Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XX (2020), 1147-1167

Compact hypersurfaces in Randers space

JINTANG LI

Abstract. Let $\left(\overline{M}^{n+1}, \overline{F}\right)$ be a Randers space and (M^n, F) a compact hypersurface of $\left(\overline{M}^{n+1}, \overline{F}\right)$. In this paper, we prove that if the second mean curvature H_2 is constant and the norm square S of the second fundamental form of M satisfies a certain inequality, then either M is a Randers space with constant flag curvature $1 + H_2$ or the equality holds.

Mathematics Subject Classification (2010): 53C60 (primary); 53C40 (secondary).

1. Introduction

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M, Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM\backslash 0$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form $g_Y = g_{ij}(x, Y)dx^i \otimes dx^j$ is positively definite, where $g_{ij} = \frac{1}{2}\partial^2(F^2)/\partial Y^i \partial Y^j$.

A Finsler metric is just a Riemannian metric without the quadratic restriction. The pair (M, F) is called a Finsler manifold.

Riemannian submanifolds are important in modern differential geometry. For a compact Riemannian hyprsurface M of the Euclidean unit sphere $S^{(n+1)}(1)$, the second fundamental form $B = h_{ij}^{n+1}\omega^i \otimes \omega^j \otimes e_{n+1}$, where $\{\omega^i\}$ is the orthonormal coframe of M. The second mean curvature H_2 of M is defined by $H_2 = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \lambda_i\lambda_j$, where λ_i are the eigenvalues of the second fundamental

This research was supported by the National Natural Science Foundation of China (No. 11871405).

Received November 14, 2017; accepted in revised form October 06, 2018.
Published online September 2020.

tensor h_{ij}^{n+1} of M. For a compact hypersurface M of the Euclidean unit sphere, the Gauss equations is $R = \sum_{i,j} R_{ijij} = n(n-1) + n(n-1)H_2$, which implies that the scalar curvature R is constant if and only if the second mean curvature H_2 is constant.

As well kown, by using Cheng-Yau's self-adjoint operator $\Box$, Li [2] proved the following:

Theorem A. *Let M^n be a compact hypersurface with constant second mean curvature H_2 in the Euclidean unit sphere. If $H_2 \geq 0$ and the norm square S of the second fundamental form of M satisfies*

$$S \leq \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big),$$

then either $S = nH_2$ and M is a totally umbilical hypersurface; or

$$S = \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big)$$

and $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, where $r = \sqrt{\frac{n-2}{n(H_2+1)}}$.

As far as we know, there are very few global rigidity results on Finsler submanifolds. In this paper, by the Gauss formula of Chern connection and defining a similar self-adjoint operator $\Box$ on Finsler manifolds, we study the hypersurfaces of Randers space with constant flag curvature and obtain the following:

Main Theorem. *Let $(\overline{M}, \overline{F})$ be a Randers space with constant flag curvature $\overline{K} = 1$ and (M^n, F) a compact hypersurface of $(\overline{M}, \overline{F})$ with constant second mean curvature H_2. If $H_2 \geq 0$ and the norm square S of the second fundamental form of M satisfies*

$$S \leq \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big), \tag{1.1}$$

then either $S = nH_2$ and M is a Randers space with constant flag curvature $K = 1 + H_2$; or

$$S = \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big)$$

and $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, where $r = \sqrt{\frac{n-2}{n(H_2+1)}}$.

Remark. This theorem generalizes theorem A from the Riemannian case to the Randers space.

2. Preliminaries

Let (M^n, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form, the fundamental tensor and the *Cartan* tensor as follows:

$$\omega = \frac{\partial F}{\partial Y^i} dx^i, \;\; g_Y = g_{ij}(x, Y) dx^i \otimes dx^j, \;\; A_Y = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \;\; A_{ijk} := \frac{F \partial g_{ij}}{2 \partial Y^k}.$$

It is well known that there exists uniquely the Chern connection ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying that

$$\begin{cases} d(dx^i) - dx^j \wedge \omega_j^i = -dx^j \wedge \omega_j^i = 0 \\ dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta Y^k}{F}, \end{cases}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F} A_{jk}^i \gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l + \frac{1}{F} P_{jkl}^i dx^k \wedge \delta Y^l,$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and hv-curvature tensor of the Chern connection, respectively.

Given a non-zero vector $U \in T_x M$, the flag curvature $K(Y, U)$ on $(x, Y) \in TM \backslash 0$ defined by

$$K(Y, U) = \frac{Y^i U^j Y^k U^l R_{jikl}}{g_Y(Y, Y) g_Y(U, U) - [g_Y(Y, U)]^2},$$

where $R_{jikl} = g_{is} R_{jkl}^s$.

$\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ is called an isometric immersion from a Finsler manifold to a Finsler manifold if $F(Y) = \overline{F}(\varphi_*(Y))$. We have that [8]

$$\begin{aligned} g_Y(U, V) = \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)), \; & A_Y(U, V, W) \\ = \overline{A}_{\varphi_*(Y)}(\varphi_*(U), & \varphi_*(V), \varphi_*(W)), \end{aligned} \tag{2.1}$$

where $Y, U, V, W \in TM$, $\overline{g}$ and $\overline{A}$ are the fundamental tensor and the *Cartan* tensor of $\overline{M}$, respectively.

In the following we simplify A_Y and g_Y to A and g, respectively. Any vector field $U \in \Gamma(TM)$ will be identified with the corresponding vector field $d\varphi(U) \in \Gamma(T\overline{M})$. We will use the following convention:

$$\begin{aligned} &1 \le i, j, \cdots \le n; \qquad && n+1 \le \alpha, \beta, \cdots \le n+p; \\ &1 \le \lambda, \mu, \cdots \le n-1; \qquad && 1 \le A, B, \cdots \le n+p. \end{aligned}$$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion. Take a $\overline{g}$-orthonormal frame form $\{e_A\}$ for each fibre of $\pi^*T\overline{M}$ and $\{\omega^A\}$ is its local dual coframe, such that $\{e_i\}$ is a frame field for each fibre of π^*TM and ω^n is the *Hilbert* form, where $\pi : TM \to M$ denotes the natural projection. Let θ_B^A and ω_j^i denote the Chern connection 1-form of $\overline{F}$ and F, respectively, *i.e.*, $\overline{\nabla} e_A = \theta_A^B e_B$ and $\nabla e_i = \omega_i^j e_j$, where $\overline{\nabla}$ and ∇ are the Chern connection of $\overline{M}$ and M, respectively. We obtain that $A(e_i, e_j, e_n) = \overline{A}(e_A, e_B, e_n) = 0$, where $e_n = \ell = \frac{Y^i}{F}\frac{\partial}{\partial x^i}$ is the natural dual of the Hilbert form ω^n. The structure equations of $\overline{M}$ are given by

$$\begin{cases} d\theta^A = -\theta_B^A \wedge \theta^B \\ d\theta_B^A = -\theta_C^A \wedge \theta_B^C + \frac{1}{2}\overline{R}_{BCD}^{A}\omega^C \wedge \omega^D + \overline{P}_{BCD}^{A}\omega^C \wedge \theta_n^D \\ \theta_B^A + \theta_A^B = -2\overline{A}_{ABC}\theta_n^C \\ \theta_n^A + \theta_A^n = 0, \quad \theta_n^n = 0. \end{cases} \tag{2.2}$$

The collection $\{e_i^H, \widehat{e}_{n+\lambda}\}$ forms an orthonormal basis on the projectivized tangent bundle PTM and $\{\omega^i, \omega_n^\lambda\}$ is its local dual coframe, where $e_i^H = u_i^j \frac{\delta}{\delta x^j} = u_i^j(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k})$ denotes the horizontal part of e_i, $\widehat{e}_{n+\lambda} = u_\lambda^j \frac{\delta}{\delta Y^j} = u_\lambda^j \frac{F\partial}{\partial Y^j}$, $\omega^i = v_j^i dx^j$ and $\omega_n^\lambda = v_j^\lambda \delta Y^j$.

We obtain that [3] $\theta_j^\alpha = h_{ij}^\alpha \omega^i$, $h_{ij}^\alpha = h_{ji}^\alpha$ and

$$\omega_i^j = \theta_i^j - \Psi_{jik}\omega^k, \tag{2.3}$$

where

$$\begin{aligned} \Psi_{jik} = {} & h_{jn}^\alpha \overline{A}_{ki\alpha} - h_{kn}^\alpha \overline{A}_{ji\alpha} - h_{in}^\alpha \overline{A}_{kj\alpha} \\ & - h_{nn}^\alpha \overline{A}_{iks}\overline{A}_{sj\alpha} + h_{nn}^\alpha \overline{A}_{ijs}\overline{A}_{sk\alpha} + h_{nn}^\alpha \overline{A}_{jks}\overline{A}_{si\alpha}. \end{aligned} \tag{2.4}$$

In particular,

$$\omega_i^n = \theta_i^n - h_{nn}^\alpha \overline{A}_{ki\alpha}\omega^k, \tag{2.5}$$

Using the almost $\overline{g}$-compatibility, we have

$$\theta_\alpha^j = (-h_{ij}^\alpha - 2h_{ni}^\beta \overline{A}_{j\alpha\beta} + 2h_{nn}^\beta \overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta})\omega^i - 2\overline{A}_{j\alpha\lambda}\omega_n^\lambda. \tag{2.6}$$

In particular, $\theta_\alpha^n = -h_{ni}^\alpha \omega^i$.

We quote the following propositions

Proposition 2.1 (Gauss equations, [3]). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{cases} P_{ik\lambda}^j = \overline{P}_{ik\lambda}^j + \Psi_{jik;\lambda} - 2\Psi_{sik}A_{js\lambda} - 2h_{ik}^\alpha \overline{A}_{j\lambda\alpha} \\ R_{ikl}^j = \overline{R}_{ikl}^j - h_{ik}^\alpha h_{jl}^\alpha + h_{il}^\alpha h_{jk}^\alpha + \Psi_{jik|l} - \Psi_{jil|k} \\ \qquad +\Psi_{sik}\Psi_{jsl} - \Psi_{sil}\Psi_{jsk} - 2h_{ik}^\alpha h_{nl}^\beta \overline{A}_{j\alpha\beta} + 2h_{il}^\alpha h_{nk}^\beta \overline{A}_{j\alpha\beta} \\ \qquad +2h_{ik}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ls\beta} - 2h_{il}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ks\beta} - h_{nn}^\alpha \overline{A}_{sl\alpha}\overline{P}_{iks}^j \\ \qquad +h_{nn}^\alpha \overline{A}_{sk\alpha}\overline{P}_{ils}^j + h_{nl}^\alpha \overline{P}_{ik\alpha}^j - h_{nk}^\alpha \overline{P}_{il\alpha}^j, \end{cases}$$

where " ; " and "|" denote the vertical and the horizontal covariant differentials with respect to the Chern connection ∇ respectively.

Proposition 2.2 (Codazzi equations, [3]). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{cases} h^{\alpha}_{ij;\lambda} = -\overline{P}^{\alpha}_{ij\lambda} \\ h^{\alpha}_{ij|k} - h^{\alpha}_{ik|j} = -\overline{R}^{\alpha}_{ijk} + h^{\beta}_{nj}\overline{P}^{\alpha}_{ik\beta} - h^{\beta}_{nk}\overline{P}^{\alpha}_{ij\beta} \\ \qquad -h^{\alpha}_{lk}\Psi_{lij} + h^{\alpha}_{lj}\Psi_{lik} - h^{\beta}_{nn}\overline{A}_{lj\beta}\overline{P}^{\alpha}_{ikl} + h^{\beta}_{nn}\overline{A}_{lk\beta}\overline{P}^{\alpha}_{ijl}. \end{cases}$$

Proposition 2.3 ([6]). *For $X = \sum_i x_i \omega^i \in \Gamma(\pi^* T^* M)$,*

$$\mathrm{div}_{\widehat{g}} X = \sum_i x_{i|i} + \sum_{\mu,\lambda} x_\mu P^n_{\lambda\lambda\mu}.$$

Proposition 2.4 ([1]). *Let (M, F) be a Finsler manifold. The following statements are mutually equivalent:*

(1) *(M, F) has constant flag curvature $K = \lambda$;*
(2) *$R^i_{njn} = (\delta_{ij} - \delta_{in}\delta_{jn})\lambda, \forall i, j$;*
(3) *The hh-curvature tensor R^i_{jkl} of the Chern connection has the formula*

$$R^i_{jkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\lambda - (A_{ijk}\delta_{ln} - A_{ijl}\delta_{kn})\lambda - \sum_s (A_{iks|n}A_{sjl|n} - A_{ils|n}A_{sjk|n})\lambda.$$

And, we have

$$A_{ijk|n|l} - A_{ijl|n|k} = (A_{ijl}\delta_{kn} - A_{ijk}\delta_{ln})\lambda. \tag{2.7}$$

Lemma 2.5 ([7]). *Let B be a real symmetric matrix with $tr B = 0$, then*

$$|tr B^3| \le \frac{n-2}{\sqrt{n(n-1)}}(tr B^2)^{\frac{3}{2}}.$$

Let $\phi = \sum_{i,j} \phi_{ij}\omega^i \otimes \omega^j$ be a symmetric tensor defined on the sphere bundle SM and $\psi = \sum_i \psi_i \omega^i \in \Gamma(\pi^* T^* M)$. Now we can define an operator associated to ϕ by

$$\Box f = \sum_{i,j} \phi_{ij} f_{|i|j} + \sum_{i,\lambda,\mu} \phi_{i\lambda} f_{|i} P^n_{\mu\mu\lambda} + \sum_i \psi_i f_{|i}, \quad \forall f \in C^\infty(SM). \tag{2.8}$$

Proposition 2.6 ([4]). *Let (M, F) be a compact Finsler manifold. Then the operator $\Box$ is self-adjoint if and only if $\sum_j \phi_{ij|j} - \psi_i = 0$.*

3. Hypersurfaces of Randers spaces

Proposition 3.1. *Let* (M, F) *be a Randers space. Then*

$$\begin{cases} A_{ijk} = 0 & \text{for } i, j, k \text{ distinct from each other} \\ A_{\lambda\lambda i} = A_{\mu\mu i} & \text{for } i, \lambda, \mu \text{ distinct from each other} \\ A_{\lambda\lambda\lambda} = 3A_{\mu\mu\lambda} & \text{for } \lambda \neq \mu. \end{cases}$$

Proof. For the Randers space (M, F) with $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}Y^iY^j}$ is a Riemannian metric and $\beta = b_iY^i$ is a 1-form, we have that [1]

$$\begin{aligned} &A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \\ &= \frac{1}{2}\left[\eta_{ij}\left(b_k - \frac{\beta}{\alpha}\widetilde{\ell}_k\right) + \eta_{jk}\left(b_i - \frac{\beta}{\alpha}\widetilde{\ell}_i\right) + \eta_{ki}\left(b_j - \frac{\beta}{\alpha}\widetilde{\ell}_j\right)\right], \end{aligned} \tag{3.1}$$

where $\ell_i = g_{ij}\frac{Y^j}{F}$, $\eta_{ij} = g_{ij} - \ell_i\ell_j$ and $\widetilde{\ell}_i = \ell_i - b_i$.

Since $u^i_k u^j_l \eta_{ij} = \delta_{kl} - \delta_{kn}\delta_{ln}$, where $e_i = u^j_i \frac{\partial}{\partial x^j}$ and $u^j_n = \frac{Y^j}{F}$, we obtain by (3.1)

$$\begin{aligned} A_{ijk} &= u^s_i u^t_j u^r_k A\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^t}, \frac{\partial}{\partial x^r}\right) \\ &= \frac{1}{2}\left[u^r_k\left(b_r - \frac{\beta}{\alpha}\widetilde{\ell}_r\right)(\delta_{ij} - \delta_{in}\delta_{jn}) + u^s_i\left(b_s - \frac{\beta}{\alpha}\widetilde{\ell}_s)(\delta_{jk} - \delta_{kn}\delta_{jn}\right)\right. \\ &\quad \left. + u^t_j\left(b_t - \frac{\beta}{\alpha}\widetilde{\ell}_t\right)(\delta_{ik} - \delta_{kn}\delta_{in})\right]. \end{aligned} \tag{3.2}$$

We obtain Proposition 3.1 immediately from (3.2). □

Proposition 3.2. *Let* (M, F) *be a Randers space. Then*

$$\begin{cases} A_{ijk|l} = 0 & \text{for } i, j, k \text{ distinct from each other} \\ A_{\mu\nu\sigma;\lambda} = 0 & \text{for } \mu, \nu, \sigma, \lambda \text{ distinct from each other} \\ A_{\nu\nu\mu;\lambda} = 4A_{\nu\nu\mu}A_{\nu\nu\lambda} & \text{for } \nu, \mu, \lambda \text{ distinct from each other.} \end{cases}$$

Proof. By Proposition 3.1, we have for λ, μ, ν distinct from each other

$$\begin{aligned} dA_{\lambda\mu\nu} =& A_{\lambda\mu\nu|l}\omega^l + A_{\lambda\mu\nu;\tau}\omega^\tau_n + A_{\sigma\mu\nu}\omega^\sigma_\lambda + A_{\lambda\sigma\nu}\omega^\sigma_\mu + A_{\lambda\mu\sigma}\omega^\sigma_\nu \\ =& A_{\lambda\mu\nu|l}\omega^l + A_{\lambda\mu\nu;\tau}\omega^\tau_n + A_{\mu\mu\nu}\omega^\mu_\lambda + A_{\nu\mu\nu}\omega^\nu_\lambda \\ &+ A_{\lambda\lambda\nu}\omega^\lambda_\mu + A_{\lambda\nu\nu}\omega^\nu_\mu + A_{\lambda\mu\lambda}\omega^\lambda_\nu + A_{\lambda\mu\mu}\omega^\mu_\nu \\ =& A_{\lambda\mu\nu|l}\omega^l + A_{\lambda\mu\nu;\tau}\omega^\tau_n - 2A_{\mu\mu\nu}A_{\mu\lambda\tau}\omega^\tau_n - 2A_{\nu\nu\mu}A_{\nu\lambda\tau}\omega^\tau_n \\ &- 2A_{\lambda\nu\nu}A_{\mu\nu\tau}\omega^\tau_n. \end{aligned} \tag{3.3}$$

It can be seen from (3.3) and Proposition 2.1 that

$$\begin{cases} A_{ijk|l} = 0 \\ A_{\lambda\mu\nu;\tau} = 2A_{\mu\mu\nu}A_{\mu\lambda\tau} + 2A_{\nu\nu\mu}A_{\nu\lambda\tau} + 2A_{\lambda\nu\nu}A_{\mu\nu\tau}. \end{cases} \tag{3.4}$$

We obtain Proposition 3.2 immediately from (3.4) and Proposition 3.1. □

Proposition 3.3. *Let* (M, F) *be a Randers space. Then*

$$\begin{cases} A_{\nu\nu\lambda;\lambda} = A_{\nu\nu\mu;\mu} + 4A^2_{\nu\nu\lambda} - 4A^2_{\nu\nu\mu} & \textit{for } \lambda, \mu\ \nu \textit{ distinct from each other} \\ A_{\nu\nu\mu|l} = A_{\lambda\lambda\mu|l} & \textit{for } \lambda, \mu, \nu \textit{ distinct from each other.} \end{cases}$$

Proof. For $\nu \neq \mu$, we have by Proposition 3.1 that

$$\begin{aligned} dA_{\nu\nu\mu} =& A_{\nu\nu\mu|l}\omega^l + A_{\nu\nu\mu;\lambda}\omega^\lambda_n + 2A_{s\nu\mu}\omega^s_\nu + A_{\nu\nu s}\omega^s_\mu \\ =& A_{\nu\nu\mu|l}\omega^l + A_{\nu\nu\mu;\lambda}\omega^\lambda_n + 2A_{\nu\nu\mu}\omega^\nu_\nu + 2A_{\mu\nu\mu}\omega^\mu_\nu \\ & + A_{\nu\nu\nu}\omega^\nu_\mu + A_{\nu\nu\tau}\omega^\tau_\mu + \sum_{s\neq\nu,\tau} A_{\nu\nu s}\omega^s_\mu \\ =& A_{\nu\nu\mu|l}\omega^l + A_{\nu\nu\mu;\lambda}\omega^\lambda_n - 2A_{\nu\nu\mu}A_{\nu\nu\lambda}\omega^\lambda_n - 4A_{\mu\nu\mu}A_{\mu\nu\lambda}\omega^\lambda_n \\ & + A_{\mu\mu\nu}\omega^\nu_\mu + A_{\nu\nu\tau}\omega^\tau_\mu + \sum_{s\neq\nu,\tau} A_{\nu\nu s}\omega^s_\mu. \end{aligned} \tag{3.5}$$

And for $\tau \neq \mu$, we have

$$\begin{aligned} dA_{\tau\tau\mu} =& A_{\tau\tau\mu|l}\omega^l + A_{\tau\tau\mu;\lambda}\omega^\lambda_n + 2A_{s\tau\mu}\omega^s_\tau + A_{\tau\tau s}\omega^s_\mu \\ =& A_{\tau\tau\mu|l}\omega^l + A_{\tau\tau\mu;\lambda}\omega^\lambda_n - 2A_{\tau\tau\mu}A_{\tau\tau\lambda}\omega^\lambda_n - 4A_{\mu\tau\mu}A_{\mu\tau\lambda}\omega^\lambda_n \\ & + A_{\mu\mu\tau}\omega^\tau_\mu + A_{\tau\tau\nu}\omega^\nu_\mu + \sum_{s\neq\tau,\nu} A_{\tau\tau s}\omega^s_\mu. \end{aligned} \tag{3.6}$$

It follows from (3.5), (3.6) and Proposition 3.1 that

$$\begin{cases} A_{\nu\nu\mu|l} = A_{\tau\tau\mu|l} \\ A_{\nu\nu\mu;\lambda} = A_{\tau\tau\mu;\lambda} + 2A_{\nu\nu\mu}A_{\nu\nu\lambda} + 4A_{\mu\nu\mu}A_{\mu\nu\lambda} \\ \qquad\qquad -2A_{\tau\tau\mu}A_{\tau\tau\lambda} - 4A_{\mu\tau\mu}A_{\mu\tau\lambda}. \end{cases} \tag{3.7}$$

We obtain Proposition 3.3 immediately from (3.7) and Proposition 3.1. □

Proposition 3.4. *Let* (M, F) *be a Randers space. Then*

$$\begin{cases} A_{ii\nu;\mu|l} = 4A_{ii\nu|l}A_{ii\mu} + 4A_{ii\nu}A_{ii\mu|l} & \textit{for } i, \mu, \nu \textit{ distinct from each other} \\ A_{ii\nu;\mu;\lambda} = 4A_{ii\nu;\lambda}A_{ii\mu} + 4A_{ii\nu}A_{ii\mu;\lambda} & \\ \qquad -8A_{ii\nu}A_{ii\mu}A_{ii\lambda} & \\ \qquad +2A_{ii\nu;\nu}A_{\nu\mu\lambda} - 8A^2_{ii\nu}A_{\nu\mu\lambda} & \textit{for } i, \mu, \nu \textit{ distinct from each other.} \end{cases}$$

Proof. Exterior differentiate the third formula of Proposition 3.2, we obtain

$$\begin{aligned}
&A_{ii\nu;\mu|l}\omega^l + A_{ii\nu;\mu;\lambda}\omega_n^\lambda + 2A_{si\nu;\mu}\omega_i^s + A_{iis;\mu}\omega_\nu^s + A_{ii\nu;s}\omega_\mu^s \\
=& 4(A_{ii\nu|l}\omega^l + A_{ii\nu;\lambda}\omega_n^\lambda + 2A_{si\nu}\omega_i^s + A_{iis}\omega_j^s)A_{ii\mu} \\
&+ 4A_{ii\nu}(A_{ii\mu|l}\omega^l + A_{ii\mu;\lambda}\omega_n^\lambda + 2A_{si\mu}\omega_i^s + A_{iis}\omega_\mu^s).
\end{aligned} \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}
&2A_{si\nu;\mu}\omega_i^s - 8A_{si\nu}A_{ii\mu}\omega_i^s - 8A_{ii\nu}A_{si\mu}\omega_i^s \\
=&2A_{ii\nu;\mu}\omega_i^i + 2A_{\nu i\nu;\mu}\omega_i^\nu + 2A_{\mu i\nu;\mu}\omega_i^\mu + 2\sum_{s\neq i,\nu,\mu} A_{si\nu;\mu}\omega_i^s \\
&- 8A_{ii\nu}A_{ii\mu}\omega_i^i - 8A_{\nu i\nu}A_{ii\mu}\omega_i^\nu \\
&- 8A_{ii\nu}A_{ii\mu}\omega_i^i - 8A_{ii\nu}A_{\mu i\mu}\omega_i^\mu \\
=&- 8A_{ii\nu}A_{ii\mu}\omega_i^i = 8A_{ii\nu}A_{ii\mu}A_{ii\lambda}\omega_n^\lambda,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&A_{iis;\mu}\omega_\nu^s + A_{ii\nu;s}\omega_\mu^s - 4A_{iis}A_{ii\mu}\omega_\nu^s - 4A_{ii\nu}A_{iis}\omega_\mu^s \\
=& A_{iii;\mu}\omega_\nu^i + A_{ii\nu;\mu}\omega_\nu^\nu + A_{ii\mu;\mu}\omega_\nu^\mu + \sum_{s\neq i,\nu,\mu} A_{iis;\mu}\omega_\nu^s \\
&+ A_{ii\nu;i}\omega_k^i + A_{ii\nu;\nu}\omega_\mu^\nu + A_{ii\nu;\mu}\omega_\mu^\mu + \sum_{s\neq i,j,\mu} A_{ii\nu;s}\omega_\mu^s \\
&- 4A_{iii}A_{ii\mu}\omega_\nu^i - 4A_{ii\nu}A_{ii\mu}\omega_\nu^\nu - 4A_{ii\mu}A_{ii\mu}\omega_\nu^\mu \\
&- 4\sum_{s\neq i,\nu,\mu} A_{iis}A_{ii\mu}\omega_\nu^s - 4A_{ii\nu}A_{iii}\omega_\mu^i - 4A_{ii\nu}A_{ii\nu}\omega_\mu^\nu \\
&- 4A_{ii\nu}A_{ii\mu}\omega_\mu^\mu \quad - 4\sum_{s\neq i,\nu,\mu} A_{ii\nu}A_{iis}\omega_\mu^s \\
=& A_{ii\mu;\mu}\omega_\nu^\mu + A_{ii\nu;\nu}\omega_\mu^\nu - 4A_{ii\mu}A_{ii\mu}\omega_\nu^\mu - 4A_{ii\nu}A_{ii\nu}\omega_\mu^\nu \\
=& (A_{ii\nu;\nu} + 4A_{ii\mu}^2 - 4A_{ii\nu}^2)\omega_\nu^\mu + A_{ii\nu;\nu}\omega_\mu^\nu - 4A_{ii\mu}A_{ii\mu}\omega_\nu^\mu \\
&- 4A_{ii\nu}A_{ii\nu}\omega_\mu^\nu \\
=& -2A_{ii\nu;\nu}A_{\nu\mu\lambda}\omega_n^\lambda + 8A_{ii\nu}^2 A_{\nu\mu\lambda}\omega_n^\lambda.
\end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$\begin{aligned}
&A_{ii\nu;\mu|l}\omega^l + A_{ii\nu;\mu;\lambda}\omega_n^\lambda + 8A_{ii\nu}A_{ii\mu}A_{ii\lambda}\omega_n^\lambda \\
&- 2A_{ii\nu;\nu}A_{\nu\mu\lambda}\omega_n^\lambda + 8A_{ii\nu}^2 A_{\nu\mu\lambda}\omega_n^\lambda \\
=& 4A_{ii\nu|l}A_{ii\mu}\omega^l + 4A_{ii\nu;\lambda}A_{ii\mu}\omega_n^\lambda + 4A_{ii\nu}A_{ii\mu|l}\omega^l + 4A_{ii\nu}A_{ii\mu;\lambda}\omega_n^\lambda.
\end{aligned} \tag{3.11}$$

We obtain Proposition 3.4 immediately from (3.11). $\square$

Let (M, F) be a hypersurface of Randers space $\left(\overline{M}^{n+1}, \overline{F}\right)$. Then we have that [1] (M, F) is a Randers manifold. We have:

Proposition 3.5 ([5]). *Let (M,F) be a hypersurface of a Randers space $(\overline{M}^{n+1}, \overline{F})$. Then*

$$\overline{A}_{n+1n+1n+1} = \overline{A}_{\lambda\lambda n+1} = 0, \quad \forall\lambda \quad \textit{and} \quad \Psi_{ijk} = 0, \forall i, j, k.$$

Proposition 3.6. *Let (M, F) be a hypersurface of a Randers space $\left(\overline{M}^{n+1}, \overline{F}\right)$. Then*

$$h^{n+1}_{ni}\overline{A}_{n+1n+1\lambda} = 0.$$

Proof. Taking the exterior differentiation of $\overline{A}_{\lambda\lambda n+1} = 0$, we obtain that

$$\begin{aligned}
0 &= \overline{A}_{\lambda\lambda n+1|l}\omega^l + \overline{A}_{\lambda\lambda n+1;\mu}\theta^\mu_n + \overline{A}_{\lambda\lambda n+1;n+1}\theta^{n+1}_n + 2\overline{A}_{n+1\lambda n+1}\theta^{n+1}_\lambda \\
&\quad + \overline{A}_{\lambda\lambda\tau}\theta^\tau_{n+1} \\
&= \Big\{\overline{A}_{\lambda\lambda n+1|l}\omega^l + h^{n+1}_{nl}\overline{A}_{\lambda\lambda n+1;n+1}\omega^l + 2h^{n+1}_{\lambda l}\overline{A}_{n+1n+1\lambda}\omega^l \\
&\quad - \sum_\tau h^{n+1}_{\tau l}\overline{A}_{\lambda\lambda\tau}\omega^l - \sum_\tau 2h^{n+1}_{nl}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau}\omega^l\Big\}(\mathrm{mod}(\omega^\nu_n)).
\end{aligned} \tag{3.12}$$

This implies that

$$\begin{aligned}
&\overline{A}_{\lambda\lambda n+1|l} + h^{n+1}_{nl}\overline{A}_{\lambda\lambda n+1;n+1} - \sum_\tau h^{n+1}_{\tau l}\overline{A}_{n+1n+1\tau} \\
&\quad - \sum_\tau 2h^{n+1}_{nl}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} = 0.
\end{aligned} \tag{3.13}$$

Setting $l = n$ in (3.13) and then taking the exterior differentiation of this formula, we obtain

$$\begin{aligned}
0 = \Big\{&(\overline{A}_{\lambda\lambda n+1|n;\mu}\omega^\mu_n + 2\overline{A}_{\lambda\lambda n+1|n}\theta^\lambda_\lambda + \overline{A}_{\lambda\lambda n+1|n}\theta^{n+1}_{n+1} + \overline{A}_{\lambda\lambda n+1|\mu}\omega^\mu_n) \\
&+ \Big[(h^{n+1}_{nn;\mu}\omega^\mu_n + 2h^{n+1}_{n\mu}\omega^\mu_n - h^{n+1}_{nn}\theta^{n+1}_{n+1})\overline{A}_{\lambda\lambda n+1;n+1} \\
&+ h^{n+1}_{nn}(\overline{A}_{\lambda\lambda n+1;n+1;\mu}\omega^\mu_n + 2\overline{A}_{\sigma\lambda n+1;n+1}\theta^\sigma_\lambda + 2\overline{A}_{\lambda\lambda n+1;n+1}\theta^{n+1}_{n+1})\Big] \\
&- \Big[\sum_\tau(h^{n+1}_{\tau n;\mu}\omega^\mu_n + h^{n+1}_{nn}\omega^n_\tau + h^{n+1}_{\sigma n}\omega^\sigma_\tau + h^{n+1}_{\tau\mu}\omega^\mu_n - h^{n+1}_{\tau n}\theta^{n+1}_{n+1})\overline{A}_{n+1n+1\tau} \\
&+ \sum_\tau h^{n+1}_{\tau n}(\overline{A}_{n+1n+1\tau;\mu}\omega^\mu_n + 2\overline{A}_{n+1n+1\tau}\theta^{n+1}_{n+1} + \overline{A}_{n+1n+1\sigma}\theta^\sigma_\tau)\Big] \\
&- \Big[\sum_\tau 2(h^{n+1}_{nn;\mu}\omega^\mu_n + 2h^{n+1}_{n\mu}\omega^\mu_n - h^{n+1}_{nn}\theta^{n+1}_{n+1})\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
&+ \sum_\tau 2h^{n+1}_{nn}(\overline{A}_{\lambda\lambda\tau;\mu}\omega^\mu_n + 2\overline{A}_{\sigma\lambda\tau}\theta^\sigma_\lambda + \overline{A}_{\lambda\lambda\sigma}\theta^\sigma_\tau)\overline{A}_{n+1n+1\tau} \\
&+ \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}(\overline{A}_{n+1n+1\tau;\mu}\omega^\mu_n + 2\overline{A}_{n+1n+1\tau}\theta^{n+1}_{n+1} + \overline{A}_{n+1n+1\sigma}\theta^\sigma_\tau)\Big]\Big\}(\mathrm{mod}(\omega^l))
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{ \Big(\overline{A}_{\lambda\lambda n+1|n;\mu} + \overline{A}_{\lambda\lambda n+1|\mu} + 2h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda n+1;n+1} + h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1;\mu} \\
&\quad - \sum_\tau h^{n+1}_{\tau n;\mu}\overline{A}_{n+1n+1\tau} + h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} - \sum_\tau h^{n+1}_{\tau\mu}\overline{A}_{n+1n+1\tau} \\
&\quad - \sum_\tau h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau;\mu} - \sum_\tau 4h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
&\quad - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau} - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu}\Big)\omega^\mu_n \\
&\quad + 2\overline{A}_{\lambda\lambda n+1|n}\omega^\lambda_\lambda + \Big(2h^{n+1}_{nn}\overline{A}_{\sigma\lambda n+1;n+1} - \sum_\tau 4h^{n+1}_{nn}\overline{A}_{\sigma\lambda\tau}\overline{A}_{n+1n+1\tau}\Big)\omega^\sigma_\lambda \\
&\quad - \Big(\sum_\tau h^{n+1}_{\sigma n}\overline{A}_{n+1n+1\tau} + \sum_\tau h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\sigma}\overline{A}_{n+1n+1\tau} \\
&\quad + \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\omega^\sigma_\tau \Big\} (\mathrm{mod}(\omega^l)) \\
&= \Big\{ \Big(\overline{A}_{\lambda\lambda n+1|n;\mu} + h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda n+1;n+1} + h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1;\mu} \\
&\quad + \sum_\tau \overline{P}^{n+1}_{\tau n\mu}\overline{A}_{n+1n+1\tau} \\
&\quad + h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} - \sum_\tau h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau;\mu} - \sum_\tau 2h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
&\quad - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau} - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu}\Big)\omega^\mu_n \\
&\quad + 2\overline{A}_{\lambda\lambda n+1|n}\omega^\lambda_\lambda + \Big(2h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1} - \sum_\tau 4h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau}\Big)\omega^\lambda_\lambda \\
&\quad - \sum_{\tau,\sigma}\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)(\omega^\sigma_\tau + \omega^\tau_\sigma)\Big\} (\mathrm{mod}(\omega^l)) \\
&= \Big\{ \Big(\overline{A}_{\lambda\lambda n+1|n;\mu} + h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda n+1;n+1} + h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1;\mu} + \overline{P}^{n+1}_{\mu n\mu}\overline{A}_{n+1n+1\mu} \\
&\quad + h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} - \sum_\tau h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau;\mu} - \sum_\tau 2h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
&\quad - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau} - \sum_\tau 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu}\Big)\omega^\mu_n \\
&\quad + \sum_\tau 2h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau}\omega^\lambda_\lambda \\
&\quad + \sum_{\tau,\sigma} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\tau\mu}\omega^\mu_n\Big\} (\mathrm{mod}(\omega^l))
\end{aligned}
$$

$$
\begin{aligned}
= \Big\{ & \overline{A}_{\lambda\lambda n+1|n;\mu} + h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda n+1;n+1} + h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1;\mu} \\
& + \overline{P}^{n+1}_{\mu n\mu}\overline{A}_{n+1n+1\mu} \\
& + h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} - \sum_{\tau} h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau;\mu} - \sum_{\tau} 2h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
& - \sum_{\tau} 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau} - \sum_{\tau} 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu} \\
& - \sum_{\tau} 2h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau} A_{\lambda\lambda\mu} \\
& + \sum_{\tau,\sigma} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\tau\mu}\Big\}\omega^{\mu}_{n}(\mathrm{mod}(\omega^{l})).
\end{aligned}
\tag{3.14}
$$

This implies that

$$
\begin{aligned}
& \overline{A}_{\lambda\lambda n+1|n;\mu} + h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda n+1;n+1} + h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1;\mu} + \overline{P}^{n+1}_{\mu n\mu}\overline{A}_{n+1n+1\mu} \\
& + h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} - \sum_{\tau} h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau;\mu} - \sum_{\tau} 2h^{n+1}_{n\mu}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \\
& - \sum_{\tau} 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau} - \sum_{\tau} 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu} \\
& - \sum_{\tau} 2h^{n+1}_{\tau n}\overline{A}_{n+1n+1\tau}\overline{A}_{\lambda\lambda\mu} \\
& + \sum_{\tau,\sigma} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\tau\mu} = 0.
\end{aligned}
\tag{3.15}
$$

It follows from Proposition 3.1 and Proposition 3.2 that for $\lambda \neq \mu$

$$
\begin{aligned}
& \sum_{\tau,\sigma} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\tau\mu} \\
&= \sum_{\tau} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\mu} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\mu}\Big)\overline{A}_{\mu\tau\mu} \\
&\quad + \sum_{\tau}\sum_{\sigma\neq\mu} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\tau\mu} \\
&= \Big[\sum_{\tau\neq\lambda,\mu} 2\Big(h^{n+1}_{\tau n}\overline{A}_{n+1n+1\mu} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\mu}\Big)\overline{A}_{\mu\tau\mu} \\
&\quad + 2\Big(h^{n+1}_{\lambda n}\overline{A}_{n+1n+1\mu} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\lambda}\overline{A}_{n+1n+1\mu}\Big)\overline{A}_{\mu\lambda\mu} \\
&\quad + 2\Big(h^{n+1}_{\mu n}\overline{A}_{n+1n+1\mu} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\mu}\Big)\overline{A}_{\mu\mu\mu}\Big] \\
&\quad + \sum_{\sigma\neq\mu} 2\Big(h^{n+1}_{\mu n}\overline{A}_{n+1n+1\sigma} + 2h^{n+1}_{nn}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\mu\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\sigma\neq\mu}2\Big(h_{\sigma n}^{n+1}\overline{A}_{n+1n+1\sigma}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\sigma}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\sigma\mu}\\
&=\sum_{\tau\neq\lambda,\mu}2\Big(h_{\tau n}^{n+1}\overline{A}_{n+1n+1\mu}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\mu}\Big)\overline{A}_{\mu\mu\tau}\\
&\quad+2h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda}+12h_{nn}^{n+1}\overline{A}_{\mu\mu\lambda}^{2}\overline{A}_{n+1n+1\mu}\\
&\quad+6h_{\mu n}^{n+1}\overline{A}_{n+1n+1\mu}^{2}+12h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}^{3}\\
&\quad+\Big[\sum_{\sigma\neq\lambda,\mu}2\Big(h_{\mu n}^{n+1}\overline{A}_{n+1n+1\sigma}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\mu\mu}\\
&\quad+2\Big(h_{\mu n}^{n+1}\overline{A}_{n+1n+1\lambda}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\lambda}\Big)\overline{A}_{\lambda\mu\mu}\Big]\\
&\quad+\Big[\sum_{\sigma\neq\lambda,\mu}2\Big(h_{\sigma n}^{n+1}\overline{A}_{n+1n+1\sigma}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\sigma}\overline{A}_{n+1n+1\sigma}\Big)\overline{A}_{\sigma\sigma\mu}\\
&\quad+2\Big(h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\lambda}+2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\lambda}\overline{A}_{n+1n+1\lambda}\Big)\overline{A}_{\lambda\lambda\mu}\Big]\\
&=4h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda}+28h_{nn}^{n+1}\overline{A}_{\mu\mu\lambda}^{2}\overline{A}_{n+1n+1\mu}\\
&\quad+2h_{\mu n}^{n+1}\overline{A}_{n+1n+1\lambda}^{2}+6h_{\mu n}^{n+1}\overline{A}_{n+1n+1\mu}^{2}+12h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}^{3}\\
&\quad+\sum_{\tau\neq\lambda,\mu}\Big(4h_{\tau n}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{\mu\mu\tau}+2h_{\mu n}^{n+1}\overline{A}_{\lambda\lambda\tau}^{2}+12h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}^{2}\overline{A}_{\lambda\lambda\mu}\Big).
\end{aligned}
\tag{3.16}
$$

Similarly, we have also

$$
\begin{aligned}
&-\sum_{\tau}h_{\tau n}^{n+1}\overline{A}_{n+1n+1\tau;\mu}-\sum_{\tau}2h_{n\mu}^{n+1}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau}\\
&-\sum_{\tau}2h_{\tau n}^{n+1}\overline{A}_{n+1n+1\tau}\overline{A}_{\lambda\lambda\mu}\\
&=-6h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda}-6h_{\mu n}^{n+1}\overline{A}_{n+1n+1\lambda}^{2}-4h_{\mu n}^{n+1}\overline{A}_{n+1n+1\mu}^{2}\\
&-h_{n\mu}^{n+1}\overline{A}_{n+1n+1\mu;\mu}\\
&+\sum_{\tau\neq\lambda,\mu}\Big(-6h_{\tau n}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{\mu\mu\tau}-2h_{\mu n}^{n+1}\overline{A}_{\lambda\lambda\tau}^{2}\Big).
\end{aligned}
\tag{3.17}
$$

And

$$
\begin{aligned}
&-\sum_{\tau}2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau;\mu}\overline{A}_{n+1n+1\tau}-\sum_{\tau}2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau;\mu}\\
&=-48h_{nn}^{n+1}\overline{A}_{\mu\mu\lambda}^{2}\overline{A}_{n+1n+1\mu}-2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu;\mu}\overline{A}_{n+1n+1\mu}\\
&-2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\mu;\mu}-\sum_{\tau\neq\lambda,\mu}16h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}^{2}\overline{A}_{\lambda\lambda\mu}.
\end{aligned}
\tag{3.18}
$$

Substituting (3.16), (3.17) and (3.18) into (3.15) yields

$$
\begin{aligned}
&\overline{A}_{\lambda\lambda n+1|n;\mu} + h_{nn}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1;\mu} + \overline{P}_{\mu n\mu}^{n+1}\overline{A}_{n+1n+1\mu} + h_{nn}^{n+1}\overline{A}_{n+1n+1\mu}\\
&- 2h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda} - 4h_{\mu n}^{n+1}\overline{A}_{n+1n+1\lambda}^2 + 2h_{\mu n}^{n+1}\overline{A}_{n+1n+1\mu}^2\\
&- 20h_{nn}^{n+1}\overline{A}_{\mu\mu\lambda}^2\overline{A}_{n+1n+1\mu} + 12h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}^3\\
&+ h_{n\mu}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1} - h_{n\mu}^{n+1}\overline{A}_{n+1n+1\mu;\mu}\\
&- 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu;\mu}\overline{A}_{n+1n+1\mu} - 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\mu;\mu}\\
&+ \sum_{\tau\neq\lambda,\mu}\Big(- 2h_{\tau n}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{\mu\mu\tau} - 4h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}^2\overline{A}_{\lambda\lambda\tau}\Big)\\
&= 0.
\end{aligned}
\tag{3.19}
$$

It can be seen from the second formula of Proposition 3.4 that

$$
\begin{aligned}
h_{nn}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1;\mu} &= h_{nn}^{n+1}\overline{A}_{\lambda\lambda n+1;\mu;n+1}\\
&= 6h_{nn}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1}\overline{A}_{\lambda\lambda\mu}.
\end{aligned}
\tag{3.20}
$$

It follows from the first formula of Proposition 3.3 that

$$
\begin{aligned}
&h_{n\mu}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1} - h_{n\mu}^{n+1}\overline{A}_{n+1n+1\mu;\mu}\\
&\quad - 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu;\mu}\overline{A}_{n+1n+1\mu} - 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\mu;\mu}\\
&= h_{n\mu}^{n+1}\overline{A}_{\lambda\lambda n+1;n+1} - h_{n\mu}^{n+1}(\overline{A}_{n+1n+1\lambda;\lambda} + 4\overline{A}_{n+1n+1\mu}^2 - 4\overline{A}_{n+1n+1\lambda}^2)\\
&\quad - 2h_{nn}^{n+1}(\overline{A}_{\lambda\lambda n+1;n+1} + 4\overline{A}_{\lambda\lambda\mu}^2 - 4\overline{A}_{\lambda\lambda n+1}^2)\overline{A}_{n+1n+1\mu}\\
&\quad - 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}(\overline{A}_{n+1n+1\lambda;\lambda} + 4\overline{A}_{n+1n+1\mu}^2 - 4\overline{A}_{n+1n+1\lambda}^2)\\
&= -4h_{n\mu}^{n+1}\overline{A}_{n+1n+1\mu}^2 + 4h_{n\mu}^{n+1}\overline{A}_{n+1n+1\lambda}^2\\
&\quad - 16h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}^3 + 8h_{nn}^{n+1}\overline{A}_{n+1n+1\lambda}^2\overline{A}_{\lambda\lambda\mu} - 4h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\lambda;\lambda}.
\end{aligned}
\tag{3.21}
$$

Substituting (3.20) and (3.21) into (3.19), we obtain

$$
\begin{aligned}
&\overline{A}_{\lambda\lambda n+1|n;\mu} + \overline{P}_{\mu n\mu}^{n+1}\overline{A}_{n+1n+1\mu} + h_{nn}^{n+1}\overline{A}_{n+1n+1\mu}\\
&- 2h_{\lambda n}^{n+1}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda} - 2h_{\mu n}^{n+1}\overline{A}_{n+1n+1\mu}^2\\
&- 12h_{nn}^{n+1}\overline{A}_{\mu\mu\lambda}^2\overline{A}_{n+1n+1\mu} - 4h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}^3 + 2h_{nn}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{n+1n+1\lambda;\lambda}\\
&+ \sum_{\tau\neq\lambda,\mu}\Big(- 2h_{\tau n}^{n+1}\overline{A}_{\lambda\lambda\mu}\overline{A}_{\mu\mu\tau} - 4h_{nn}^{n+1}\overline{A}_{\lambda\lambda\tau}^2\overline{A}_{\lambda\lambda\tau}\Big)\\
&= 0.
\end{aligned}
\tag{3.22}
$$

On the other hand, by $\overline{A}_{\lambda\lambda n+1} = 0$, (3.13) and Proposition 3.4, using the Ricci identity, we have

$$
\begin{aligned}
&\overline{A}_{\lambda\lambda n+1|n;\mu} + \overline{P}^{n+1}_{\mu n\mu}\overline{A}_{n+1n+1\mu} \\
&= \overline{A}_{\lambda\lambda n+1;\mu|n} + \overline{A}_{\lambda\lambda\mu}\overline{P}^{\mu}_{n+1n\mu} + \overline{P}^{n+1}_{\mu n\mu}\overline{A}_{n+1n+1\mu} \\
&= 2\overline{A}_{\lambda\lambda\mu}\overline{A}_{\lambda\lambda n+1|n} \\
&= -2h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1} + 2\sum_{\tau} h^{n+1}_{\tau l}\overline{A}_{n+1n+1\tau} \\
&\quad + \sum_{\tau} 4h^{n+1}_{nn}\overline{A}_{\lambda\lambda\tau}\overline{A}_{n+1n+1\tau} \qquad (3.23) \\
&= -2h^{n+1}_{nn}\overline{A}_{\lambda\lambda n+1;n+1} + 2h^{n+1}_{\lambda n}\overline{A}_{n+1n+1\mu}\overline{A}_{\mu\mu\lambda} + 2h^{n+1}_{\mu n}\overline{A}^{2}_{n+1n+1\mu} \\
&\quad + 12h^{n+1}_{nn}\overline{A}^{2}_{\mu\mu\lambda}\overline{A}_{n+1n+1\mu} + 4h^{n+1}_{nn}\overline{A}^{3}_{\lambda\lambda\mu} \\
&\quad + \sum_{\tau\neq\lambda,\mu}\left(2h^{n+1}_{\tau n}\overline{A}_{\lambda\lambda\mu}\overline{A}_{\mu\mu\tau} + 4h^{n+1}_{nn}\overline{A}^{2}_{\lambda\lambda\tau}\overline{A}_{\lambda\lambda\tau}\right).
\end{aligned}
$$

Combining (3.22) and (3.23), we get

$$h^{n+1}_{nn}\overline{A}_{n+1n+1\mu} = 0. \qquad (3.24)$$

Taking the exterior differentiation of (3.24), we have that $h^{n+1}_{ni}\overline{A}_{n+1n+1\mu} = 0$. This completes the proof of Proposition 4.2. □

Proposition 3.7. *Let $\left(\overline{M}^{n+1}, \overline{F}\right)$ be a Randers space with constant flag curvature $\overline{K} = 1$. If (M, F) is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$. Then $h^{n+1}_{ni}\overline{P}^{n+1}_{jkn+1} = h^{n+1}_{ni}\overline{P}^{l}_{jkn+1} = h^{n+1}_{ni}\overline{P}^{n+1}_{jkl} = 0$.*

Proof. In the following, the computation is pointwisely estimated.

In case $h^{n+1}_{ni} = 0, \forall i$. Obviously Proposition 3.7 is true. In case $h^{n+1}_{ni} \neq 0$. This is a neighborhood U such that $h^{n+1}_{ni} \neq 0$ on U. Then we have that $\overline{A}_{n+1n+1\lambda} = 0$ on U by Proposition 3.6. Exterior differentiate $\overline{A}_{n+1n+1n+1} = 0$, we obtain that $\overline{A}_{n+1n+1n+1;\lambda} = 0$, *i.e.*, $\overline{A}_{n+1n+1\lambda;n+1} = 0$. Now exterior differentiate $\overline{A}_{n+1n+1\lambda} = 0$, we get that

$$\overline{A}_{n+1n+1\lambda|l} = \overline{A}_{n+1n+1\lambda;\mu} = 0. \qquad (3.25)$$

It can be seen from (3.25) and Proposition 3.5 that

$$\overline{A}_{\lambda\lambda n+1|l} = \overline{A}_{\lambda\lambda n+1;\mu} = 0. \qquad (3.26)$$

By (3.25), (3.26), Proposition 2.4, Proposition 3.4 and Proposition 3.5, using the Ricci identity, we have that for $\forall\lambda \neq \mu$

$$
\begin{aligned}
\overline{A}_{\lambda\lambda n+1|n|\mu} &= \overline{A}_{\lambda\mu n+1|n|\lambda} = h^{n+1}_{n\lambda}\overline{A}_{\lambda\mu n+1|n;n+1} \\
&= h^{n+1}_{n\lambda}\overline{A}_{\lambda\mu n+1;n+1|n} \qquad (3.27) \\
&= 0,
\end{aligned}
$$

and

$$\overline{A}_{\lambda n+1n+1|n|\mu} = 0. \tag{3.28}$$

Exterior differentiate (3.25) and (3.26), combining (3.27) and (3.28), we obtain that

$$\overline{A}_{\lambda\lambda n+1|n+1}\theta_n^{n+1} = \overline{A}_{\lambda n+1n+1|n+1}\theta_n^{n+1} = 0. \tag{3.29}$$

This implies that $h_{ni}^{n+1}\overline{P}^{n+1}_{jkn+1} = h_{ni}^{n+1}\overline{P}^{n+1}_{jkl} = h_{ni}^{n+1}\overline{P}^{l}_{jkn+1} = 0.$ □

Proposition 3.8. *Let $\left(\overline{M}^{n+1}, \overline{F}\right)$ be a Randers space with constant flag curvature $\overline{K} = 1$. If (M, F) is a hypersurface of $\left(\overline{M}^{n+1}, \overline{F}\right)$. Then $\overline{A}_{ijn+1|n} = 0$.*

Proof. In the following, the computation is pointwisely estimated.

It follows from Proposition 3.6 that

$$h_{nl}^{n+1} = 0 \qquad or \qquad \overline{A}_{n+1n+1\lambda} = 0. \tag{3.30}$$

(1) In case $\overline{A}_{n+1n+1\lambda} = 0$. It can be seen from (3.26) and the first formula of Proposition 3.2 that $\overline{A}_{ijn+1|n} = 0$;

(2) In case $h_{ni}^{n+1} = 0$.

It follows from (3.13) that

$$\overline{A}_{\lambda\lambda n+1|l} - \sum_{\tau} h_{\tau l}^{n+1}\overline{A}_{n+1n+1\tau} = 0. \tag{3.31}$$

Setting $l = n$ in (3.31), we get that $\overline{A}_{ijn+1|n} = 0$. □

4. Proof of the main theorem

Let

$$\Box f = \sum_{i,j}(nH_1\delta_{ij} - h_{ij}^{n+1}) f_{|i|j} + \sum_{i,\lambda,\mu}(nH_1\delta_{i\lambda} - h_{i\lambda}^{n+1}) f_{|i} P^n_{\mu\mu\lambda}, \tag{4.1}$$

where $nH_1 = \sum_i h_{ii}^{n+1}$.

For Randers space $(\overline{M}, \overline{F})$ with constant flag curvature $\overline{K} = 1$, using (2.7) and the first formula of Proposition 3.2, we get

$$\begin{aligned} \overline{R}^{n+1}_{jij} &= -\sum_s(\overline{A}_{n+1is|n}\overline{A}_{sjj|n} - \overline{A}_{n+1js|n}\overline{A}_{sji|n}) \\ &= -(\overline{A}_{n+1ii|n}\overline{A}_{ijj|n} - \overline{A}_{n+1jj|n}\overline{A}_{jji|n}) \\ &= 0. \end{aligned} \tag{4.2}$$

It can be seen from the second formula of Proposition 2.2, Proposition 3.5, Proposition 3.7 and (4.2) that $h^{n+1}_{ji|j} = h^{n+1}_{jj|i}$. So we can obtain

$$\sum_j (nH_1\delta_{ij} - h^{n+1}_{ij})_{|j} = 0,$$

which together with Proposition 2.6 implies that the operator $\Box$ is self-adjoint.

It follows from Proposition 2.2, Proposition 2.4, Proposition 3.5, Proposition 3.7 and Proposition 3.8 that

$$h^{n+1}_{ij|k} - h^{n+1}_{ik|j} = -\overline{R}^{n+1}_{ijk} = 0, \tag{4.3}$$

and

$$\begin{aligned}\sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{ki;s} R^s_{njk} &= -\sum_{i,j,k} h^{n+1}_{ij} \overline{P}^{n+1}_{kis}(\delta_{sj}\delta_{nk} - \delta_{sk}\delta_{nj}) \\ &= -\sum_{i,j} h^{n+1}_{ij} \overline{P}^{n+1}_{nij} + \sum_{i,k} h^{n+1}_{in} \overline{P}^{n+1}_{kik} = 0.\end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned}&\sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{ij|k|k} \\ &= \sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{ik|j|k} \\ &= \sum_{i,j,k} h^{n+1}_{ij} (h^{n+1}_{ki|k|j} + h^{n+1}_{ki;s} R^s_{njk} + h^{n+1}_{si} R^s_{kjk} + h^{n+1}_{ks} R^s_{ijk} - h^{n+1}_{ki} \overline{R}^{n+1}_{n+1jk}) \\ &= \sum_{i,j,k} h^{n+1}_{ij} (h^{n+1}_{kk|i|j} + h^{n+1}_{si} R^s_{kjk} + h^{n+1}_{ks} R^s_{ijk}).\end{aligned} \tag{4.5}$$

It follows from Proposition 2.4, Proposition 3.7 and (4.5) that

$$\begin{aligned}&\frac{1}{2}\mathrm{div}_{\hat{g}}(S_{|i}\omega^i) \\ &= \sum_k \frac{1}{2} S_{|k|k} + \frac{1}{2}\sum_{\lambda,\mu} S_{|\mu} P^n_{\lambda\lambda\mu} \\ &= \sum_{i,j,k} (h^{n+1}_{ij|k})^2 + \sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{ij|k|k} + \sum_{i,j,\lambda,\mu} h^{n+1}_{ij} h^{n+1}_{ij|\mu} P^n_{\lambda\lambda\mu} \\ &= \sum_{i,j,k} (h^{n+1}_{ij|k})^2 + \sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{kk|i|j} \\ &\quad + \sum_{i,j,k} h^{n+1}_{ij} (h^{n+1}_{si} R^s_{kjk} + h^{n+1}_{ks} R^s_{ijk}) \\ &\quad + \sum_{i,j,\lambda,\mu} h^{n+1}_{ij} h^{n+1}_{ij|\mu} P^n_{\lambda\lambda\mu}.\end{aligned} \tag{4.6}$$

Since the second mean curvature $H_2 = \frac{1}{n(n+1)}[(nH_1)^2 - S]$ is constant, we have that $(nH_1)^2_{|i} = S_{|i}$. It follows from (4.6) that

$$\begin{aligned}
&\frac{1}{2}\mathrm{div}_{\widehat{g}}((nH_1)^2_{|i}\omega^i)\\
&=\sum_{i,j,k}(h^{n+1}_{ij|k})^2 + \sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{kk|i|j}\\
&\quad+\sum_{i,j,k} h^{n+1}_{ij}(h^{n+1}_{si}R^s_{kjk} + h^{n+1}_{ks}R^s_{ijk}) + \sum_{i,j,\lambda,\mu} h^{n+1}_{ij}h^{n+1}_{ij|\mu}P^n_{\lambda\lambda\mu}.
\end{aligned}\tag{4.7}$$

It follows from (4.1) and (4.7) that

$$\begin{aligned}
\Box(nH_1) &= \sum_i nH_1(nH_1)_{|i|i} - \sum_{i,j} h^{n+1}_{ij}(nH_1)_{|i|j}\\
&\quad+\sum_{i,\lambda,\mu}(nH_1\delta_{i\lambda} - h^{n+1}_{i\lambda})(nH_1)_{|i}P^n_{\mu\mu\lambda}\\
&=\frac{1}{2}\mathrm{div}_{\widehat{g}}[(nH_1)^2_{|i}\omega^i] - \sum_i (nH_{1|i})^2\\
&\quad-\sum_{i,\lambda,\mu} nH_1(nH_1)_{|\mu}P^n_{\lambda\lambda\mu} - \sum_{i,j} h^{n+1}_{ij}(nH_1)_{|i|j}\\
&\quad+\sum_{i,\lambda,\mu}(nH_1\delta_{i\lambda} - h^{n+1}_{i\lambda})(nH_1)_{|i}P^n_{\mu\mu\lambda}\\
&=\sum_{i,j,k}(h^{n+1}_{ij|k})^2 - \sum_i (nH_{1|i})^2\\
&\quad+\sum_{i,j,k} h^{n+1}_{ij}(h^{n+1}_{si}R^s_{kjk} + h^{n+1}_{ks}R^s_{ijk})\\
&\quad+\sum_{i,j,\lambda,\mu} h^{n+1}_{ij}h^{n+1}_{ij|\mu}P^n_{\lambda\lambda\mu} - \sum_{i,\lambda,\mu} h^{n+1}_{i\lambda}(nH_1)_{|i}P^n_{\mu\mu\lambda}.
\end{aligned}\tag{4.8}$$

Lemma 4.1. *Let $(\overline{M}, \overline{F})$ be a Randers space with constant flag curvature $\overline{K} = 1$ and (M^n, F) a compact hypersurface of $(\overline{M}, \overline{F})$ with constant second mean curvature $H_2 \geq 0$. Then $\sum_{i,j,k}(h^{n+1}_{ij|k})^2 - \sum_i (nH_{1|i})^2 \geq 0$.*

Proof. Since H_2 is constant, exterior differentiate $H_2 = \frac{1}{n(n+1)}[(nH_1)^2 - S]$, we have that

$$n^2 H_1 H_{1|k} = \sum_{ij} h^{n+1}_{ij}h^{n+1}_{ij|k},$$

which implies that

$$n^4 H_1^2 (H_{1|k})^2 = \left(\sum_{ij} h^{n+1}_{ij}h^{n+1}_{ij|k}\right)^2 \leq \sum_{ij}(h^{n+1}_{ij})^2 \sum_{i,j}(h^{n+1}_{ij|k})^2. \tag{4.9}$$

On the other hand, it follows from $H_2 \geq 0$ that $(nH_1)^2 \geq S^2$. Then it is easy to see from (4.9) that $\sum_{i,j,k}(h^{n+1}_{ij|k})^2 - \sum_i (nH_{1|i})^2 \geq 0$. □

Lemma 4.2. *Let $\left(\overline{M}^{n+1}, \overline{F}\right)$ be a Randers space with constant flag curvature $\overline{K} = 1$. If (M, F) is a hypersurface of $\left(\overline{M}^{n+1}, \overline{F}\right)$. Then $h^{n+1}_{ii|j}\overline{P}^n_{\lambda\lambda\mu} = 0$.*

Proof. In the following, the computation is pointwisely estimated.

(1) In case $\overline{A}_{n+1n+1\lambda} = 0$. By (3.25) and Proposition 3.3, we have

$$\overline{P}^n_{\lambda\lambda\mu} = \overline{A}_{n+1n+1\mu|n} = 0; \tag{4.10}$$

(2) In case $h^{n+1}_{ni} = 0$. Exterior differentiate $h^{n+1}_{n\lambda} = 0$, we obtain

$$\begin{cases} h^{n+1}_{\lambda\mu} = 0 & \forall \lambda \neq \mu \\ h^{n+1}_{\lambda\lambda} = -h^{n+1}_{n\lambda;\lambda} = -\overline{A}_{\lambda\lambda n+1|n}. \end{cases} \tag{4.11}$$

For a Randers space $\left(\overline{M}^{n+1}, \overline{F}\right)$ with constant flag curvature $\overline{K} = 1$, (2.8) implies that

$$\overline{A}_{ijk|n|l} - \overline{A}_{ijl|n|k} = -\overline{A}_{ijk}\delta_{nl} + \overline{A}_{ijl}\delta_{nk}. \tag{4.12}$$

It follows from (4.12) and Proposition 3.2 that

$$\overline{A}_{\mu\mu n+1|n|\lambda} = \overline{A}_{\lambda\mu n+1|n|\mu} = 0, \quad \forall \lambda \neq \mu. \tag{4.13}$$

By the second formula of (4.11) and (4.13), we get

$$\begin{cases} h^{n+1}_{\lambda\lambda|\mu} = -A_{\lambda\lambda n+1|n|\nu} = 0 & \forall \lambda \neq \mu \\ h^{n+1}_{\lambda\lambda|\lambda} = -A_{\mu\mu n+1|n|\lambda} = 0 & \forall \lambda \neq \mu. \end{cases} \tag{4.14}$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space with constant flag curvature $\overline{K} = 1$. If (M, F) is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$. Then*

$$\begin{aligned} &\sum_{i,j,k,s,t} h^{n+1}_{ij}h^{n+1}_{si}(\overline{A}_{sjt|n}\overline{A}_{tkk|n} - \overline{A}_{skt|n}\overline{A}_{tkj|n}) \\ &+ \sum_{i,j,k,s,t} h^{n+1}_{ij}h^{n+1}_{ks}(\overline{A}_{sjt|n}\overline{A}_{tik|n} - \overline{A}_{skt|n}\overline{A}_{tij|n}) = 0. \end{aligned}$$

Proof. It follows from (4.11) that

$$\begin{aligned}
&\sum_{i,j,k,s,t} h^{n+1}_{ij} h^{n+1}_{si} (\overline{A}_{sjt|n}\overline{A}_{tkk|n} - \overline{A}_{skt|n}\overline{A}_{tkj|n}) \\
&\quad + \sum_{i,j,k,s,t} h^{n+1}_{ij} h^{n+1}_{ks} (\overline{A}_{sjt|n}\overline{A}_{tik|n} - \overline{A}_{skt|n}\overline{A}_{tij|n}) \\
&= \sum_{i,k,t} h^{n+1}_{ii} h^{n+1}_{ii} (\overline{A}_{iit|n}\overline{A}_{tkk|n} - \overline{A}_{ikt|n}\overline{A}_{tki|n}) \qquad (4.15)\\
&\quad + \sum_{i,k,t} h^{n+1}_{ii} h^{n+1}_{kk} (\overline{A}_{kit|n}\overline{A}_{tik|n} - \overline{A}_{kkt|n}\overline{A}_{tii|n}) \\
&= 0.
\end{aligned}$$

□

Now we can prove the following:

Main Theorem. *Let $(\overline{M}, \overline{F})$ be a Randers space with constant flag curvature $\overline{K} = 1$ and (M^n, F) a compact hypersurface of $(\overline{M}, \overline{F})$ with constant second mean curvature H_2. If $H_2 \geq 0$ and the norm square S of the second fundamental form of M satisfies*

$$S \leq \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big), \qquad (4.16)$$

then either $S = nH_2$ and M is a Randers space with constant flag curvature $K = 1 + H_2$; or

$$S = \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big)$$

and $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, where $r = \sqrt{\frac{n-2}{n(H_2+1)}}$.

Proof. It can be seen from Proposition 2.2, Proposition 2.4, Proposition 3.5 ∼ Proposition 3.8 that

$$\begin{aligned}
&\sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{si} R^s_{kjk} \\
&= \sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{si} [\overline{R}^s_{kjk} + h^{n+1}_{sj} h^{n+1}_{kk} - h^{n+1}_{sk} h^{n+1}_{kj}] \\
&= \sum_{i,j,k,s} h^{n+1}_{ij} h^{n+1}_{si} [(\delta_{sj}\delta_{kk} - \delta_{sk}\delta_{kj}) - (\overline{A}_{skj}\delta_{kn} - \overline{A}_{skk}\delta_{jn}) \\
&\quad - \sum_t (\overline{A}_{sjt|n}\overline{A}_{tkk|n} - \overline{A}_{skt|n}\overline{A}_{tkj|n}) + h^{n+1}_{sj} h^{n+1}_{kk} - h^{n+1}_{sk} h^{n+1}_{kj}] \qquad (4.17)\\
&\geq (n-1)S + 3nH_1^2 S - 2(nH_1^2)^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H_1|[S - nH_1^2]^{\frac{3}{2}} \\
&\quad - \sum_{i,j,k,s} h^{n+1}_{ij} h^{n+1}_{si} h^{n+1}_{sk} h^{n+1}_{kj} \\
&\quad - \sum_{i,j,k,s,t} h^{n+1}_{ij} h^{n+1}_{si} (\overline{A}_{sjt|n}\overline{A}_{tkk|n} - \overline{A}_{skt|n}\overline{A}_{tkj|n}).
\end{aligned}$$

And
$$\begin{aligned}&\sum_{i,j,k} h_{ij}^{n+1} h_{ks}^{n+1} R_{ijk}^{s}\\&=S-(nH_1)^2-S^2+\sum_{i,j,k,s} h_{ij}^{n+1} h_{si}^{n+1} h_{sk}^{n+1} h_{kj}^{n+1}\\&\quad-\sum_{i,j,k,s,t} h_{ij}^{n+1} h_{ks}^{n+1} (\overline{A}_{sjt|n}\overline{A}_{tik|n}-\overline{A}_{skt|n}\overline{A}_{tij|n}).\end{aligned}\tag{4.18}$$

It follows from (4.17), (4.18) and Lemma 4.3 that
$$\begin{aligned}&\sum_{i,j,k} h_{ij}^{n+1} h_{si}^{n+1} R_{kjk}^{s}+\sum_{i,j,k} h_{ij}^{n+1} h_{ks}^{n+1} R_{ijk}^{s}\\&\geq nS-(nH_1)^2-S^2+3nH_1^2S-2(nH_1^2)^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H_1|[S-nH_1^2]^{\frac{3}{2}}.\end{aligned}\tag{4.19}$$

Substituting (4.19), Lemma 4.1 and Lemma 4.2 into (4.8), we obtain
$$\begin{aligned}&\Box(nH_1)\\&\geq nS-(nH_1)^2-S^2+3nH_1^2S-2(nH_1^2)^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H_1|[S-nH_1^2]^{\frac{3}{2}}\\&\geq (S-nH_1^2)\Big\{n+nH_1^2-S-\frac{n(n-2)}{\sqrt{n(n-1)}}|H_1|\sqrt{S-nH_1^2}\Big\}.\end{aligned}\tag{4.20}$$

It is a direct check that our assumption condition (4.16), *i.e.*,
$$S\leq \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2+4(n-1)H_2+n\Big)$$
is equivalent to
$$n+nH_1^2-S\geq \frac{n(n-2)}{\sqrt{n(n-1)}}|H_1|\sqrt{S-nH_1^2}.\tag{4.21}$$

Therefore the right hand side of (4.20) is non-negative. Since M is compact and the operator $\Box$ is self-adjoint, we have that either $S-nH_1^2=0$ or
$$S=\frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2+4(n-1)H_2+n\Big).\tag{4.22}$$

(1) In case $S-nH_1^2=0$. We have that $h_{ii}^{n+1}=|H_1|=\sqrt{H_2}$ is constant and $h_{ij}^{n+1}=0, \forall i\neq j$.
It follows from Proposition 2.2 and Proposition 2.4 that
$$\begin{aligned}R_{njn}^{i}&=\delta_{ij}\delta_{nn}-\delta_{in}\delta_{jn}+h_{ij}^{n+1}h_{nn}^{n+1}-h_{in}^{n+1}h_{jn}^{n+1}\\&=\delta_{ij}-\delta_{in}\delta_{jn}+\delta_{ij}\sqrt{H_2}h_{nn}^{n+1}-\delta_{in}h_{nn}^{n+1}\delta_{jn}h_{nn}^{n+1}\\&=(\delta_{ij}-\delta_{in}\delta_{jn})(1+H_2).\end{aligned}\tag{4.23}$$

This together with Proposition 2.4 yields that M is a Randers space with constant flag curvature $K = 1 + H_2$;

(2) In the latter case.

We first prove that M is a Riemannian. Suppose M is not a Riemannian. Then $\overline{A}_{\lambda\lambda\mu} \neq 0$ at a point x. It can be seen from (4.11) and Proposition 3.6 that $h^{n+1}_{ni} = h^{n+1}_{\lambda\mu} = 0, \forall\lambda \neq \mu$ and $h^{n+1}_{\mu\mu} = \lambda, \forall\mu$. Then we obtain

$$H_1 = \frac{n-1}{n}\lambda, \qquad S = (n-1)\lambda^2 \quad \text{and} \quad H_2 = \frac{n-2}{n}\lambda^2. \tag{4.24}$$

It is easy to see from (4.24) that

$$S < \frac{n}{(n-2)(nH_2+2)}\Big(n(n-1)H_2^2 + 4(n-1)H_2 + n\Big), \tag{4.25}$$

which is in contradiction (4.22). So M is a Riemannian. Then $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, where $r = \sqrt{\frac{n-2}{n(H_2+1)}}$. We complete the proof of Main theorem. □

References

[1] D. BAO, S. S. CHERN and Z. SHEN, "An Introduction to Riemann-Finsler Geometry", Springer-Verlag, 2000.

[2] H. LI, *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann. **305** (1996), 665–672.

[3] J. LI, *The variation formulas of Finsler submanifolds*, J. Geom. Phys. **61** (2011), 890–898.

[4] J. LI, *A classification theorem for hypersurface with constant mean curvature in Randers spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **14** (2015), 221–231.

[5] J. LI, *Hypersurfaces of Randers spaces with constant mean curvature*, Taiwanese J. Math. **21** (2015), 979–996.

[6] X. MO, *Harmonic maps from Finsler manifolds*, Illinois J. Math. **45** (2001), 1331–1345.

[7] W. SANTOS, *Submanifolds with parallel mean curvature vector in spheres*, Tohoku Math. J. **46** (1994), 403–415.

[8] Z. SHEN, *On Finsler geometry of submanifolds*, Math. Ann. **311** (1998), 549–576.

Department of Mathematics
Xiamen University
361005 Xiamen, Fujian,China
dli66@xmu.edu.cn