscr{B}r$, $\mathscr{B}r_1$,* Pic *if char.$k = 0$).*

For any global field k, and for any connected reductive k-group G, we have a similar local-global result for the weak Brauer equivalence relation introduced in Subsection 3.1.3.

Proposition 5.2. *Let k be a global field. Then for any connected reductive k-group G, the natural homomorphism $G(k)/B_f \hookrightarrow \prod_v G(k_v)/B_f$ is injective.*

Proof. Let $1 \to F \to H \to G \to 1$ be a flasque resolution of G. Due to the functoriality, we have the following commutative diagram

$$\begin{array}{ccccc} G(k) & \times & \theta_G(\mathrm{H}^1(k, \hat{F})) & \overset{\cup}{\to} & Br(k) \\ \downarrow i_v & & \downarrow res_v & & \downarrow res_v \\ G(k_v) & \times & \theta_G(\mathrm{H}^1(k_v, \hat{F})) & \overset{\cup}{\to} & Br(k_v). \end{array}$$

Denote by $res_v : Br(k) \to Br(k_v)$ the localization map and by $\sim_{B_{f,v}}$ the weak Brauer equivalence over k_v. Let $g, g' \in G(k)$ such that $g \sim_{B_{f,v}} g'$ for all v. Then we have for any $\hat{f}_v \in \mathrm{H}^1(k_v\hat{F}))$,

$$(i_v(g), \theta_G(\hat{f}_v)) = (i_v(g'), \theta_G(\hat{f}_v)).$$

Let $\hat{f} \in \mathrm{H}^1(k, \hat{F})$ be any element. Then from the above commutative diagram we have

$$(i_v(g), res_v(\theta_G(\hat{f})) = (i_v(g'), res_v(\theta_G(\hat{f})))$$

thus also

$$res_v(g, \theta_G(\hat{f})) = res_v(g', \theta_G(\hat{f})).$$

This implies that the two elements (from $Br(k)$) $(g, \theta_G(\hat{f}))$ and $(g', \theta_G(\hat{f}))$ have the same restriction to $Br(k_v)$. By Brauer-Hasse-Noether Theorem, they are equal in $Br(k)$, thus also equal in $\theta_G(\mathrm{H}^1(k, \hat{F}))$. Therefore $g \sim_{B_f} g'$. □

The following result presents some relations between the local and global groups of weak Brauer equivalence classes and the obstruction to weak approximation. In the case of number fields, by using Theorem 3.6, we recover [43, Theorems 3.4, 3.7].

Theorem 5.3. *Let k be a global field, G a connected reductive k-group, S a finite set of places of k, $1 \to F \to H \to G \to 1$ a flasque resolution of G and let B stand for either B_f, Br, Br_1 (or $\mathscr{B}r$, $\mathscr{B}r_1$,* Pic *if char.$k = 0$).*

(a) *The following sequences are exact and functorial in G*

$$G(k)/B \overset{\gamma_{G,S}}{\to} \prod_{v\in S} G(k_v)/B \to \mathrm{A}(S, G) \to 1, \tag{5.1}$$

$$1 \to G(k)/B \overset{\gamma_G}{\to} \prod_{v} G(k_v)/B \to \mathrm{A}(G) \to 1, \tag{5.2}$$

$$1 \to G(k)/B \overset{\gamma_G}{\to} \prod_{v} \mathrm{H}^1(k_v, F) \to \mathrm{A}(G) \to 1. \tag{5.3}$$

In particular, if $x, y \in G(k)$, then $x \sim_B y \leftrightarrow x \sim_{B_v} y$ as elements in $G(k_v)$ for all v.

(b) *There is a natural commutative diagram*

$$\begin{array}{ccccccccc} 1 & \to & G(k)/B & \overset{\gamma_G}{\to} & \prod_v G(k_v)/B & \to & \mathrm{A}(G) & \to & 1 \\ & & \simeq \big\downarrow p & & \simeq \big\downarrow q & & \simeq \big\downarrow r & & \\ 1 & \to & T(k)/B & \overset{\gamma_T}{\to} & \prod_v T(k_v)/B & \to & \mathrm{A}(T) & \to & 1 \end{array}$$

where T is the torus quotient of any co-flasque z-extension H of G as in Subsection 3.1.4. *In particular, if G_1, G_2 are connected reductive k-groups, which are inner form of each other then $G_1(k)/B \simeq G_2(k)/B$ and if G is of inner type, then $G(k)/B = 1$.*

Proof. (a) Fix a flasque resolution $1 \to F \to H_1 \to G \to 1$ of G and use this to construct a co-flasque resolution $1 \to P_2 \to H \to G \to 1$ as in Subsection 3.1.4 or in the proof of Theorem 3.5. We consider only the case of weak Brauer equivalence B_f since the proof is similar in other cases. We have by definition

$$G(k)/B_f := G(k)/B_fG(k), \ \prod_{v\in S} G(k_v)/B_f := \prod_{v\in S} G(k_v)/B_fG_S.$$

By [42, Theorem 2.4] we have $RG_S \subseteq \overline{G(k)}$ (or one may use Theorem 4.7: we have $RG_S = B_fG_S = \overline{B_fG(k)} \subseteq \overline{G(k)}$). We know (see *e.g.* the proof of Theorem 4.7) that $RG_S = B_fG_S$ is an open normal subgroup of G_S. Thus $\overline{G(k)} = B_fG_SG(k)$ and we have the following exact sequence

$$G(k)/B_fG(k) \to G_S/B_fG_S \to G_S/B_sG_SG(k) \simeq G_S/\overline{G(k)} = \mathrm{A}(S, G)$$

thus (5.1) is an exact sequence. We know (*cf.* [36, Theorem 3.3] for number fields and [46, Theorem 2.3(2)] for global function fields) that for some finite (possibly empty) set S_0 of places of k, G has weak approximation with respect to any finite set S outside S_0. Thus if we take S sufficiently large to contain S_0, then $\mathrm{A}(G) = \mathrm{A}(S, G)$. Also, for almost all places v, by [8, Corollary page 205] and Lemma 4.9 we have $G(k_v)/B_f = G(k_v)/R = 1$. Therefore (5.2) follows from (5.1) and (5.3) follows from (5.2) and Lemma 4.9.

Due to Theorem 4.7 we have $\overline{B_1G(k)} = B_1G_S = RG_S$ (which is equal to the closure $\overline{\mathscr{B}G(k)} = \mathscr{B}G_S$ if G has a smooth k-compactification $\mathscr{G}$), thus in a similar way as above, the sequences (5.1) and (5.2) are also exact, where B_f is replaced by Br_1 or Br. The functoriality follows from Proposition 3.4 (b) and the last exact sequence (5.3) follows from the isomorphism $G(k_v)/R \simeq \mathrm{H}^1(k_v, F)$ (see Lemma 4.9).

(b) By functoriality we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & T(k)/B_f & \overset{\gamma_T}{\to} & \prod_v T(k_v)/B_f & \to & \mathrm{A}(T) & \to & 1 \\
 & & \simeq \uparrow p_1 & & \simeq \uparrow q_1 & & \simeq \uparrow r_1 & & \\
1 & \to & H(k)/B_f & \overset{\gamma_H}{\to} & \prod_v H(k_v)/B_f & \to & \mathrm{A}(H) & \to & 1 \\
 & & \simeq \downarrow p_2 & & \simeq \downarrow q_2 & & \simeq \downarrow r_2 & & \\
1 & \to & G(k)/B_f & \overset{\gamma_G}{\to} & \prod_v G(k_v)/B_f & \to & \mathrm{A}(G) & \to & 1.
\end{array}$$

By [43, Lemma 3.8] (characteristic 0 case) and [46, Theorem 2.3] (characteristic $p > 0$ case), r_1, r_2 are isomorphisms. By Theorem 3.5, q_1, q_2 and p_2 are isomorphisms. Therefore the homomorphism p_1 is also an isomorphism, due to the commutativity of the above diagram. The last assertions follow directly from the above. $\square$

We deduce immediately:

Corollary 5.4. *Keep the notation as in Theorem* 5.3.

(1) *If* $G(k_v)/B = 1$ *for all* $v \in S$ *(respectively all* v*) then* $\mathrm{A}(S, G) = 1$ *(respectively* $\mathrm{A}(G) = 1$*);*
(2) *If* G *is such that* $\mathrm{A}(G) \neq \{1\}$*, then for some* v*, we have* $G(k_v)/B \neq \{1\}$*. In particular, for such* k_v*,* G *is not a direct factor of a* k_v*-rational variety over* k_v*, thus neither over* k*.*

Proof. (1) and the first assertion of (2) are trivial. By Lemma 4.9 we have $G(k_v)/B \simeq G(k_v)/R$, so $G(k_v)/R \neq 1$. Recall that according to Grothendieck, G is unirational as a variety over k. By the additivity of the set $X(K)/R$ of R-equivalence classes (see [8, page 195]) and the birational invariance of $G(K)/R$ (as a set) among the connected unirational linear algebraic groups defined over an infinite field K (see [8, Corollary page 197]) G cannot be a direct factor of a k_v-rational variety over k_v, thus neither over k. □

Remark 5.5. Let X be a variety defined over a global field k, and let $B_\omega(X)$ be the set of all elements of $Br_a(X)$ which have trivial images locally for almost all $v \in V$, with respect to natural homomorphisms $Br_a(X) \to Br_a(X \times_k k_v)$. In [36, Corollary 9.4, Theorem 9.5 and their proofs], it was shown that if k is a number field (and in [46, Theorem 3.9] if k is a global function field) then for any connected reductive k-group G, there is the following exact sequence[9]

$$1 \to \mathrm{A}(G) \to B_\omega(G)^D \to Ш^1(G) \to 1, \tag{5.4}$$

which, for a smooth compactification $\mathcal{G}$ of G, is isomorphic to

$$1 \to \mathrm{A}(G) \to \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s))^D \to Ш^1(G) \to 1, \tag{5.5}$$

the sequence established by Voskresenskii [48, Theorem 11.6] for tori. One may use this sequence to get an example of a connected k-group which is non-rational over k. Namely, if $\mathrm{A}(G) \neq 1$ or $Ш^1(G) \neq 1$, then $\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \neq 1$, so G is not a direct factor of a k-rational variety (and a fortiori, G is not stably birationally trivial over k). But it a priori does not imply that G (considered as a variety over k_v) is not a direct factor of a k_v-rational variety for some v, which is a stronger fact, as it was asserted in Corollary 5.4.

5.2. Comparison of weak Brauer and other Brauer equivalence relations over global fields

Consider a global field k and a connected reductive k-group G. We show in this section that the weak Brauer equivalence, Br-equivalence and Br_1-equivalence are the

[9] One can remove the assumption that G has no factors of type E_8 in [36, Theorem 9.5] due to the validity of the Hasse principle in this case (see [34, Theorem 6.6, page 286]).

same on $G(k)$, which also coincide with the usual Brauer equivalence if a smooth k-compactification $\mathscr{G}$ exists.

Theorem 5.6. *Let k be a global field and let G be a connected reductive k-group. Then the weak Brauer equivalence B_f, the Br-equivalence and the Br_1-equivalence on $G(k)$ are the same. In particular, all assertions regarding B_f also hold if B_f is replaced by Br_1 or Br (and if char.$k = 0$, $\mathscr{B}r$, $\mathscr{B}r_1$ or* Pic *with respect to any smooth k-compactification $\mathscr{G}$ of G).*

Proof. Let B stand for either Br or Br_1. By Subsection 3.1.3, we know that B_f-equivalence is coarser than the Br-equivalence and Br_1-equivalence so the (well-defined) natural maps α, β given below are surjective by (3.6) and (4.1c). We have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & G(k)/B & \overset{\gamma_G}{\to} & \prod_v G(k_v)/B & \to & \mathrm{A}(G) & \to & 1 \\ & & \big\downarrow \alpha & & \simeq \big\downarrow \beta & & \big\downarrow id & & \\ 1 & \to & G(k)/B_f & \overset{\gamma_G}{\to} & \prod_v G(k_v)/B_f & \to & \mathrm{A}(G) & \to & 1 \end{array}$$

where the two rows are exact due to Theorem 5.1 and Theorem 5.3. Also, by Lemma 4.9, β is an isomorphism. Therefore α is also an isomorphism.

If there exists a smooth compactification $\mathscr{G}$ of G (*e.g.* if char.$k = 0$), then the same argument also works, where B now stands either for $\mathscr{B}r$, $\mathscr{B}r_1$, or Pic and we apply the surjective maps given in (4.1d) and (4.1e) again to get the surjectivity of α. ☐

Remark 5.7. (1) By Theorem 4.2, the Abelian group $G(k)/\mathscr{B}r$ is a stably birational invariant in the class of connected linear algebraic groups over a field k of characteristic 0. Here is a natural question: If G is a connected reductive k-group, is $G(k)/B_f$ also a stably birational invariant if char.$k > 0$?

(2) We do not know any examples of a field k and a connected reductive k-group G where one of the following holds:

(a) $G(k)/Br_1 \neq G(k)/B_f$;
(b) all the groups $G(k)/R, G(k)/Br_1, G(k)/B_f$ are distinct.

5.3. Some exact sequences relating weak Brauer and R-equivalence classes with Tate-Shafarevich group and defect of weak approximation

Our aim in this section is to consider some extensions of exact sequences relating the weak approximation obstruction to the group of R-equivalence classes, first established by Colliot-Thélène and Sansuc for tori, to connected reductive groups over global fields.

By Lemma 4.9, if k is a local field, then for any connected reductive k-group G, we have $G(k)/R \simeq G(k)/B_f$. It is natural to ask if it is so if k is a global field. More generally, we ask what the difference between $G(k)/B_f$ and $G(k)/R$

may be. First, in the case of tori T, we have the following exact sequence giving the difference between $G(k)/R$ and $G(k)/B_f$. (Recall that by Theorem 3.6, if char.$k = 0$, the weak Brauer equivalence coincides with the usual one, so we recover [43, Proposition 2.6] for Brauer equivalence over number fields.) For a commutative k-group scheme A, we denote $\text{Ч}^i(A) :=$ Coker $[\mathrm{H}^i_{\mathrm{fppf}}(k, A) \to \prod_v \mathrm{H}^i_{\mathrm{fppf}}(k_v, A)]$.

Proposition 5.8. *Let T be a torus over a global field k, $1 \to F \to P \to T \to 1$ a flasque resolution of T over k. Then the following sequence is exact*

$$1 \to \text{Ш}^1(F) \to T(k)/R \to T(k)/B_f \to 1. \tag{5.6}$$

In particular, we have $\mathrm{Card}(T(k)/B_f) = [T(k)/R : \text{Ш}^1(F)]$ *and if* $\text{Ш}^1(F) \neq 1$, *then* $T(k)/R \not\simeq T(k)/B_f$.

Proof. We have the following exact sequence

(*cf.* [8, Proposition 19(iR)]) *Let k be a global field, T a k-torus, F a flasque kernel of T. Then the following sequence is exact*

$$1 \to \text{Ш}^1(F) \to T(k)/R \to \prod_v T(k_v)/R \to \mathrm{A}(T) \to 1. \tag{5.7}$$

This was proved in [8] in the case of number fields, but it also holds in the case of any global field. Indeed, on the one hand, by [8, Theorem 2], we have $T(k)/R \simeq \mathrm{H}^1(k, F)$ and for any place v of k, $T(k_v)/R \simeq \mathrm{H}^1(k_v, F)$. On the other hand, by definition we have the following exact sequence

$$1 \to \text{Ш}^1(F) \to \mathrm{H}^1(k, F) \xrightarrow{\alpha} \prod_v \mathrm{H}^1(k_v, F) \to \text{Ч}^1(F) \to 0, \tag{5.8}$$

and by [5, Theorem 9.4(i)] (for number fields) and by [46, Theorem 2.3(1)] (for global function fields), we have $\mathrm{A}(T) \simeq \text{Ч}^1(F)$. Hence (5.7) holds. From this exact sequence, the same proof of [43, Proposition 2.6], combined with the exact sequence of Theorem 5.3(a) gives us the result. □

Remark 5.9. Note that over a global field k, the (weak) Brauer and R-equivalence relations are different in general, *i.e.*, in above proposition, $\text{Ш}^1(S)$ may be non-trivial, or the same, the local-global principle may not hold for R-equivalence, both for tori and their compactifications. For example, in the case $k = \mathbb{Q}$, an example of a 7-dimensional k-torus T with its flasque kernel S satisfying $\text{Ш}^1(S) \simeq \mathbb{Z}/2\mathbb{Z}$ was given in [8, Remarque R12, page 224].

Our next aim is to generalize the above exact sequences to connected reductive groups overt global fields. The sequences for tori established in [8, Proposition 19(iR)] lead to the following:

Question. *Do the sequences* (5.6) *and* (5.7) *hold for more general classes of fields and algebraic groups?*

We have the following almost complete answer to this question, which was given in [42] when char.$k = 0$ but the proof also holds in the case char.$k > 0$.

Theorem 5.10. **(*cf.* [42, Theorems 2.7, 2.8]).** *Let k be a global field, S a finite set of valuations of k, and let G be a connected reductive k-group. Then we have the following exact sequences of finite Abelian groups*

$$G(k)/R \overset{\rho_{G,S}}{\to} \prod_{v \in S} G(k_v)/R \to \mathrm{A}(S, G) \to 1, \tag{5.9}$$

$$G(k)/R \overset{\rho_G}{\to} \prod_{v} G(k_v)/R \to \mathrm{A}(G) \to 1. \tag{5.10}$$

Notice that (5.9) has been extended by Colliot-Thélène, Gille and Parimala to the case of fields of geometric type (gl) or (ll).

Let $1 \to F \to H_1 \to G \to 1$ be a flasque resolution of G,

$$1 \to Z \to H \to G \to 1$$

a z-extension co-flasque resolution of G, $T = H^{\mathrm{tor}}$. Then by (3.20), F is also a flasque kernel for T. We derive the following theorem, which makes precise the formulation of [43, Theorem 4.11]. Namely, the proof given there in reality is based on the assumption that the torus S there is a flasque kernel of T (instead of being the "Néron-Severi torus" of T as in the formulation of [43, Theorem 4.11]). In fact, only after we prove Theorem 5.11 below, we may assert that S can also be replaced by the "Néron-Severi torus" of T. The corresponding required statements are given in 5.11(3) below. Also, when $n = 1$, k is a number field and G is semisimple, the isomorphism $\mathrm{H}^1(L, S) \simeq \mathrm{H}^1(L, S_{\mathscr{G}})$ was proved in [12, Proof of Theorem III.4.3]. Here is the correct formulation of [43, Theorem 4.11].

Theorem 5.11. *Let k be a global field, G a connected reductive k-group and other notation be as above.*

(1) *For the following exact sequences*

$$1 \to \mathrm{Ker}\,(\rho_G) \to G(k)/R \overset{\rho_G}{\to} \prod_{v} G(k_v)/R \to \mathrm{A}(G) \to 1,$$

$$1 \to \mathrm{Ker}\,(\varphi_G) \to G(k)/R \overset{\varphi_G}{\to} G(k)/B_f \to 1,$$

we have $\mathrm{Ker}\,(\rho_G) \simeq \mathrm{Ker}\,(\varphi_G)$ *and* $\mathrm{Im}\,(\rho_G) \simeq \mathrm{Im}\,(\varphi_G)$. *In particular, a local-global principle holds for R-equivalence relation if and only if the R-equivalence and weak Brauer equivalence coincide on $G(k)$.*

(2) *We have* $[G(k)/R : \mathrm{Ker}\,(\rho_G)] = [T(k)/R : Ш^1(F)] = [\mathrm{H}^1(k, F) : Ш^1(F)]$.

(3) *With above notation, let $\mathscr{T}$ (respectively $\mathscr{G}$) be a smooth k-compactification of T (respectively G), $S_{\mathscr{T}}$ (respectively $S_{\mathscr{G}}$) the Néron–Severi torus for $\mathscr{T}$ (respectively $\mathscr{G}$). Then for any $n \geq 1$ and any field extension L/k, we have*

$$\mathrm{H}^n(L, F) \simeq \mathrm{H}^n(L, S_{\mathscr{T}}) \simeq \mathrm{H}^n(L, S_{\mathscr{G}}),$$

$$\mathrm{H}^n(L, \hat{F}) \simeq \mathrm{H}^n(L, \hat{S}_{\mathscr{T}}) \simeq \mathrm{H}^n(L, \hat{S}_{\mathscr{G}}) \simeq \mathrm{H}^n(L, \mathrm{Pic}(\mathscr{T}_s)) \simeq \mathrm{H}^n(L, \mathrm{Pic}(\mathscr{G}_s)).$$

In particular, we have Ker $(\rho_T) \simeq Ш^1(F) \simeq Ш^1(S_{\mathcal{T}}) \simeq Ш^1(S_{\mathcal{G}})$. *If* G_1, G_2 *are connected reductive k-groups, which are inner form of each other, then for two smooth compactifications* $\mathcal{G}_1, \mathcal{G}_2$ *of* G_1, G_2*, respectively, we have* $\mathrm{H}^n(L, \hat{S}_{\mathcal{G}_1}) \simeq \mathrm{H}^n(L, \hat{S}_{\mathcal{G}_2})$.

Proof. (1) By Theorem 5.3 we have the following commutative diagram with exact lines

$$\begin{array}{ccccccc} 1 \to \mathrm{Ker}\ (\rho_G) & \to & G(k)/R & \overset{\rho_G}{\to} & \prod_v G(k_v)/R & \to & \mathrm{A}(G) \to 1 \\ \downarrow & & \downarrow \varphi_G & & \downarrow\simeq & & \downarrow= \\ 1 & \to & G(k)/B_f & \overset{\rho_G}{\to} & \prod_v G(k_v)/B_f & \to & \mathrm{A}(G) \to 1 \end{array}$$

where the middle isomorphism follows from Lemma 4.9. From this we derive that Ker $(\varphi_G) \simeq$ Ker (ρ_G). The image of $G(k)/R$ in $\prod_v G(k_v)/R$ therefore is isomorphic to the image of $G(k)/B_f$ in $\prod_v G(k_v)/B_f$, which is also isomorphic to the image of $T(k)/B_f$ in $\prod_v T(k_v)/B_f$, where $T = H^{\mathrm{tor}}$ and H is a co-flasque z-extension of G (see Theorem 5.3(b)).

(2) By (1), Theorem 5.3(b) and Proposition 5.8, we have

$$\begin{aligned} [G(k)/R : \mathrm{Ker}\ (\rho_G)] &= [G(k)/R : \mathrm{Ker}\ (\varphi_G)] = \mathrm{Card}(G(k)/B_f) \\ &= \mathrm{Card}(T(k)/B_f) = [T(k)/R : \mathrm{Ker}\ (\rho_T)] \\ &= [T(k)/R : \mathrm{Ker}\ (\varphi_T)] = [T(k)/R : Ш^1(F)]. \end{aligned}$$

By 3.1.5.8, F is a flasque resolution of T and by [8, Theorem 2], we have $T(k)/R \simeq \mathrm{H}^1(k, F)$, hence (2) follows.

(3) By applying [5, Proposition 6.2] and [3, Theorem 3.21] to a flasque resolution $1 \to F \to H \to G \to 1$ of G and to $S_{\mathcal{G}}$, the Néron-Severi torus of $\mathcal{G}$ (notice that the proof of [3, Theorem 3.21] also works (verbatim !) in the case of positive characteristic), we have $F \times P_1 \simeq S_{\mathcal{G}} \times P_2$ for some induced k-tori P_1, P_2. The same also applies when G is replaced by T, which was first mentioned in [8, page 221]. It shows that for $n \geq 1$, we have $\mathrm{H}^n(L, F) \simeq \mathrm{H}^n(L, S_{\mathcal{T}}) \simeq \mathrm{H}^n(L, S_{\mathcal{G}})$ and similarly we have for $n \geq 1$ $\mathrm{H}^n(L, \hat{F}) \simeq \mathrm{H}^n(k, \hat{S}_{\mathcal{T}}) \simeq \mathrm{H}^n(L, \hat{S}_{\mathcal{G}})$. Therefore by applying above isomorphisms to $L = k$ and $L = k_v$, we have in particular Ker $(\rho_T) \simeq Ш^1(F) \simeq Ш^1(S_{\mathcal{T}}) \simeq Ш^1(S_{\mathcal{G}})$. If G_1, G_2 are inner form of each other, then it is clear that they share the same torus T, so the last assertion follows. ☐

Remark 5.12. If there exists a smooth k-compactification $\mathcal{G}$ of G, and k is a local field, then Lemma 4.9 combined with Theorem 5.11 says that we have $G(k)/R \simeq G(k)/Br \simeq \mathrm{H}^1(k, F) \simeq \mathrm{H}^1(k, \hat{F})^D \simeq \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s))^D$, which recovers [12, Theorem III.4.3(a)] (stated for semisimple groups over $\mathfrak{p}$-adic fields) and also [44, Theorem 2.1] (stated for connected linear algebraic groups over $\mathfrak{p}$-adic fields) and also extends [8, Corollary 1, page 217].

Definition (*cf.* [19, page 173] for the semisimple case). A central isogeny $f : H \to G$ of connected reductive groups, where H is a quasi-trivial group, all defined over a field k, is called *normal* if Ker (f) can be embedded into an induced k-torus E such that $E/$Ker (f) is also induced.

We have the following extension of some classical results (*cf.* [19, Satz 2.2], [20, Satz 2.2.4], [36, Corollary 5.4], [43, Theorem 4.6] for number fields and [6, Corollary 4.14] for semisimple groups over fields of type (gl), (ll).)

Corollary 5.13. *Let k be a global field, S a finite subset of V_k, G a connected reductive k-group. The following groups have weak approximation over k with respect to S:*

(a) *The groups which are images of normal isogenies;*
(b) *Absolutely almost simple k-groups;*
(c) *Inner forms of a connected reductive k-group G, which has weak approximation over k with respect to S. In particular, such groups G are those, which considered as a k_v-group, are split over a metacyclic extension of k_v, for all places $v \in S$.*

Proof. (a) The proof below follows an argument used in [20, Satz 2.2.4]. Let $f : H \to G$, be a normal isogeny, where H is a quasi-trivial k-group. Let $i : K := \mathrm{Ker}\,(f) \to H$ be the inclusion. Let there be an embedding of $j : K \to E$ into an induced k-torus E such that E/K is also an induced k-torus. The embedding $K \hookrightarrow H$ and $K \hookrightarrow E$ defines a commutative diagram similar to that of (3.14) and in particular, we have an embedding $H \hookrightarrow Q$, where $Q := H \times E/\varphi(K)$, where $\varphi : K \to H \times E, k \mapsto (i(k), j(k^{-1})$. We have the following commutative diagram with exact lines

$$\begin{array}{ccccccccc}
1 \to & H(k) & \to & Q(k) & \to & (E/K)(k) & \to & \mathrm{H}^1(k, H) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 \to & \prod_{v\in S} H(k_v) & \to & \prod_{v\in S} Q(k_v) & \to & \prod_{v\in S}(E/K)(k_v) & \to & \prod_{v\in S} \mathrm{H}^1(k_v, H).
\end{array}$$

We may enlarge S so as to include all archimedean places of k. If char.$k = 0$, then by [5, Proposition 9.2], s is a bijection (Hasse principle for quasi-trivial reductive groups) and H has weak approximation over k, thus also in S. The same is true if char.$k > 0$, by using [46, Proof of Proposition 2.2] (for weak approximation) and by using Harder's Theorem [21, Satz A] (for the Hasse principle for simply connected groups in char.$k > 0$). Also, E/K is an induced k-torus, so it has weak approximation property over k. Now a standard argument shows that Q has weak approximation over k, hence so does G, since we have the exact sequence

$$1 \to S \to Q \to G \to 1$$

and the projection $Q(L) \to G(L)$ is surjective over L-points for any extension L/k.

(b) The same argument as in [36, Proof of Corollary 5.6] shows that (a) implies (b).

(c) According to Corollary 4.10, Theorem 5.3 and Theorem 5.10, any connected reductive k-group G satisfying the condition of (c), has weak approximation over k. □

As shown in Theorem 5.11, to know $G(k)/R$ one may try to determine the group Ker (ρ_G) above. We have the following theorem, which extends Proposition 5.8 to the case of connected reductive groups and also extends [43, Theorem 4.12] to the case of global function fields. The proof given below also simplifies and corrects the proof of [43, Theorem 4.12].

Theorem 5.14. *Let k be a global field, G a connected reductive k-group and let notation be as in Theorem 5.11. Denote by $\tilde{G}$ the simply connected covering of the semisimple part of G.*

(1) *We have the following commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc} & & \tilde{G}(k)/R & \xrightarrow{\sim} & \tilde{G}(k)/R & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & \mathrm{Ker}\,(\rho_G) & \to & G(k)/R & \xrightarrow{\varphi_G} & G(k)/B_f & \to & 1 \\ & & \downarrow p & & \downarrow q & & r \downarrow \simeq & & \\ 1 & \to & Ш^1(S_{\mathscr{T}}) & \to & T(k)/R & \to & T(k)/B_f & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{5.11}$$

where T is the torus quotient of any co-flasque z-extension H of G, $\mathscr{T}$ is a smooth k-compactification of T, $S_{\mathscr{T}}$ is the Néron–Severi torus for $\mathscr{T}$.

(2) *If $\tilde{G}(k)/R = 1$ (which is the case, if either k has no real places or G has no anisotropic almost simple factors of type* E_6*), then $G(k)/R \simeq T(k)/R$ and we have the following exact sequence*

$$1 \to Ш^1(S_{\mathscr{T}}) \to G(k)/R \xrightarrow{\rho_G} \prod_v G(k_v)/R \to \mathrm{A}(G) \to 1, \tag{5.12}$$

which is functorial in G.

(3) *Under the assumption of* (2)*, if G has a smooth k-compactification $\mathscr{G}$, with the Néron–Severi torus $S_{\mathscr{G}}$, then we have the following Colliot-Thélène–Sansuc exact sequence*

$$1 \to Ш^1(S_{\mathscr{G}}) \to G(k)/R \xrightarrow{\rho_G} \prod_v G(k_v)/R \to \mathrm{A}(G) \to 1, \tag{5.13}$$

which is functorial in G and does not depend on the choice of the smooth compactification $\mathscr{G}$.

(4) *Under the assumption of* (2) *and the notation of* (3)*, we have the following exact sequence*

$$1 \to Ш^1(S_{\mathscr{G}}) \to G(k)/R \to G(k)/B_f \to 1, \tag{5.14}$$

which is functorial in G and does not depend on the choice of the smooth compactification $\mathscr{G}$.

(5) *With notation as in* (3)*, we have the following commutative diagram with exact rows and the vertical maps are isomorphisms*

$$\begin{array}{ccccccccc} 1 & \to & Ш^1(S_{\mathscr{G}}) & \to & G(k)/R & \overset{\varphi_G}{\to} & G(k)/B_f & \to & 1 \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 1 & \to & Ш^1(S_{\mathscr{T}}) & \to & T(k)/R & \overset{\varphi_T}{\to} & T(k)/B_f & \to & 1. \end{array}$$

Proof. If H is a z-extension, then we have an isomorphism $H(L)/R \simeq G(L)/R$ (see [3, Corollary 4.16]) and $H(L)/B_f \simeq G(L)/B_f$ (see Corollary 3.2.2), for any field extension L/k. Also, by [3, Lemma 5.5] or [42, Lemma 2.1], we have $\mathrm{A}(G) \simeq \mathrm{A}(H)$. If $\mathscr{G}$, $\mathscr{H}$ are smooth compactifications of G, H, respectively, then they are stably birationally k-equivalent, since so are G and H. Then according to [9, Proposition 2A1, Appendix 2A] (see also [46, Proof of Theorem 3.7.1]), the Picard groups of $\mathscr{G}$ and $\mathscr{H}$ are isomorphic up to a permutation summand, thus their Néron–Severi tori have isomorphic Galois cohomology in degree 1. In particular, $Ш^1(S_{\mathscr{G}}) \simeq Ш^1(S_{\mathscr{H}})$. Thus we may and will assume that the semisimple part of G is simply connected and we set $\tilde{G} := G^{ss}$.

(1) By the functoriality, from the compositions (of natural homomorphisms)

$$q : G(k)/R \simeq H(k)/R \to T(k)/R,$$
$$r : G(k)/B_f \simeq H(k)/B_f \to T(k)/B_f,$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & \mathrm{Ker}\ (p) & \overset{s}{\to} & \mathrm{Ker}\ (q) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & \mathrm{Ker}\ (\varphi_G) & \to & G(k)/R & \overset{\varphi_G}{\to} & G(k)/B_f & \to & 1 \\ & & \downarrow p & & \downarrow q & & r\downarrow & & \\ 1 & \to & \mathrm{Ker}\ (\varphi_T) & \to & T(k)/R & \overset{\varphi_T}{\to} & T(k)/B_f & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1. & & & & \end{array} \tag{5.15}$$

By Theorem 3.5, in the diagram (5.15), r is an isomorphism and by Theorem 4.3, Ker $(q) \simeq \tilde{G}(k)/R$ and q is surjective. It clearly implies that so is p, and the diagram chase shows that s is an isomorphism, so Ker $(p) \simeq \tilde{G}(k)/R$. Let F be a flasque kernel of G. By (3.21), we know that F is also a flasque kernel of T. Then by Proposition 5.8, we have Ker $(\varphi_T) \simeq Ш^1(F)$ and by Theorem 5.11 (2), we have $Ш^1(F) \simeq Ш^1(S_{\mathscr{T}})$, where $\mathscr{T}$ is a smooth compactification of T (which exists by [CTHS, Corollary 1]). Also, by Theorem 5.11 (2), we have Ker $(\varphi_G) \simeq$ Ker (ρ_G), so (1) is proven.

(2) Assume that $\tilde{G}(k)/R = 1$. From the proof of Theorem 4.3 it implies that we have $G(k)/R \simeq T(k)/R$, $G(k_v)/R \simeq T(k_v)/R$ for all places v of k and from [43, Lemma 3.8] (number field case) and [46, Theorem 2.3] (global function field case),

we have $\mathrm{A}(G) \simeq \mathrm{A}(T)$, thus also the following commutative diagram where the second row is exact (Proposition 5.8) and all vertical maps are isomorphisms (see above)

$$\begin{array}{ccccccccc} 1 \to & Ш^1(S_{\mathcal{T}}) & \to & G(k)/R & \overset{\rho_G}{\to} & \prod_v G(k_v)/R & \to & \mathrm{A}(G) & \to 1 \\ & \downarrow = & & \simeq\downarrow q & & \simeq\downarrow q' & & \downarrow\simeq & \\ 1 \to & Ш^1(S_{\mathcal{T}}) & \to & T(k)/R & \overset{\rho_T}{\to} & \prod_v T(k_v)/R & \to & \mathrm{A}(T) & \to 1. \end{array}$$

From this, the exactness of the first row also follows. To prove the functoriality in G, consider connected reductive k-groups G_1, G_2 with $\tilde{G}_1(k)/R = \tilde{G}_2(k)/R = 1$ and let $f : G_1 \to G_2$ be a k-morphism. We have the following natural commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 \to & \mathrm{Ker}\,(\rho_{G_1}) & \to & G_1(k)/R & \overset{\rho_{G_1}}{\to} & \prod_v G_1(k_v)/R & \to & \mathrm{A}(G_1) & \to 1 \\ & \downarrow r_R & & \downarrow f_R & & \downarrow f''_R & & \downarrow f'''_R & \\ 1 \to & \mathrm{Ker}\,(\rho_{G_2}) & \to & G_2(k)/R & \overset{\rho_{G_2}}{\to} & \prod_v G_2(k_v)/R & \to & \mathrm{A}(G_2) & \to 1. \end{array}$$

By (1) and Theorem 5.11, for $i = 1, 2$, we have the following isomorphisms, p_i : $\mathrm{Ker}\,(\rho_{G_i}) \simeq Ш^1(S_{\mathcal{T}_i})$, where $T_i := H_i/[H_i, H_i]$ and H_i is a co-flasque resolution z-extension of G_i as above and $\mathcal{T}_i$ a smooth compactification of T_i. Thus we have also a commutative diagram

$$\begin{array}{ccccccccc} 1 \to & Ш^1(S_{\mathcal{T}_1}) & \to & G_1(k)/R & \overset{\rho_{G_1}}{\to} & \prod_v G_1(k_v)/R & \to & \mathrm{A}(G_1) & \to 1 \\ & \downarrow s_R & & \downarrow f_R & & \downarrow f''_R & & \downarrow f'''_R & \\ 1 \to & Ш^1(S_{\mathcal{T}_2}) & \to & G_2(k)/R & \overset{\rho_{G_2}}{\to} & \prod_v G_2(k_v)/R & \to & \mathrm{A}(G_2) & \to 1 \end{array}$$

where s_R comes from r_R. Recall that by Corollary 4.5, $\tilde{G}(k)/R = 1$ if k has no real places or $\tilde{G}$ has no anisotropic factors of type E_6. Hence (5.12) holds.

(3) Follows from (2) above and Theorem 5.11(3). The last statement follows from the fact that if we consider another smooth compactification $\mathcal{G}$ of G, there are permutation Γ-modules P, P' such that $\mathrm{Pic}(\mathcal{G}_s) \oplus P \simeq \mathrm{Pic}(\mathcal{G}_s) \oplus P'$, according to [9, Proposition 2A1, Appendix 2A]. Then clearly we have $\mathrm{H}^n(L, S_{\mathcal{G}}) \simeq \mathrm{H}^n(L, S_{\mathcal{G}})$ for any field extension L/k and in particular, $Ш^1(S_{\mathcal{G}}) \simeq Ш^1(S_{\mathcal{G}})$. From this we obtain (5.13).

(4) Follows from (1) and Theorem 5.11.

(5) The isomorphism of the two exact sequences follows from the isomorphism between the corresponding sequences for G and H

$$\begin{array}{ccccccc} 1 \to & Ш^1(S_{\mathcal{G}}) & \to & G(k)/R & \overset{\varphi_G}{\to} & G(k)/B_f & \to 1 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\ 1 \to & Ш^1(S_{\mathcal{H}}) & \to & H(k)/R & \overset{\varphi_H}{\to} & H(k)/B_f & \to 1 \end{array}$$

which follows from $Ш^1(S_{\mathscr{G}}) \simeq Ш^1(S_{\mathscr{H}})$ (above), $G(k)/R \simeq H(k)/R$ [3, Corollary 4.16], $G(k)/B_f \simeq H(k)/B_f$ (Theorem 3.5 (a)), and the following isomorphism between the corresponding sequences for H and T

$$\begin{array}{ccccccccc} 1 & \to & Ш^1(S_{\mathscr{H}}) & \to & H(k)/R & \xrightarrow{\varphi_H} & H(k)/B_f & \to & 1 \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 1 & \to & Ш^1(S_{\mathscr{T}}) & \to & T(k)/R & \xrightarrow{\varphi_T} & T(k)/B_f & \to & 1 \end{array}$$

which follows from Theorems 5.14, 4.3 and 3.5. □

We derive immediately the following:

Corollary 5.15. *Let notation be as above, k a global field and let G_1, G_2 be connected reductive k-groups which are inner form of each other. Assume that $\tilde{G}_1(k)/R = \tilde{G}_2(k)/R = 1$. Then we have $G_1(k)/R \simeq G_2(k)/R$.*

Proof. By assumption, G_1 and G_2 share the same (up to k-isomorphism) torus T. Thus by Theorem 5.14, we have $G_1(k)/R \simeq T(k)/R \simeq G_2(k)/R$. □

5.4. Tate-Shafarevich kernel, Brauer groups and R-equivalence

In [8, Proposition 19(ii)], there has been established yet another exact sequence connecting various cohomological, geometric invariants and R-equivalence class group for tori. Namely, if k is a number field, $\mathscr{T}$ a smooth k-compactification of a k-torus T, then the following sequence is exact

$$1 \to Ш^1(T)^D \to Br_a(\mathscr{T}) \xrightarrow{\mu} \prod_v Br_a(\mathscr{T}_v) \xrightarrow{\tau} (T(k)/R)^D \to Ш^1(S_{\mathscr{T}})^D \to 1. \quad (*)$$

Question. *Does the exact sequence $(*)$ hold for more general class of fields and connected algebraic groups? Does it hold for any global field k and connected reductive k-group G?*

In this section we aim to answer this question. Also, in the spirit of [8, Proposition 19], we combine the two exact sequences proven so far to extend our exact sequence (5.6) and the exact sequence of [8, Proposition 19(ii)] (established for tori) to the case of connected reductive groups over global fields, which connecting Tate-Shafarevich kernels, Brauer groups and the groups of weak Brauer and R-equivalence classes.

Theorem 5.16. **(*cf.* [8, Proposition 19(ii)] for tori).** *Let k be a global field, G a connected reductive k-group. Let $\tilde{G}$ be the simply connected covering of G^{ss}, and fix a flasque resolution $1 \to F \to H_1 \to G \to 1$ of G and let T be the torus quotient of any co-flasque z-extension H of G. Assume that $\tilde{G}(k)/R = 1$ (which is the case if either k has no real places, or if the semisimple part of G contains no anisotropic almost simple factors of type E_6).*

(1) *The following sequence connecting Tate-Shafarevich kernels, the cohomology of flasque kernel and the group of R-equivalence classes is exact*

$$1 \to Ш^1(G)^D \to \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F}) \to (G(k)/R)^D \to Ш^1(F)^D \to 1.$$

(2) *Assume further that $\mathcal{G}$ (respectively $\mathcal{T}$) is a smooth compactification of G (respectively T) with the Néron-Severi torus $S_{\mathcal{G}}$ (respectively $S_{\mathcal{T}}$). Then in the above sequence one can replace F by $S_{\mathcal{G}}$ or $S_{\mathcal{T}}$, which also takes the following forms depending only on G and $\mathcal{G}$*

$$1 \to Ш^1(G)^D \to \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \to \prod_v \mathrm{H}^1(k_v, \mathrm{Pic}(\mathcal{G}_s))$$
$$\to (G(k)/R)^D \to Ш^1(S_{\mathcal{G}})^D \to 1,$$
$$1 \to Ш^1(G)^D \to Br_a(\mathcal{G}) \xrightarrow{\mu} \prod_v Br_a(\mathcal{G}_v) \xrightarrow{\tau} (G(k)/R)^D$$
$$\to Ш^1(S_{\mathcal{G}})^D \to 1.$$

Proof. (1) Recall that if T is a torus over a global field k, $\mathcal{T}$ a smooth k-compactification of T, $S_{\mathcal{T}}$ the Néron-Severi torus of $\mathcal{T}$, $1 \to F \to P \to T \to 1$ a flasque resolution of T, then by [8, Proposition 19], we have the following natural exact sequences

$$1 \to Ш^1(F) \to T(k)/R \to \prod_v T(k_v)/R \to \mathrm{A}(T) \to 1,$$

$$1 \to \mathrm{A}(T) \to \mathrm{H}^1(k, \hat{F})^D \to Ш^1(T) \to 1.$$

From these sequences we obtain the following exact sequences

$$1 \to Ш^1(T)^D \to \mathrm{H}^1(k, \hat{F}) \to \mathrm{A}(T)^D \to 1, \tag{5.16}$$

$$1 \to \mathrm{A}(T)^D \to \left(\prod_v T(k_v)/R\right)^D \to \big(T(k)/R\big)^D \to Ш^1(F)^D \to 1. \tag{5.17}$$

Since $T(L)/R \simeq \mathrm{H}^1(L, F)$ for any field extension L/k [8, Theorem 2] and for all archimedean places and for almost all places we have $\mathrm{H}^1(k_v, F) \simeq T(k_v)/R = 1$, so from Tate-Nakayama duality for tori, so by combining the sequences (5.16)-(5.17) we deduce the following exact sequence

$$1 \to Ш^1(T)^D \to \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F}) \to (T(k)/R)^D \to Ш^1(F)^D \to 1.$$

By [43, Proposition 4.2.3] (for the case of number fields) and by [46, Theorem 3.7] (for the case of global function fields), we have $Ш^1(G) \simeq Ш^1(H) \simeq Ш^1(T)$ and by Theorem 4.3 we have $G(k)/R \simeq T(k)/R$, so (1) follows.

(2) We have an isomorphism (*cf.* [8, Lemma 15], [36, Lemma 6.3(iii)]) $Br_a(\mathcal{G}_L) \simeq \mathrm{H}^1(L, \mathrm{Pic}(\mathcal{G}_s))$ for any field extension L/k. The exact sequences stated in the part 2 of the theorem now follow from (1), from [5, Theorem 7.1] (that $Br_a(\mathcal{G}_L) \simeq \mathrm{H}^1(L, \hat{F})$ for any field extension L/k) and from the isomorphism $Ш^1(F) \simeq Ш^1(S_{\mathcal{G}})$ (Theorem 5.11(3)). □

5.5. Some formulas for computing $G(k)/R$ and $G(k)/B_f$

There were given in [8, Corollary 1, page 217], [8, Proposition 19, page 220] some formulas for computing $T(k)/Br$ where T is a torus defined over a number field k and there was some extension of these in [44, Theorem 2.1] to the case of an arbitrary connected linear algebraic group over a number field k. We have the following formulas giving some explicit computations of the group $G(k)/B_f$, which are similar to that of [8, Proposition 19]. We will give yet another exact sequence relating $G(k)/R$ and the cohomological invariants of the Néron-Severi torus $S_{\mathcal{G}}$ of a smooth compactification $\mathcal{G}$ of G, extending the formula given for tori over number fields given in [8, Corollary 5, page 201].

5.5.1.

Let $1 \to F \to H_1 \to G \to 1$ be a flasque resolution of G and then choose, $1 \to Z \to H \to G \to 1$ a z-extension and co-flasque resolution of $G, T := H^{\mathrm{tor}}$. Assume that $\mathcal{G}$ (respectively $\mathcal{T}$) is a smooth k-compactification of G (respectively T), $\hat{S}_{\mathcal{G}} = \mathrm{Pic}(\mathcal{G}_s)$ (respectively $\hat{S}_{\mathcal{T}} = \mathrm{Pic}(\mathcal{T}_s)$). Let $\Delta : \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k, \hat{F})$ be the product of identity maps, $\lambda : \prod_v \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F})$ the diagonal localization map and let $\mu = \lambda \circ \Delta$. Then for any extension field L/k, according to Theorem 5.11(3), we have isomorphisms

$$\begin{aligned}\mathrm{H}^n(L, \hat{F}) &\simeq \mathrm{H}^n(L, \hat{\mathcal{T}}) \simeq \mathrm{H}^n(L, \hat{\mathcal{G}}) \simeq \mathrm{H}^n(L, \mathrm{Pic}(\mathcal{T} \times_k L_s)) \\ &\simeq \mathrm{H}^n(L, \mathrm{Pic}(\mathcal{G} \times_k L_s)).\end{aligned}$$

By [8, Lemme 15], we have

$$Br_a(\mathcal{G} \times_k L) \simeq \mathrm{H}^1(L, \mathrm{Pic}(\mathcal{G} \times_k L_s)), Br_a(\mathcal{T} \times_k L) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{T} \times_k L_s))$$

for any field extension L/k.

Therefore, from isomorphisms $\mathrm{H}^1(k, \hat{F}) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\mathcal{G}_s)) \simeq Br_a(\mathcal{G})$ and from λ above we have a homomorphism $\lambda_{\mathcal{G}} : \prod_v Br_a(\mathcal{G}) \to \prod_v Br_a(\mathcal{G}_v)$. From [8, page 221] we have the following exact sequence

$$1 \to Ш^1(T)^D \to \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F}) \xrightarrow{\tau} (T(k)/R)^D \to Ш^1(F)^D \to 1.$$

By [26, Lemma 4.3.1], [43, Proposition 4.2.3], [46, Theorem 3.7] and Theorem 5.11(3), we may write this sequence as follows

$$1 \to \text{Ш}^1(G)^D \to \mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F}) \xrightarrow{\tau} \mathrm{H}^1(k, F)^D \xrightarrow{\psi} \text{Ш}^1(F)^D \to 1,$$

and if $\mathscr{G}$ exists, also in the form

$$1 \to \text{Ш}^1(G)^D \to Br_a(\mathscr{G}) \to \prod_v Br_a(\mathscr{G}_v) \xrightarrow{\tau} \mathrm{H}^1(k, F)^D \xrightarrow{\psi} \text{Ш}^1(F)^D \to 1.$$

Theorem 5.17. *Let k be a global field, G a connected reductive k-group, $\tilde{G}$ the simply connected covering of G^{ss} and keep the notation as in Subsection* 5.5.1.

(1) *(*cf. [8, Corollary 5(iii), page 201]*for tori) Assume that $\tilde{G}(k)/R = 1$. Then we have the following exact sequence*

$$1 \to \text{Ш}^2(\hat{S}_{\mathscr{G}})^D \to G(k)/R \to \text{Ч}^1(\hat{S}_{\mathscr{G}})^D \to 1. \tag{5.18}$$

(2) *(Formulas for $G(k)/B_f$) We have the following canonical isomorphisms*

$$G(k)/B_f \simeq (\mathrm{Im}\,\lambda/\mathrm{Im}\,\mu)^D. \tag{5.19}$$

$$(G(k)/B_f)^D \simeq \mathrm{Im}\,(\tau) \simeq \mathrm{Im}\,(\tau \circ \lambda_{\mathscr{G}}), \tag{5.20}$$

where in (5.20) *we assume that $\mathscr{G}$ exists.*

Proof. (1) By [8, Corollary 5, page 201], for any k-torus T we have

$$1 \to \text{Ш}^2(\hat{S})^D \to T(k)/R \to \text{Ч}^1(\hat{S})^D \to 1$$

where S is a flasque kernel of T. Notice that if $\mathscr{T}$ is a smooth k-compactification of T (see [7, Corollary 1] for the existence), $S_{\mathscr{T}}$ the Néron-Severi torus for $\mathscr{T}$, then by [8, page 221] or by the proof of Theorem 5.11, we have $S \sim S_{\mathscr{T}}$, where $\sim$ denotes the similarity relation. Hence above exact sequence can be written as

$$1 \to \text{Ш}^2(\hat{S}_{\mathscr{T}})^D \to T(k)/R \to \text{Ч}^1(\hat{S}_{\mathscr{T}})^D \to 1\,.$$

Let H be a co-flasque z-extension of G and set $T := H^{\mathrm{tor}}$. Since $\tilde{G}(k)/R = 1$, by Theorem 5.14(2) we have $G(k)/R \simeq T(k)/R$ and by Theorem 5.11(3), we have $\text{Ш}^2(\hat{S}_{\mathscr{T}}) \simeq \text{Ш}^2(\hat{S}_{\mathscr{G}})$, $\text{Ч}^1(\hat{S}_{\mathscr{T}}) \simeq \text{Ч}^1(\hat{S}_{\mathscr{G}})$, so we have the exact sequence (5.18).

(2) First we show the formula in the case $G = T$ is a torus. With notation as above, F is a flasque kernel of T, and consider the exact sequence

$$\mathrm{H}^1(k, F) \to \mathrm{H}^1(k, F(\mathbb{A}_s)) \to \mathrm{H}^1(k, F(\mathbb{A}_s)/F(k_s)).$$

By using the local and global Tate-Nakayama duality and by taking the dual of this sequence, it gives the following dual exact sequence

$$\mathrm{H}^1(k, \hat{F}) \to \prod_v \mathrm{H}^1(k_v, \hat{F}) \to \mathrm{H}^1(k, F)^D.$$

By Proposition 3.2 (c), $T(k)/B_f \simeq \mathrm{Im}\ (\omega)$, where $\omega : \mathrm{H}^1(k, F) \to \mathrm{Hom}(\mathrm{H}^1(k, \hat{F}), Br(k))$ is the natural homomorphism. Since $Br(k) \hookrightarrow \bigoplus_v Br(k_v)$, so $T(k)/B_f \simeq \mathrm{Im}\ (j \circ \omega)$, where

$$j : \mathrm{Hom}(\mathrm{H}^1(k, \hat{F}), Br(k)) \hookrightarrow \mathrm{Hom}(\mathrm{H}^1(k, \hat{F}), \bigoplus_v Br(k_v)))$$

is an injection. Then in the same manner as in [8, pages 217-218], we have $T(k)/B_f \simeq (\mathrm{Im}\ (\lambda)/\mathrm{Im}\ (\mu))^D$.

Now we assume that G is not a torus. Let $T = H^{\mathrm{tor}}$, where H is a z-extension of G as above. By Theorem 5.3(b), we have an isomorphism $G(k)/B_f \simeq T(k)/B_f$. Since G and T share the same flasque kernel, so the isomorphism

$$G(k)/B_f \simeq (\mathrm{Im}\ (\lambda)/\mathrm{Im}\ (\mu))^D$$

follows from the case of tori.

(3) By Proposition 5.8, we have the following exact sequence

$$1 \to Ш^1(F) \to T(k)/R \to T(k)/B_f \to 1$$

and by taking its dual and using the isomorphism $T(k)/R \simeq \mathrm{H}^1(k, F)$ [8, Theorem 2] we have the exact sequence

$$1 \to (T(k)/B_f)^D \to (T(k)/R)^D \xrightarrow{\psi} (Ш^1(F))^D \to 1.$$

From the exact sequence (5.6.1.1) it follows that $\mathrm{Im}\ (\tau) = \mathrm{Ker}\ (\psi)$, hence $(T(k)/B_f)^D \simeq \mathrm{Im}\ (\tau)$. By Theorem 5.3(b), we have an isomorphism $G(k)/B_f \simeq T(k)/B_f$, so finally we have

$$(G(k)/B_f)^D \simeq \mathrm{Im}\ (\tau).$$

The second isomorphism is obtained by first proving its validity for tori, which follows from the same proof as in [8, Poposition 19] and then by using again the isomorphism $G(k)/B_f \simeq T(k)/B_f$, so we omit the details. □

5.6. Some remarks and questions

Let k be a field. The following questions regarding Brauer equivalence relations remain open. We keep the notation as in 3.1.

(1) Assume that char.$k > 0$. For which Brauer equivalence relation B, the set $X(k)/B$ is a (stably) birational invariant in the class of all smooth k-varieties? Similar question for algebraic groups?

(2) What can one say about the discussed exact sequences connecting various arithmetic, geometric and cohomological invariants if we go further, beyond the class of connected reductive groups?

The following principle has been put forward by Conrad [10, Section 1.1]: To prove a theorem for all connected linear algebraic groups over non-perfect fields, it is sufficient to prove the same thing for all solvable groups over the field k and for all semisimple groups over finite extensions of k.

Thus the next step is to study the case of connected solvable groups defined over a global function field. One important class is that of unipotent groups. Recall that there exist connected unipotent groups G defined over a global function field k, where $G(k)$ is finite. For such groups G, the obstruction $\mathrm{A}(G)$ is infinite, by contrast to the case of connected reductive groups. On the one hand, it was known (*cf.* [47, Proposition 5.1]) that if G is a k-unirational connected unipotent k-group, then the obstruction to weak approximation is always *finite*. On the other hand, it is known (*cf.* [10, Theorem 1.3.3]), that (for a global function field k) the Tate-Shafarevich kernel of an affine group k-scheme of finite type is always finite. So if one wishes to have an analog of the exact sequences treated in the present paper as sequences of *finite Abelian groups*, perhaps one should restrict to the case of connected linear algebraic groups with finite obstruction to weak approximation $\mathrm{A}(G)$. The questions come as follows.

(3) Is it true that for a connected linear algebraic group G defined over a global function field k, the obstruction to weak approximation $\mathrm{A}(G)$ is finite if and only if G is unirational over k?

Further questions are:

(4) Do the exact sequences treated in the present paper hold for any connected solvable linear algebraic group G over any global field k?

(5) Do the exact sequences considered above, after a suitable modification, (*e.g.*, replacing smooth compactifications by regular compactifications) hold true without requiring the finiteness of the invariants treated? Do they also hold true in the case of geometric fields of any characteristic considered in [6,33]?

(6) Do the exact sequences considered above, after a suitable modification (*e.g.*, replacing smooth compactifications by regular compactifications), or similar sequences as in [36, Theorem 9.5, (V), (M), (C)], also hold for connected algebraic groups, not necessary linear?

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