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Pathologies on Mori fibre spaces in positive characteristic

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Abstract. We show that there exist Mori fibre spaces whose total spaces are klt but bases are not. We also construct Mori fibre spaces which have relatively non-trivial torsion line bundles.

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1. Introduction

Given an algebraic variety X, the minimal model conjecture implies that X is birational to either a minimal model or a Mori fibre space. The purpose of this paper is to find some phenomena on Mori fibre spaces that occur only in positive characteristic. Originally, the advantage of Mori fibre spaces is their simple structure, which allows us to reduce some problems to the study of their fibres and bases. For instance, given a Mori fibre space $f : X \to S$ from a klt variety X in characteristic zero, it is known that its base space S is also klt (*cf.* [1, Theorem 0.2], [8, Corollary 4.6]). Unfortunately, the same statement is no longer true in positive characteristic.

Theorem 1.1. *Let k be an algebraically closed field whose characteristic is two or three. Then there exists a projective k-morphism $f : V \to W$ of normal k-varieties that satisfies the following properties:*

(1) *V is a 4-dimensional $\mathbb{Q}$-factorial klt variety over k;*
(2) *W is a 3-dimensional $\mathbb{Q}$-factorial variety over k that is not klt;*
(3) *$f_*\mathcal{O}_V = \mathcal{O}_W$, $\rho(V/W) = 1$, $-K_V$ is f-ample;*
(4) *Any fibre of f is an irreducible scheme of dimension one, and there is a non-empty open subset W^0 of W such that the fibre $V \times_W \operatorname{Spec} k(w)$ is isomorphic to $\mathbb{P}^1_{k(w)}$ for any point $w \in W^0$, where $k(w)$ denotes the residue field at w.*

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A prominent property of Mori fibre spaces in characteristic zero is that any relatively numerically trivial Cartier divisor is trivial (*cf.* [17, Lemma 3-2-5(2)]). We construct an example in positive characteristic that violates this property.

Theorem 1.2. *Let k be an algebraically closed field whose characteristic p is two or three. Then there exists a projective k-morphism $f : V \to W$ of normal k-varieties that satisfies the following properties:*

(1) *V is a 3-dimensional $\mathbb{Q}$-factorial klt variety over k;*
(2) *W is a smooth curve over k;*
(3) *$f_*\mathcal{O}_V = \mathcal{O}_W$, $\rho(V/W) = 1$, $-K_V$ is f-ample, and*
(4) *there is a Cartier divisor D on V such that $D \not\sim_f 0$ and $pD \sim_f 0$.*

Remark 1.3. Since [17, Lemma 3-2-5(2)] is a formal consequence of the relative Kawamata–Shokurov base point free theorem [17, Theorem 3-1-1], the same statement as in [17, Theorem 3-1-1] does not hold in positive characteristic.

1.1. Construction of examples

Let us overview how to construct the examples appearing in Theorem 1.1 and Theorem 1.2.

1.1.1. *Pathological surfaces over imperfect fields*

To find examples appearing in Theorem 1.1 and Theorem 1.2, we first construct log del Pezzo surfaces over imperfect fields satisfying pathological properties as follows.

Theorem 1.4. *Let k be an imperfect field whose characteristic p is two or three. Then there exists a k-morphism $\rho : S \to C$ that satisfies the following properties:*

(1) *S is a projective regular surface over k and there is an effective $\mathbb{Q}$-divisor Δ_S such that (S, Δ_S) is klt and $-(K_S + \Delta_S)$ is ample;*
(2) *C is a projective regular curve over k with $K_C \sim 0$;*
(3) *ρ is a $\mathbb{P}^1$-bundle, and*
(4) *There is a Cartier divisor L on C such that $L \not\sim 0$ and $pL \sim 0$.*

The surface S in Theorem 1.4 is a log Fano variety dominating a Calabi–Yau variety. Such an example does not exist in characteristic zero (*cf.* [25, Lemma 2.8], [9, Theorem 5.1]). For some related results in positive characteristic, we refer to [7].

Let us overview the construction of $\rho : S \to C$ appearing in Theorem 1.4. We take a regular cubic curve C that is not smooth and has a k-rational point P around which C is smooth over k. For example, if k is the function field of a curve over an algebraically closed field, then C is nothing but the generic fibre of a quasi-elliptic fibration equipped with a section. Since we have that $H^1(C, \mathcal{O}_C(-P)) \neq 0$ by Serre duality, a nonzero element ξ of $H^1(C, \mathcal{O}_C(-P))$ induces a locally free sheaf E of rank two. Then S is the $\mathbb{P}^1$-bundle defined as $\mathbb{P}(E)$. In order to show that S

is log del Pezzo, one of the essential facts is that we can find a purely inseparable field extension $k \subset k'$ of degree p such that $C \times_k k'$ is an integral but non-normal scheme and that its normalisation is isomorphic to $\mathbb{P}^1_{k'}$. Since the scheme-theoretic inverse image $\varphi^{-1}(P)$ of the k-rational point P is a k'-rational point, we have that

$$H^1(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(-\varphi^* P)) = H^1(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(-1)) = 0,$$

where $\varphi^* P$ denotes the pull-back of the Cartier divisor P. This implies that the pull-back $\varphi^*\xi$ is zero. This property plays a crucial role in our construction. For more details, see Section 3.

1.1.2. *Proofs of the theorems*

Let us overview some of the ideas of the proofs of Theorem 1.1 and Theorem 1.2.

First let us treat the latter one: Theorem 1.2. This is a consequence of Theorem 1.4. Indeed, for an algebraically closed field k whose characteristic is two or three, it follows from Theorem 1.4 that we get a log del Pezzo surface (S, Δ_S) over $k(t)$ which has a non-trivial p-torsion Cartier divisor. Then we can spread it out over some non-empty open subset W of $\mathbb{A}^1_k$, *i.e.*, there is a morphism $V \to W$ with $V \times_W \operatorname{Spec} k(t) = S$. Although this example does not satisfy the property $\rho(V/W) = 1$, we may assume this condition by contracting an appropriate curve on S in advance. For more details, see Subsection 4.1.

Second, let us overview the proof of Theorem 1.1. To this end, we first find a similar example over imperfect fields (*cf.* Theorem 4.4). The basic idea is to take cones over $\rho : S \to C$. However, there is no morphism between cones. What we will actually do is to take $\mathbb{P}^1$-bundles functorially for an ample divisor M_C on C:

$$X := \mathbb{P}_S(\mathcal{O} \oplus \mathcal{O}(\rho^* M_C)) \to \mathbb{P}_C(\mathcal{O} \oplus \mathcal{O}(M_C)) =: W_0.$$

Let $W_0 \to W$ be the birational contraction of the section C^- of $W_0 \to C$ with negative self-intersection number. Since $K_C \sim 0$, W is not klt.

If there was a divisorial contraction whose exceptional locus is the pull-back of C^-, then the resulting variety would be what we are looking for. Although we can not hope this, we will get close to this situation by running a suitable minimal model program. To this end, we first construct a minimal model program after taking a purely inseparable cover of X, and descend it to X after that. For more details, see Subsection 4.2.

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2. Preliminaries

2.1. Notation

In this subsection we summarise the notation used in this paper.

- We will freely use the notation and terminology in [15] and [20].
- For a scheme X, its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced morphism $X_{\mathrm{red}} \to X$ is surjective.
- For an integral scheme X, we define the *function field* $K(X)$ of X as $\mathcal{O}_{X,\xi}$ for the generic point ξ of X.
- For a field k, X is a *variety over* k or a k*-variety* if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* over k or a k*-curve* (respectively a *surface* over k or a k*-surface*, respectively a *threefold* over k) if X is a k-variety of dimension one (respectively two, respectively three).
- We say that two schemes X and Y over a field k are k*-isomorphic* if there exists an isomorphism $\theta \colon X \to Y$ of schemes such that both θ and θ^{-1} commute with the structure morphisms: $X \to \operatorname{Spec} k$ and $Y \to \operatorname{Spec} k$.

Definition 2.1. Let k be a field.

1. Let C be a proper curve over k. Let M be an invertible sheaf on C. It is well-known that
$$\chi(C, mM) = \dim_k(H^0(C, mM)) - \dim_k(H^1(C, mM)) \in \mathbb{Z}[m]$$
and that the degree of this polynomial is at most one (*cf.* [19, Chapter I, Section 1, Theorem on page 295]). We define the *degree* of M over k, denoted by $\deg_k M$ or $\deg M$, as the coefficient of m;
2. Let X be a separated scheme of finite type over k, let L be an invertible sheaf on X, and let $C \hookrightarrow X$ be a closed immersion over k from a proper k-curve C. We define the *intersection number* over k, denoted by $L \cdot_k C$ or $L \cdot C$, as $\deg_k(L|_C)$.

2.2. Properties spreading out from the generic fibre

In this subsection we summarise some properties extendable from the generic fibre: Lemma 2.2. Also, we give a criterion for projective morphisms of generically relative Picard number one (Lemma 2.5). To this end, we establish two auxiliary lemmas: Lemma 2.3 and Lemma 2.4.

Lemma 2.2. *Let k be a field. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

(1) *Assume that k is algebraically closed. Then the generic fibre $X_{K(Y)}$ is $\mathbb{Q}$-factorial if and only if there is a non-empty open subset Y' of Y such that $X \times_Y Y'$ is $\mathbb{Q}$-factorial;*

(2) *Let Δ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that there is a log resolution of $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$. Then $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$ is klt (respectively log canonical) if and only if there is a non-empty open subset Y' of Y such that $(X \times_Y Y', \Delta|_{X \times_Y Y'})$ is klt (respectively log canonical).*

Proof. Assertion (1) holds by [3, the third Theorem in the Introduction]. We now show (2). After shrinking Y, we may assume that there is a log resolution $g : Z \to X$ of (X, Δ). We define a $\mathbb{Q}$-divisor Δ_Z on Z by $K_Z + \Delta_Z = g^*(K_X + \Delta)$. Then [20, Corollary 2.13(2)] implies the following:

(a) (X, Δ) is klt if and only if all the coefficients of Δ_Z are less than 1;
(b) $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$ is klt if and only if all the coefficients of $\Delta_Z|_{Z_{K(Y)}}$ are less than 1.

Shrinking Y again, we may assume that any irreducible component of Δ_Z dominates Y. Then all the coefficients of Δ_Z are less than 1 if and only if all the coefficients of $\Delta_Z|_{Z_{K(Y)}}$ are less than 1. Thanks to (a) and (b), (X, Δ) is klt if and only if $(X_{K(Y)}, \Delta|_{X_{K(Y)}})$ is klt. $\square$

Lemma 2.3. *Let k be a field. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties. Assume that the generic fibre $X_{K(Y)}$ is $K(Y)$-isomorphic to $\mathbb{P}^n_{K(Y)}$ for some non-negative integer n. Then there exists a non-empty open subset Y' of Y such that the fibre X_y is $k(y)$-isomorphic to $\mathbb{P}^n_{k(y)}$ for any point $y \in Y'$.*

Proof. Replacing Y by a non-empty open subset, we may assume that the following properties hold:

(1) f is a smooth morphism and $f_*\mathcal{O}_X = \mathcal{O}_Y$;
(2) $-K_X$ is f-ample;
(3) the tangent bundle T_{X_y} is ample for any $y \in Y$ (*cf.* [14, Proposition 4.4], [21, Proposition 6.1.9]), and
(4) there is a section of f, *i.e.*, there exists a closed immersion $j : Y_1 \to X$ such that the composite morphism $Y_1 \to X \to Y$ is an isomorphism.

Fix $y \in Y$ and let $X_{\overline{k(y)}}$ be the base change of the fibre X_y to its algebraic closure $\overline{k(y)}$. Since $X_{\overline{k(y)}}$ is a smooth projective variety whose tangent bundle $T_{X_{\overline{k(y)}}}$ is ample, we have that $X_{\overline{k(y)}}$ is $\overline{k(y)}$-isomorphic to $\mathbb{P}^n_{\overline{k(y)}}$ by [23, Theorem 8]. This implies that X_y is a Severi–Brauer variety. Since X_y has a $k(y)$-rational point by (4), we have that X_y is $k(y)$-isomorphic to $\mathbb{P}^n_{k(y)}$ by [10, Theorem 5.1.3]. $\square$

Lemma 2.4. *Let k be a field of characteristic $p > 0$. Consider a commutative diagram of projective k-morphisms of normal k-varieties*

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\beta} & Y, \end{array}$$

where α and β are finite universal homeomorphisms. Then $\rho(X/Y) = \rho(X'/Y')$.

Proof. As X and Y are normal, the e-th iterated absolute Frobenius morphisms $F_X^e : X \to X$ and $F_Y^e : Y \to Y$ can be considered as the normalisation of X in $K(X)^{1/p^e}$ and Y in $K(Y)^{1/p^e}$, respectively. Since there is a positive integer e such that $K(X)^{1/p^e} \supset K(X') \supset K(X)$ and $K(Y^{1/p^e}) \supset K(Y') \supset K(Y)$, the e-th iterated absolute Frobenius morphisms $F_X^e : X \to X$ and $F_Y^e : Y \to Y$ factor through α and β respectively:

$$F_X^e : X \overset{\widetilde{\alpha}}{\to} X' \overset{\alpha}{\to} X, \quad F_Y^e : Y \overset{\widetilde{\beta}}{\to} Y' \overset{\beta}{\to} Y.$$

Then we have that $\rho(X/Y) \le \rho(X'/Y')$. The opposite inequality follows from the fact that the e-th iterated absolute Frobenius morphisms $F_{X'}^e : X' \to X'$ and $F_{Y'}^e : Y' \to Y'$ factor through $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. $\square$

Lemma 2.5. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that there exists a finite universal homeomorphism $\varphi : \mathbb{P}_L^n \to X_{K(Y)}$ over $K(Y)$ for some finite purely inseparable extension $K(Y) \subset L$. Then there exists a non-empty open subset Y' of Y such that $\rho(X'/Y') = 1$ for $X' := X \times_Y Y'$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} X_1 := \mathbb{P}_L^n \arrow[r, "\varphi"] \arrow[d, "f_1"] & X_{K(Y)} \arrow[r] \arrow[d, "f_{K(Y)}"] & X \arrow[d, "f"] \\ Y_1 := \operatorname{Spec} L \arrow[r] & \operatorname{Spec} K(Y) \arrow[r] & Y, \end{tikzcd}$$

where the right square is cartesian. All the schemes in the left square are projective over $K(Y)$, hence after shrinking Y, we can find a commutative diagram of projective k-morphisms of normal k-varieties

$$\begin{tikzcd} X_2 \arrow[r, "\alpha"] \arrow[d, "f_2"] & X \arrow[d, "f"] \\ Y_2 \arrow[r, "\beta"] & Y, \end{tikzcd}$$

where α and β are finite universal homeomorphisms, $K(Y_2) = L$ and the generic fibre of f_2 is $K(Y_2)$-isomorphic to $\mathbb{P}_{K(Y_2)}^n$. Then the assertion follows from Lemma 2.3 and Lemma 2.4. $\square$

2.3. Varieties of Fano type

In this subsection we recall the definition of varieties of Fano type and one of thein basic properties (Lemma 2.8).

Definition 2.6. Let k be a field. A projective normal k-variety X is *of Fano type* if there is an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. In this case, (X, Δ) is called *log Fano*. We say that (X, Δ) is *log del Pezzo* if X is log Fano and $\dim X = 2$.

Lemma 2.7. *Let k be a field of characteristic $p > 0$ such that $[k : k^p] < \infty$. Let (X, Δ) be a projective klt pair over k such that $\dim X \leq 3$. If A is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A' such that $A \sim_{\mathbb{Q}} A'$ and $(X, \Delta + A)$ is klt.*

Proof. Thanks to the assumptions $[k : k^p] < \infty$ and $\dim X \leq 3$, we may freely use log resolutions by [5,6]. Then we can apply the same argument as in [11, Lemma 2.8]. $\square$

Lemma 2.8. *Let k be a field of characteristic $p > 0$. Let X and Y be projective normal varieties over k. Assume that a rational map $f : X \dashrightarrow Y$ over k satisfies one of the following properties:*

(1) *f is a birational morphism;*
(2) *f is a birational map which is an isomorphism in codimension one.*

If X is of Fano type, $[k : k^p] < \infty$ and $\dim X \leq 3$, then Y is of Fano type.

Proof. For both the cases, we can apply the same argument as in [2, Lemma 2.4]. However, we give a proof only for the case (1) since our setting differs from [2, Lemma 2.4]. Thanks to the assumptions $[k : k^p] < \infty$ and $\dim X \leq 3$, we may freely use log resolutions by [5,6].

Since X is of Fano type, we can find an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. By Lemma 2.7, we can find an effective $\mathbb{Q}$-divisor A_X on X such that $-(K_X + \Delta) \sim_{\mathbb{Q}} A_X$ and $(X, \Delta + A_X)$ is klt. Taking the push-forward by f, we have that

$$-(K_Y + f_*\Delta + f_*A_X) \sim_{\mathbb{Q}} 0.$$

Then it holds that

$$K_X + \Delta + A_X = f^*(K_Y + f_*\Delta + f_*A_X),$$

hence $(Y, f_*\Delta + f_*A_X)$ is klt. Since f_*A_X is big, we can write

$$f_*A_X = A_Y + E$$

for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A_Y and an effective $\mathbb{Q}$-divisor E. Therefore, it holds that

$$-(K_Y + f_*\Delta + (1-\epsilon) f_*A_X + \epsilon E) \sim_{\mathbb{Q}} \epsilon A_Y$$

for any rational number ϵ. Since log resolutions exist, the pair

$$(Y, f_*\Delta + (1-\epsilon) f_*A_X + \epsilon E)$$

is klt if ϵ is a sufficiently small rational number. Hence, Y is of Fano type. $\square$

Remark 2.9. The assumptions $[k : k^p] < \infty$ and $\dim X \leq 3$ in Lemma 2.8 is used only to assure the existence of log resolutions [5,6].

2.4. Jacobian criterion for regularity

For later use, we summarise results for regularity of some explicit varieties that follow from the Jacobian criterion.

Lemma 2.10. *Let k be a field of characteristic $p > 0$. Take elements $s, t \in k \setminus k^p$. Then the following hold:*

(1) *If $p = 2$, then* $\operatorname{Spec} k[x, y]/(x^2 + ty^2)$ *is regular outside the origin* $\{(0, 0)\}$*;*
(2) *If $p = 2$, then $k[x, y]/(tx^2 + 1)$ is regular;*
(3) *If $p = 2$ and $[k(s^{1/2}, t^{1/2}) : k] = 4$, then $k[x, y]/(sx^2 + ty^2 + 1)$ is regular;*
(4) *If $p = 2$, then* $\operatorname{Proj} k[x, y, z]/(y^2z + x^3 + sxz^2)$ *is regular;*
(5) *If $p = 3$, then* $\operatorname{Proj} k[x, y, z]/(-y^2z + x^3 + sz^3)$ *is regular.*

Proof. Since all the proofs are quite similar, we only prove (1). We consider the following open subset of $\operatorname{Spec} k[x, y]/(x^2 + ty^2)$:

$$D(y) = \operatorname{Spec} k[x, y, z]/(x^2 + ty^2, zy + 1) \simeq \operatorname{Spec} k[x, y, y^{-1}]/(x^2y^{-2} + t).$$

We can find an $\mathbb{F}_2$-derivation D_1 of $k[x, y, z]$ with $D_1(t) = 1$ by $t \notin k^2$ and [22, Theorem 26.5]. We have that the ring $k[x, y, z]/(x^2 + ty^2, zy + 1)$ is regular by applying the Jacobian criterion [12, Proposition 22.6.7(iii)] for $k_0 := \mathbb{F}_2$, $B = k[x, y, z]$, $\mathfrak{q}$ is a prime ideal of B containing $(x^2 + ty^2, zy + 1)$, $f_1 := x^2 + ty^2$, and the $\mathbb{F}_2$-derivation D_1 defined above. It follows from the same argument that also the open subset $D(x)$ of $\operatorname{Spec} k[x, y]/(x^2 + ty^2)$ is regular. To summarise, $\operatorname{Spec} k[x, y]/(x^2 + ty^2)$ is regular outside the origin $(0, 0)$. □

2.5. Slc-ness of conics

The purpose of this subsection is to show that any plane conic curve is semi log canonical (Lemma 2.12). We start with a typical case in characteristic two.

Lemma 2.11. *Let k be a field of characteristic two with an element $t \in k$ such that $t \notin k^2$. Let*

$$Z := \operatorname{Spec} k[x, y]/(x^2 + ty^2).$$

Then Z is a semi log canonical curve.

Proof. By Lemma 2.10(1), Z is regular outside the origin. By using the assumption: $t \notin k^2$, we can directly check that Z has a node at the origin (*cf.* [20, 1.41]). It follows from [27, Proposition 3.6] that Z is semi log canonical. □

Lemma 2.12. *Let k be a field and let $Z =$* $\operatorname{Spec} k[x, y]/(f)$*, where Z is reduced and f is a polynomial of degree two. Then Z is semi log canonical.*

Proof. We only treat the case where the characteristic of k is two, since otherwise the problem is easier. We may assume that k is separably closed. We can write

$$f = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

where $a_{ij} \in k$.

Step 1. If $a_{11} \neq 0$, then Z is semi log canonical.

Proof. (of Step 1) Assume that $a_{11} \neq 0$. Since k is separably closed, we can write $a_{20}x^2 + a_{11}xy + a_{02}y^2 = l_1 l_2$ for some homogeneous polynomials l_1 and l_2 of degree one with $(l_1, l_2) = (x, y)$. In particular, we may assume that $a_{20} = a_{02} = 0$. After applying a suitable linear transform, we may assume that $f = xy + b$ for some $b \in k$. Then Z is semi log canonical. This completes the proof of Step 1. $\square$

Step 2. If $a_{11} = 0$ and $a_{10} \neq 0$, then Z is smooth over k.

Proof. (of Step 2) We first prove that we may assume that $a_{00} = 0$. If $a_{20} = 0$, then we are reduced to the case where $a_{00} = 0$, after applying the linear transform: $a_{10}x + a_{00} \mapsto x$, $y \mapsto y$. Assume that $a_{20} \neq 0$. Since k is separably closed, $a_{20}x^2 + a_{10}x + a_{00} = 0$ has two distinct solutions if $a_{20} \neq 0$. In particular, applying the linear transform $x + \alpha \mapsto x$, $y \mapsto y$ for a solution $\alpha \in k$, we may assume that $a_{00} = 0$.

We assume that $a_{00} = 0$. Thus, we can write

$$f = a_{20}x^2 + a_{02}y^2 + a_{10}x + a_{01}y.$$

Since $a_{10} \neq 0$, we have that $Z \simeq k[x, y]/(x + bx^2 + cy^2)$ for some $b, c \in k$. It follows from the Jacobian criterion for smoothness that Z is smooth. This completes the proof of Step 2. $\square$

Step 3. If $a_{11} = a_{10} = a_{01} = 0$, then Z is semi log canonical.

Proof. (of Step 3) If $a_{00} = 0$, then we may assume that $a_{20} = 1$ by symmetry, *i.e.*, $f = x^2 + a_{02}y^2$. Since Z is reduced, we have that $a_{02} \notin k^2$. Thus Z is semi log canonical by Lemma 2.11.

Thus we may assume that $a_{00} \neq 0$. Replacing f by f/a_{00} we get

$$f = a_{20}x^2 + a_{02}y^2 + 1.$$

Since f is reduced, we may assume that $a_{20} \notin k^2$, hence $[k(a_{20}^{1/2}) : k] = 2$. This implies that $[k(a_{20}^{1/2}, a_{02}^{1/2}) : k]$ is equal to either 4 or 2. If $[k(a_{20}^{1/2}, a_{02}^{1/2}) : k] = 4$, then Z is regular by Lemma 2.10(3).

Thus we may assume that $[k(a_{20}^{1/2}, a_{02}^{1/2}) : k] = 2$. We have that $a_{02} \in k^2(a_{20}) = k^2 \oplus k^2 a_{20}$, hence $a_{02} = b^2 + c^2 a_{20}$ for some $b, c \in k$. We get

$$f = a_{20}x^2 + a_{02}y^2 + 1 = a_{20}x^2 + (b^2 + c^2 a_{20})y^2 + 1 = a_{20}(x + cy)^2 + (by + 1)^2.$$

After replacing $x + cy$ by x, we can write

$$f = a_{20}x^2 + (by + 1)^2.$$

If $b = 0$, then Z is regular by Lemma 2.10(2). If $b \neq 0$, then we get $k[x, y]/(f) \simeq k[x, y]/(a_{20}x^2 + y^2)$. It follows from Lemma 2.11 that Z is semi log canonical. This completes the proof of Step 3. $\square$

Step 1, Step 2 and Step 3 complete the proof of Lemma 2.12. $\square$

3. Pathological surfaces over imperfect fields

3.1. Construction in a general setting

In this subsection we give a criterion to find a log Fano variety dominating a Calabi–Yau variety (Proposition 3.5). Although our construction is analogous to a standard one over an algebraically closed field (*cf*. [30], [24]), we give details of them because our setting is more general and our base field is not necessarily algebraically closed.

Notation 3.1. Let k be a field of characteristic $p > 0$. Assume that there exist a k-morphism $\varphi : C' \to C$ of regular projective k-varieties and a Cartier divisor D on C which satisfy the following properties.

(1) φ is a finite universal homeomorphism of degree p;
(2) There is a nonzero element $\xi \in H^1(C, \mathcal{O}_C(D))$ whose pull-back $\varphi^*(\xi) \in H^1(C', \mathcal{O}_{C'}(\varphi^* D))$ is zero.

The element ξ induces a locally free sheaf E of rank two on C equipped with the following exact sequence that does not split:

$$0 \to \mathcal{O}_C(D) \xrightarrow{\alpha} E \xrightarrow{\beta} \mathcal{O}_C \to 0. \tag{3.1}$$

By our assumption (2), the pull-back of this sequence to C' splits: $\varphi^* E \simeq \mathcal{O}_{C'} \oplus \mathcal{O}_{C'}(\varphi^* D)$. We set

$$S := \mathbb{P}_C(E), \quad S' := \mathbb{P}_{C'}(\varphi^* E)$$

and obtain a cartesian diagram:

$$\begin{tikzcd} S' \arrow[r, "\psi"] \arrow[d, "\rho'"] & S \arrow[d, "\rho"] \\ C' \arrow[r, "\varphi"] & C \end{tikzcd}$$

The surjection $\beta : E \to \mathcal{O}_C$ in (3.1) induces a section C_1 of ρ. We set $C'_1 := \psi^* C_1$ which is a section of ρ'. We have another section C'_2 of ρ' corresponding to the surjection:

$$\varphi^* E \simeq \mathcal{O}_{C'} \oplus \mathcal{O}_{C'}(\varphi^* D) \to \mathcal{O}_{C'}(\varphi^* D),$$

where the latter homomorphism is the natural projection. We set C_2 to be the reduced closed subscheme of S that is set-theoretically equal to $\psi(C'_2)$.

Lemma 3.2. *We use Notation* 3.1. *Then C_2 is an integral scheme and the induced morphism $\rho_{C_2} : C_2 \to C$ is a finite universal homeomorphism of degree p.*

Proof. Since ψ is a universal homeomorphism, we have that C_2 is an integral scheme. Since the induced composite morphism

$$C_2' \xrightarrow{\psi} C_2 \xrightarrow{\rho_{C_2}} C$$

is a finite universal homeomorphism of degree p, the degree of the latter morphism $\rho_{C_2} : C_2 \to C$ is equal to either 1 or p. It suffices to show that the latter case actually happens.

Assuming that $\rho_{C_2} : C_2 \to C$ is of degree one, *i.e.*, birational, let us derive a contradiction. Since $\rho_{C_2} : C_2 \to C$ is a finite birational morphism and C is normal, it follows that ρ_{C_2} is an isomorphism. Thus C_2 is a section of ρ, hence it is corresponding to a surjective $\mathcal{O}_C$-module homomorphism

$$\gamma : E \to \mathcal{O}_C(\widetilde{D})$$

for some Cartier divisor $\widetilde{D}$ on C. Since $C_2' = C_2 \times_S S'$, we have that the pull-back $\varphi^*(\gamma \circ \alpha)$ is an isomorphism. It follows from the faithfully flatness of φ that $\gamma \circ \alpha$ is an isomorphism. This implies that the sequence (3.1) splits, which is a contradiction. □

Lemma 3.3. *We use Notation* 3.1. *Then the following hold:*

(1) $\mathcal{O}_{S'}(C_1')|_{C_1'} \simeq (\rho'_{C_1'})^*\mathcal{O}_{C'}(-\varphi^*D)$, *where* $\rho'_{C_1'} : C_1' \to C'$ *is the induced morphism;*
(2) $\mathcal{O}_{S'}(C_2')|_{C_2'} \simeq (\rho'_{C_2'})^*\mathcal{O}_{C'}(\varphi^*D)$, *where* $\rho'_{C_2'} : C_2' \to C'$ *is the induced morphism.*

Proof. We can apply the same argument as in [15, Chapter V, Proposition 2.6]. □

Lemma 3.4. *We use Notation* 3.1. *Then the following* $\mathbb{Q}$*-linear equivalence holds:*

$$-K_S \sim_{\mathbb{Q}} \frac{2}{p}C_2 + \rho^*(-D - K_C).$$

Proof. Since the induced morphism $\rho_{C_2} : C_2 \to C$ is of degree p by Lemma 3.2, we have that

$$K_{S/C} + \frac{2}{p}C_2 \sim_{\mathbb{Q}} \rho^*(L) \tag{3.2}$$

for some $\mathbb{Q}$-divisor L on C. Taking the pull-back ψ^* of (3.2), we get

$$K_{S'/C'} + \frac{2}{p}\psi^*C_2 \sim_{\mathbb{Q}} \rho'^*\varphi^*(L). \tag{3.3}$$

Since $K_{S'} + C_1' + C_2' + \sim \rho'^*K_{C'}$, we have that

$$-C_1' - C_2' + \frac{2}{p}\psi^*C_2 \sim_{\mathbb{Q}} \rho'^*\varphi^*(L). \tag{3.4}$$

It holds that

$$(\rho'_{C'_1})^* \varphi^* D \sim_{\mathbb{Q}} -C'_1|_{C'_1} \sim_{\mathbb{Q}} (\rho'_{C'_1})^* \varphi^*(L),$$

where the first $\mathbb{Q}$-linear equivalence follows from Lemma 3.3(1) and the second one is obtained by restricting (3.4) to C'_1. Since $\rho'_{C'_1} : C'_1 \to C'$ is an isomorphism and the absolute Frobenius morphism $F_C : C \to C$ factors through $\varphi : C' \to C$, we get the $\mathbb{Q}$-linear equivalence $pD = F_C^*(D) \sim_{\mathbb{Q}} F_C^*(L) = pL$, which implies

$$D \sim_{\mathbb{Q}} L. \tag{3.5}$$

Substituting (3.5) in (3.2), we get

$$-K_S \sim_{\mathbb{Q}} \frac{2}{p} C_2 + \rho^*(-L - K_C) \sim_{\mathbb{Q}} \frac{2}{p} C_2 + \rho^*(-D - K_C),$$

as desired. □

Proposition 3.5. *We use Notation* 3.1. *If $-D - K_C$ is ample and $(S, \frac{2}{p}C_2)$ is log canonical, then there exists an effective $\mathbb{Q}$-divisor Δ on S such that (S, Δ) is klt and $-(K_S + \Delta)$ is ample.*

Proof. Since $-D - K_C$ is ample, so is

$$-\left(K_S + \left(\frac{2}{p} - \epsilon\right) C_2\right) \sim_{\mathbb{Q}} \epsilon C_2 + \rho^*(-D - K_C)$$

for some rational number ϵ with $0 < \epsilon < \frac{2}{p}$, where the $\mathbb{Q}$-linear equivalence follows from Lemma 3.4. Set $\Delta := (\frac{2}{p} - \epsilon)C_2$. Since S is regular and $(S, \frac{2}{p}C_2)$ is log canonical, we have that (S, Δ) is klt. □

3.2. Non-smooth K-trivial curves

We summarise the properties of K-trivial curves that we will need later.

Proposition 3.6. *Let k be an imperfect field whose characteristic p is two or three. Then there exists a projective regular curve C over k that satisfies the following properties:*

1. (1) *$K_C \sim 0$;*
2. (2) *the number of the k-rational points of C is at least three;*
3. (3) *there is a purely inseparable field extension $k \subset k'$ of degree p such that $C \times_k k'$ is an integral scheme which has a unique non-regular point Q;*
4. (4) *the normalisation C' of $C \times_k k'$ is k'-isomorphic to $\mathbb{P}^1_{k'}$, and*
5. (5) *there is a Cartier divisor L on C such that $L \not\sim 0$ and $pL \sim 0$.*

Proof. Since k is not perfect, we can find an element $t \in k$ with $t \notin k^p$.

First we treat the case where $p = 2$. Consider the following equation, which is taken from [16, Table 1 in page 243]:

$$C := \operatorname{Proj} k[x, y, z]/(y^2z + x^3 + (t^3 + t)xz^2).$$

We have that C is regular by Lemma 2.10(4). By the adjunction formula, (1) holds. The property (2) holds since all of $[0 : 1 : 0]$, $[0 : 0 : 1]$ and $[t+1 : t^2+1 : 1]$ are k-rational points on C. Let $k' := k(\sqrt{t^3 + t})$. The Jacobian criterion for smoothness implies that $C \times_k k'$ is smooth over k' away from $Q := [\sqrt{t^3 + t} : 0 : 1]$. Thus (3) holds. We can check that for the open set

$$D_+(z) = \operatorname{Spec} k'[x, y]/(y^2 + x^3 + (t^3 + t)x)$$

of $C \times_k k'$, its normalisation is isomorphic to $\mathbb{A}^1_{k'}$. Indeed, the integral closure of $k'[x, y]/(y^2 + x^3 + (t^3 + t)x) = k'[\overline{x}, \overline{y}]$ is equal to $k'[\overline{y}/(\overline{x} + \sqrt{t^3 + t})]$, where $\overline{x}$ and $\overline{y}$ are the images of x and y, respectively. Thus (4) holds. The property (5) holds by setting $L := P_1 - P_2$, where P_1 and P_2 are k-rational points around which C is smooth.

Second we assume that $p = 3$. Let

$$C := \operatorname{Proj} k[x, y, z]/(-y^2z + x^3 + t^2z^3).$$

All of $[0 : 1 : 0]$, $[0 : t : 1]$ and $[0 : -t : 1]$ are k-rational points on C. By Lemma 2.10(5), C is regular. We omit the remaining proof since it is similar to but easier than the one for the case where $p = 2$. $\square$

3.3. Log del Pezzo surfaces

Notation 3.7. Let k be an imperfect field whose characteristic p is two or three. Let C be a projective regular curve over k as in Proposition 3.6. By Proposition 3.6(3)(4), there is a purely inseparable field extension $k \subset k'$ of degree p such that $C \times_k k'$ is integral and its normalisation C' of $C \times_k k'$ is k'-isomorphic to $\mathbb{P}^1_{k'}$:

$$\varphi : C' \to C \times_k k' \to C.$$

In particular, φ is a finite universal homeomorphism of degree p. By Proposition 3.6(2)(3)(4), we can find a k-rational point P around which C is smooth over k. We set $D := -P$ and let $\xi \in H^1(C, \mathcal{O}_C(D))$ be a nonzero element whose existence is guaranteed by Serre duality. Since $C' \simeq \mathbb{P}^1_{k'}$ and $P' := \varphi^*P$ is a k'-rational point, we have that $H^1(C', \mathcal{O}_{C'}(\varphi^*D)) = 0$ by Serre duality. Therefore, we can apply the construction as in Subsection 3.1 (*cf.* Notation 3.1). Then we obtain a cartesian diagram of regular projective k-varieties:

$$\begin{tikzcd} S' \arrow[r, "\psi"] \arrow[d, "\rho'"] & S \arrow[d, "\rho"] \\ C' \arrow[r, "\varphi"] & C. \end{tikzcd}$$

Lemma 3.8. *We use Notation* 3.7. *Then the following hold.*

(1) *If $p = 2$, then C_2 is k-isomorphic to a conic curve in $\mathbb{P}^2_k$ or $\mathbb{P}^2_{k'}$;*
(2) *If $p = 3$, then C_2 is k'-isomorphic to $\mathbb{P}^1_{k'}$.*

Proof. We have that

$$(K_{S/C} + C_2) \cdot_k C_2 = \deg_k(\omega_{C_2}) - \deg_k(\rho^*\omega_C|_{C_2}) = \deg_k(\omega_{C_2}).$$

Moreover, it follows that

$$\begin{aligned}\psi^*(K_{S/C} + C_2) \cdot_k C_2' &= (K_{S'/C'} + pC_2') \cdot_k C_2' \\ &= (p-1)C_2' \cdot_k C_2' = (p-1)\deg_k(\varphi^* D) = -p(p-1),\end{aligned}$$

where the first equation holds by Lemma 3.2, the third one by Lemma 3.3 and the last one by $D = -P$. Since $\psi|_{C_2'} : C_2' \to C_2$ is birational, it follows that

$$\deg_k(\omega_{C_2}) = (K_{S/C} + C_2) \cdot_k C_2 = \psi^*(K_{S/C} + C_2) \cdot_k C_2' = -p(p-1).$$

Since C_2 is a projective curve such that $\omega_{C_2}^{-1}$ is an ample invertible sheaf, it holds that $H^1(C_2, \mathcal{O}_{C_2}) = 0$. Then, it follows from [20, Lemma 10.6] that

- C_2 is K-isomorphic to $\mathbb{P}^1_K$, or
- C_2 is K-isomorphic to a conic curve on $\mathbb{P}^2_K$,

where $K = H^0(C_2, \mathcal{O}_{C_2})$. In any case, it holds that $\deg_K(\omega_{C_2}) = -2$. The natural morphisms $C' \simeq C_2' \to C_2 \to C$ induce field extensions

$$k = H^0(C, \mathcal{O}_C) \subset H^0(C_2, \mathcal{O}_{C_2}) \subset H^0(C_2', \mathcal{O}_{C_2'}) = k',$$

which implies that $K = H^0(C_2, \mathcal{O}_{C_2})$ is either k or k'. Thus (1) holds.

We show (2). Since $p = 3$, we get $\deg_k(\omega_{C_2}) = -6$. Thus we have that $k' = H^0(C_2, \mathcal{O}_{C_2})$ and $\deg_{k'}(\omega_{C_2}) = -2$. Since $\psi|_{C_2'} : C_2' \to C_2$ is birational and $\deg_{k'}(\omega_{C_2'}) = \deg_{k'}(\omega_{C_2})$, $\psi|_{C_2'}$ is an isomorphism. Hence (2) holds. □

Theorem 3.9. *We use Notation* 3.7. *Then there exists an effective $\mathbb{Q}$-divisor Δ_S on S such that (S, Δ_S) is klt and $-(K_S + \Delta_S)$ is ample.*

Proof. By Proposition 3.5, it suffices to show that $-D - K_C$ is ample and $(S, \frac{2}{p}C_2)$ is log canonical. The ampleness of $-D - K_C$ follows from $D = -P$ and $K_C \sim 0$. If $p = 3$, then it follows from Lemma 3.8 that $(S, \frac{2}{3}C_2)$ is log canonical. If $p = 2$, then C_2 is semi log canonical by Lemma 2.12 and Lemma 3.8. Therefore, (S, C_2) is log canonical by inversion of adjunction (*cf.* [28, Theorem 5.1]). □

Theorem 3.10. *Let k be an imperfect field whose characteristic p is two or three. Then there exists a projective $\mathbb{Q}$-factorial klt surface T over k with $k = H^0(T, \mathcal{O}_T)$ that satisfies the following properties:*

(1) $-K_T$ *is ample;*
(2) $\rho(T) = 1$*;*
(3) *There is a Cartier divisor* M *such that* $M \not\sim 0$ *and* $pM \sim 0$*, and*
(4) *There exists a finite universal homeomorphism* $\mathbb{P}^2_{k'} \to T$*, where* $k \subset k'$ *is a purely inseparable extension of degree* p*.*

Proof. We use Notation 3.7. There is a $\mathbb{P}^1$-bundle structure $\rho : S \to C$. Since $S' = \mathbb{P}_{\mathbb{P}^1_{k'}}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, we have the blow-down $f' : S' \to \mathbb{P}^2_{k'} =: T'$ contracting C'_2. Thus, we get a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\psi} & S \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{\psi_T} & T, \end{array}$$

where ψ_T is a finite universal homeomorphism of degree p and f is the birational morphism to a projective normal surface T satisfying $\mathrm{Ex}(f) = C_2$. Note that we can find such a surface T as the Stein factorisation of $S \xrightarrow{\widetilde{\psi}} S' \xrightarrow{f'} T'$, where $\widetilde{\psi}$ is obtained by the factorisation $F^e_{S'} : S' \xrightarrow{\psi} S \xrightarrow{\widetilde{\psi}} S'$ of the e-th iterated absolute Frobenius morphism $F^e_{S'}$ for some e. Thus (4) holds. Since f can be considered as the contraction of a $(K_S + \Delta_S)$-negative extremal ray, we have that $(T, f_*\Delta_S)$ is klt and T is $\mathbb{Q}$-factorial (*cf*. [28, Theorem 4.4]). In particular, T is klt. By [28, Theorem 4.4], the assertions (1) and (2) hold. We get (3) by Proposition 3.6(5) and [28, Theorem 4.4]. □

Remark 3.11. The surface T constructed in the proof of Theorem 3.10 has a unique singular point t. We proved that the singular point t is klt in the proof of Theorem 3.10. On the other hand, we see that t is not a canonical singularity as follows. Let a be the rational number defined by the following equation

$$K_S = f^*K_T + aC_2.$$

It suffices to prove that $a < 0$. The proof of Lemma 3.8 implies that

$$(K_S + C_2) \cdot_k C_2 = \deg_k(\omega_{C_2}) = -p(p-1).$$

On the other hand, it holds that

$$\begin{aligned} C_2 \cdot_k C_2 &= \deg_k(\mathcal{O}_S(C_2)|_{C_2}) = \deg_k(\mathcal{O}_{S'}(pC'_2)|_{C'_2}) \\ &= p\deg_k(\mathcal{O}_{\mathbb{P}^1_{k'}}(-1)) = -p^2, \end{aligned}$$

where the second equation holds by Lemma 3.2 and the third one follows from Lemma 3.3 and Notation 3.7. To summarise, we obtain $K_S \cdot_k C_2 = p$ and $a = -1/p$, as desired.

4. Pathological Mori fibre spaces

4.1. Mori fibre spaces with non-trivial torsion divisors

Proof. (of Theorem 1.2) By Theorem 3.10, there exist a projective $\mathbb{Q}$-factorial klt surface T over $k(t)$ with $H^0(T, \mathcal{O}_T) = k(t)$ and a Cartier divisor M on T which satisfy the properties (1)–(4) in Theorem 3.10. We can find a projective morphism $f : V \to W$ and a Cartier divisor D on V, where W is a non-empty open subset W of $\operatorname{Spec} k[t]$, $V \times_W \operatorname{Spec} k(t) = T$, and $D|_T = M$. In particular, the property (2) in the statement holds. After possibly shrinking W, we may assume that V is normal, $f_*\mathcal{O}_V = \mathcal{O}_W$, $pD \sim 0$, and $-K_V$ is an f-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Since $D|_T = M$, the property (4) in the statement holds. After possibly shrinking again, the property (1) (respectively (3)) in the statement holds by Lemma 2.2 (respectively Lemma 2.5). $\square$

4.2. Mori fibre spaces with non-klt bases

The main purpose of this subsection is to show Theorem 4.4, since it directly implies one of our main results: Theorem 1.1. In Part 4.2.1, we summarise notation. In Part 4.2.2 and Part 4.2.3, we run a suitable minimal model program that will be needed in the proof of Theorem 4.4. In Part 4.2.4, we prove Theorem 4.4 and Theorem 1.1.

4.2.1. *Setup*

We use Notation 3.7. Assume that $[k : k^p] < \infty$. Let $M_C := \mathcal{O}_C(P)$, where P is a k-rational point on C around which C is smooth over k. Let $M_S := \rho^* M_C$, $M_{C'} := \varphi^* M_C$, and $M_{S'} := \psi^* \rho^* M_C$. We set

$$X := \mathbb{P}_S(\mathcal{O}_S \oplus M_S), \qquad R := \mathbb{P}_C(\mathcal{O}_C \oplus M_C)$$
$$X' := \mathbb{P}_{S'}(\mathcal{O}_{S'} \oplus M_{S'}), \quad R' := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus M_{C'})$$

and obtain a cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & R \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_R} \\ S & \xrightarrow{\rho} & C, \end{array}$$

whose base change by $(-) \times_C C'$ can be written as

$$\begin{array}{ccc} X' & \xrightarrow{\rho'_X} & R' \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'_R} \\ S' & \xrightarrow{\rho'} & C'. \end{array}$$

Let $C^{\pm}$ be the sections of π_R corresponding to the direct sum decomposition $\mathcal{O}_C \oplus M_C$ such that $\mathcal{O}_R(C^{\pm})|_{C^{\pm}} = \pm M_C$ if we identify C with $C^{\pm}$. We set $S^{\pm}$, $C'^{\pm}$ and $S'^{\pm}$ to be the pull-backs of $C^{\pm}$ to X, R' and X', respectively.

Since $R' \simeq \mathbb{P}_{\mathbb{P}^1_{k'}}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, we have that C'^{-} is a (-1)-curve on R', *i.e.*, $K_{R'} \cdot_{k'} C'^{-} = C'^{-} \cdot_{k'} C'^{-} = -1$. Let

$$\theta' : R' \to \mathbb{P}^2_{k'} =: Q'$$

be the blow-down contracting C'^{-}.

Corresponding to θ', we can find a birational morphism $\theta : R \to Q$ to a projective surface Q such that $\theta_* \mathcal{O}_R = \mathcal{O}_Q$ and $\mathrm{Ex}(\theta) = C^{-}$. Indeed, for a positive integer e such that the e-th iterated absolute Frobenius morphism $F^e : R' \to R' =: R'^{[p^e]}$ factors through the induced morphism $R' \to R$, we define Q as the normalisation of $Q'^{[p^e]}$ in $K(R)$, where $\theta'^{[p^e]} : R'^{[p^e]} \to Q'^{[p^e]}$ is defined as the same morphism as θ'.

Let $q := \theta(\mathrm{Ex}(\theta))$ and $q' := \theta'(\mathrm{Ex}(\theta'))$. In the proof of Theorem 4.4, we will run an S^{-}-MMP over Q

$$X =: X_0 \overset{f_0}{\dashrightarrow} X_1 \xrightarrow{f_1} X_2,$$

consisting of two steps: f_0 is a flip and f_1 is a divisorial contraction. To this end, we construct the corresponding S'^{-}-MMP in Part 4.2.2 and Part 4.2.3.

4.2.2. *The first step: flip*

We use the same notation as in Part 4.2.1. Let $H_{X'}$ be an ample Cartier divisor on X' and we define λ' as $\lambda' := \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid H_{X'} + \lambda S'^{-} \text{ is nef}\}$. Set

$$L' := H_{X'} + \lambda' S'^{-}.$$

Since S'^{-} is a smooth projective surface such that $-K_{S'^{-}}$ is ample,

- any nef Cartier divisor N on S'^{-} is semi-ample (*cf.* [26, Theorem 1.3]) and
- there are finitely many curves $\gamma_1, \cdots, \gamma_r$ on S' such that $\mathrm{NE}(S') = \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[\gamma_i]$ (*cf.* [28, Theorem 2.14]).

Hence, λ' is a positive rational number and $L'|_{S'^{-}}$ is semi-ample. It follows from Keel's theorem ([18, Proposition 1.6]) that L' is semi-ample. Let

$$g' : X' \to Z'$$

be the birational contraction with $g'_* \mathcal{O}_{X'} = \mathcal{O}_{Z'}$ induced by L'. Since $L'|_{S'^{-}}$ is semi-ample and $(L'|_{S'^{-}}) \cdot (\pi'^* \rho'^* (c')|_{S'^{-}}) > 0$ for a closed point of $c' \in C'$, we have that $\mathrm{Ex}(g')$ is equal to the (-1)-curve Γ' on S'^{-}. In particular, g' is a small birational morphism.

We construct a flip of g'. Let

$$h' : Y' \to X'$$

be the blowup along Γ'. We have that $E' := \mathrm{Ex}(h')$ is isomorphic to $\mathbb{P}_{\Gamma'}(N_{\Gamma'/X'})$, where $N_{\Gamma'/X'}$ is the normal bundle, which is an extension of $N_{S'^-/X'}|_{\Gamma'}$ and N_{Γ'/S'^-}. Since

$$S'^- \cdot_{k'} \Gamma' = -1, \quad (\Gamma' \text{ in } S'^-) \cdot_{k'} (\Gamma' \text{ in } S'^-) = -1,$$

the locally free sheaf $N_{\Gamma'/X'}$ is corresponding to an extension class $\alpha \in \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(-1), \mathcal{O}(-1)) = 0$. Therefore, we get $N_{\Gamma'/X'} \simeq \mathcal{O}_{\Gamma'}(-1) \oplus \mathcal{O}_{\Gamma'}(-1)$. It follows that $E' \simeq \mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'}$. Let T' be the proper transform of S'^- on Y'.

Lemma 4.1. *The following hold:*

(1) $K_{X'} \cdot \Gamma' = 0$;
(2) $\mathcal{O}_{Y'}(E')|_{E'} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ *if we identify* E' *with* $\mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'}$;
(3) $\mathcal{O}_{Y'}(T')|_{E'} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)$ *if we identify* E' *and* $E' \to \Gamma'$ *with* $\mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'}$ *and its first projection, respectively.*

Proof. The assertion (1) follows from $S'^- \cdot_{k'} \Gamma' = -1$ and

$$(K_{X'} + S'^-) \cdot_{k'} \Gamma' = K_{S'^-} \cdot_{k'} \Gamma' = -1,$$

where the latter equation follows from the fact that Γ' is a (-1)-curve.

We show (2). Since $K_{Y'} = h'^* K_{X'} + E'$, we have that

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \simeq \mathcal{O}_{Y'}(K_{Y'} + E')|_{E'} = \mathcal{O}_{Y'}(h'^* K_{X'} + 2E')|_{E'} \simeq \mathcal{O}_{Y'}(2E')|_{E'},$$

where the last isomorphism holds because (1) implies $K_{X'}|_{\Gamma'} \sim 0$. Thus (2) holds.

The assertion (3) follows from $h'^* S'^- = T' + E'$, (2) and $\mathcal{O}_X(S'^-)|_{\Gamma'} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. □

Lemma 4.2. *Let* k_1 *be a field. Let* $\zeta : Y_1 \to Z_1$ *be a birational* k_1*-morphism of projective normal* k_1*-varieties. For any* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* N *on* Y_1 *and an ample* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* H *on* Z_1*, there exists a positive integer* m *such that* $(N + m\zeta^* H)|_S$ *is big for any integral closed subscheme* S *on* Y_1 *such that* $S \not\subset \mathrm{Ex}(\zeta)$*. In particular, if* N *is* ζ*-nef and* $\zeta(\mathrm{Ex}(\zeta))$ *is one point, then* $N + m\zeta^* H$ *is nef.*

Proof. Let

$$I := \{S \mid S \text{ is an integral closed subscheme of } Y_1 \text{ such that } S \not\subset \mathrm{Ex}(\zeta)\}.$$

If $S \in I$, the induced morphism $S \to \zeta(S)$ is birational. Therefore, for any $S \in I$, there is $n_S \in \mathbb{Z}_{>0}$ such that $(N + n_S \zeta^* H)|_S$ is big. Let $n_1 := n_{Y_1}$. By Kodaira's lemma, we may write $N + n_1 \zeta^* H = A + D$ where A is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and D is an effective $\mathbb{Q}$-divisor on Y_1. Let $D = \sum e_j D_j$ be the decomposition into the irreducible components and we set $n_2 := \max_{D_j \in I}\{n_1, n_{D_j}\}$. For any $D_j \in I$, let D_j^N be its normalisation and we again apply Kodaira's lemma to $(N + n_2 \zeta^* H)|_{D_j^N}$. Repeating the same procedure finitely many times, we can find $n \in \mathbb{Z}_{>0}$ such that $(N + n\zeta^* H)|_S$ is big for every $S \in I$. □

For an ample Cartier divisor $H_{Z'}$ on Z' and a sufficiently large integer m, we set

$$M' := T' + mh'^* g'^* H_{Z'}.$$

We have that M' is nef and big by Lemma 4.1(3) and Lemma 4.2. It follows from Lemma 4.2 that $\mathbb{E}(M') = E'$, where we refer to [18, Definition 0.1] for the definition of $\mathbb{E}(M')$. By [18, Theorem 0.2], we get the birational morphism

$$h'_1 : Y' \to X'_1$$

with $(h'_1)_* \mathcal{O}_{Y'} = \mathcal{O}_{X'_1}$ induced by M'. By our construction, we have that X'_1 is $\mathbb{Q}$-factorial, $\rho(X'_1) = \rho(X'_0) = 3$ (*cf.* [4, Lemma 2.1]), $Y' \to Z'$ factors through h'_1, the fibre of the induced morphism $X'_1 \to Q'$ over q' is set-theoretically equal to $S'^-_1 \cup \Gamma'_1$, and $\Gamma'_1 \not\subset S'^-_1$, where $\Gamma'_1 := h'_1(E')$ and S'^-_1 is the proper transform of S'^-. In particular, S'^-_1 is ample over Z', hence $X'_1 \to Z'$ is a flip of $X' \to Z'$.

4.2.3. *The second step: divisorial contraction*

We use the same notation as in Part 4.2.1 and Part 4.2.2.

Lemma 4.3. *The following hold:*

(1) *The normalisation of S'^-_1 is a universal homeomorphism;*
(2) $-S'^-_1|_{S'^-_1}$ *is ample.*

Proof. We show (1). Let $S'^-_{Y'}$ be the proper transform of S'^- on Y'. Note that $\widetilde{h'} : S'^-_{Y'} \xrightarrow{\simeq} S'^-$ and the exceptional locus of $\widetilde{h'_1} : S'^-_{Y'} \to S'^-_1$ is equal to $\Gamma'_{Y'}$, where $\Gamma'_{Y'} := (\widetilde{h'})^{-1}(\Gamma')$. Since $\Gamma'_{Y'} \simeq \Gamma' \simeq \mathbb{P}^1_{k'}$, we have that $\widetilde{h'_1}(\Gamma'_{Y'})$ is a k'-rational point and the induced morphism $\Gamma'_{Y'} \to \widetilde{h'_1}(\Gamma'_{Y'})$ is the same as the structure morphism $\mathbb{P}^1_{k'} \to \operatorname{Spec} k'$. In particular, any fibre of $\widetilde{h'_1} : S'^-_{Y'} \to S'^-_1$ is geometrically connected. Since $\widetilde{h'_1}$ factors through the normalisation $\nu_{S'^-_1} : (S'^-_1)^N \to S'^-_1$ of S'^-_1, any fibre of $\nu_{S'^-_1}$ is geometrically connected. Hence, (1) holds.

We show (2). Take a curve B' on Q' passing through q' such that $|B'|$ is base point free. Then the inverse image D' to X'_1 can be written as

$$D' = aS'^-_1 + F'$$

where $a > 0$ and F' is a nonzero effective $\mathbb{Q}$-divisor with $S'^-_1 \not\subset \operatorname{Supp} F'$. Take a general curve G' on S'^-_1. Since $D' \cdot G' = 0$ and $F' \cdot G' > 0$, we have that $S'^-_1 \cdot G' < 0$. Thus (2) holds by $\rho(S'^-_1) = 1$. □

Let $H_{X'_1}$ be an ample Cartier divisor on X'_1 and we define ν' by $\nu' := \sup\{\nu \in \mathbb{R}_{\geq 0} \mid H_{X'_1} + \nu S'_1 \text{ is nef}\}$. We have that ν' is a positive rational number and let

$$N' := H_{X'_1} + \nu' S'^-_1.$$

Since we can find a positive integer m such that $\mathcal{O}_{X_1'}(mN')|_{S_1'^-} \simeq \mathcal{O}_{S_1'^-}$ by Lemma 4.3, we have that N' is semi-ample by Keel's theorem [18, Proposition 1.6]. Let

$$f_1' : X_1' \to X_2'$$

be the birational morphism induced by N' with $(f_1')_*\mathcal{O}_{X_1'} = \mathcal{O}_{X_2'}$. We also get a morphism $\alpha' : X_2' \to Q'$.

4.2.4. *Proof of Theorem* 1.1

Theorem 4.4. *Let k be an imperfect field whose characteristic p is two or three. If $[k : k^p] < \infty$, then there exists a k-morphism $\alpha : X_2 \to Q$ of projective normal k-varieties, with $\alpha_*\mathcal{O}_{X_2} = \mathcal{O}_Q$ and $H^0(Q, \mathcal{O}_Q) = k$, that satisfies the following properties:*

(1) *X_2 is a $\mathbb{Q}$-factorial threefold of Fano type;*
(2) *Q is a projective $\mathbb{Q}$-factorial log canonical surface which is not klt;*
(3) *Any fibre of α is geometrically irreducible of dimension one, a general fibre of α is $\mathbb{P}^1$, and*
(4) *$\rho(X_2/Q) = 1$.*

Proof. We use the same notation as in Part 4.2.1, Part 4.2.2 and Part 4.2.3. We get the rational maps

$$X =: X_0 \overset{f_0}{\dashrightarrow} X_1 \xrightarrow{f_1} X_2 \xrightarrow{\alpha} Q$$

corresponding to

$$X' =: X_0' \overset{f_0'}{\dashrightarrow} X_1' \xrightarrow{f_1'} X_2' \xrightarrow{\alpha'} Q',$$

where $X_0 \overset{f_0}{\dashrightarrow} X_1 \xrightarrow{f_1} X_2$ is an S^--MMP over Q. Indeed, for a positive integer e such that the e-th iterated absolute Frobenius morphism $F^e : X' \to X' =: X'^{[p^e]}$ factors through the induced morphism $X' \to X$, we define X_i as the normalisation of $X_i'^{[p^e]}$ in $K(X)$, where $X'^{[p^e]} \dashrightarrow X_i'^{[p^e]}$ is the same birational map as $X' \dashrightarrow X_i'$.

Since X_0' is of Fano type, f_0 is small and f_1 is birational, we have that also X_2 is of Fano type by Lemma 2.8. Thus (1) holds. Since $Q' \simeq \mathbb{P}^2_{k'}$ is $\mathbb{Q}$-factorial, so is Q (*cf.* [29, Lemma 2.5]). We can write $K_R + bC^- = \theta^* K_Q$ for some $b \in \mathbb{Q}$. By $\theta^* K_Q \cdot_k C^- = 0$, $(C^-)^2 < 0$ and

$$(K_R + C^-) \cdot_k C^- = \deg_k(\omega_{C^-}) = 0,$$

we have that $b = 1$. It follows from [20, Corollary 2.13] that Q is not klt but log canonical. Thus (2) holds. The assertion (3) follows from the construction, because the fibre of $X' \to Q'$ over q' is an image of $\mathrm{Ex}(h') \simeq \mathbb{P}^1_{k'} \times_{k'} \mathbb{P}^1_{k'}$ and hence geometrically irreducible. Thanks to (3), we have that $\rho(X_2'/Q') = 1$. The assertion (4) holds by $\rho(X_2'/Q') = 1$ and Lemma 2.4. $\square$

Proof of Theorem 1.1. We apply Theorem 4.4 for a field $k(t)$. Then there exists a $k(t)$-morphism $\alpha : X_2 \to Q$ of projective normal $k(t)$-varieties, with $\alpha_* \mathcal{O}_{X_2} = \mathcal{O}_Q$ and $H^0(Q, \mathcal{O}_Q) = k(t)$, satisfying the properties (1)–(4) in Theorem 4.4. We can find projective k-morphisms

$$V \xrightarrow{f} W \xrightarrow{g} T$$

of normal k-varieties such that T is a non-empty open subset of Spec $k[t]$ and $f \times_T \operatorname{Spec} k(t) = \alpha$. After possibly shrinking T, we may assume that

- V and W are $\mathbb{Q}$-factorial by Lemma 2.2,
- V is klt by Lemma 2.2,
- W is not klt by Lemma 2.2, and
- $f_* \mathcal{O}_V = \mathcal{O}_W$.

We set W_1 to be the subset of W consisting of the points $w \in W$ such that V_w is geometrically irreducible and of dimension one. By [13, 9.5.5 and 9.7.7], W_1 is a constructible subset of W.

Claim 4.5. There exists an open subset W_2 of W such that $W_\eta \subset W_2 \subset W_1$, where η is the generic point of T.

Proof of Claim 4.5. For a subset B of a set A, let $B^c := A \setminus B$. By Theorem 4.4(3), we have that $W_\eta \subset W_1$. This inclusion implies that $\eta \notin g(W_1^c)$, hence the constructible subset $g(W_1^c)$ of a curve T is a proper closed subset of T. Thus the inclusions $W_\eta \subset W_2 \subset W_1$ hold for $W_2 := g^{-1}(g(W_1^c)^c)$. □

After replacing W by W_2, we may assume that the fibre V_w over any point $w \in W$ is geometrically irreducible and of dimension one. In particular, $\rho(V/W) = 1$. This completes the proof of Theorem 1.1. □

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