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The Dirichlet-to-Neumann operator on $C(\partial\Omega)$

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary Γ. Let D_V be the Dirichlet-to-Neumann operator with respect to a purely second-order symmetric divergence form operator with real Lipschitz continuous coefficients and a positive potential V. We show that the semigroup generated by $-D_V$ leaves $C(\Gamma)$ invariant and that the restriction of this semigroup to $C(\Gamma)$ is a C_0-semigroup. We investigate positivity and spectral properties of this semigroup. We also present results where V is allowed to be negative. Of independent interest is a new criterium for semigroups to have a continuous kernel.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary Γ. The Dirichlet-to-Neumann operator D_0 is the self-adjoint operator that is defined in $L_2(\Gamma)$ as follows. Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in \mathrm{dom}(D_0)$ and $D_0\varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ such that $\Delta u = 0$ weakly on Ω, with $\mathrm{Tr}\, u = \varphi$ and the weak normal derivative exists with $\partial_\nu u = \psi$. It turns out that the semigroup S generated by $-D_0$ is submarkovian. Hence it extends consistently to a contraction semigroup $S^{(p)}$ on $L_p(\Gamma)$ for all $p \in [1,\infty]$ and it is a C_0-semigroup if $p \in [1,\infty)$. By elliptic regularity the semigroup S leaves the Banach space $C(\Gamma)$ of continuous functions on Γ invariant. Hence it is a natural question whether the restriction of S to $C(\Gamma)$ is a C_0-semigroup. As a special case of Theorem 5.3, we prove the following theorem.

Theorem 1.1. *Let S be the semigroup generated by the Dirichlet-to-Neumann operator on an open bounded set with Lipschitz boundary Γ. Then S leaves $C(\Gamma)$ invariant and the restriction of S to $C(\Gamma)$ is a C_0-semigroup.*

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If Ω has a C^∞-boundary, then Theorem 1.1 has been proved by Escher [21] and Engel [20].

Although S leaves $C(\Gamma)$ invariant and S is submarkovian, these two facts do not imply that the restriction T of S to $C(\Gamma)$ is a C_0-semigroup, since $C(\Gamma)$ is not reflexive. One needs in addition that the generator of the restriction T is densely defined. This is the major problem that we solve in this paper.

Actually we prove several extensions of Theorem 1.1. The first extension is that we replace the Laplacian by a divergence form operator A with real symmetric Lipschitz continuous coefficients. The second extension is that we add a potential $V \in L_\infty(\Omega, \mathbb{R})$ to the divergence form operator and consider cases where the potential is negative (but still assuming the Dirichlet problem has a unique solution). This means that given $\varphi \in L_2(\Gamma)$ we now solve the Dirichlet problem

$$\left[\begin{array}{l} (A+V)u = 0 \ \text{ weakly on } \Omega \\ \operatorname{Tr} u = \varphi, \end{array}\right.$$

and define the Dirichlet-to-Neumann operator D_V by $D_V \varphi = \partial_\nu u$ on a suitable domain. Using form methods one obtains that $-D_V$ generates a C_0-semigroup S on $L_2(\Gamma)$ (see [8]). The main point in this paper is to prove that the part of D_V in $C(\Gamma)$ is densely defined in $C(\Gamma)$. We prove this for all $V \in L_\infty(\Omega, \mathbb{R})$, without any sign condition on V (except assuming that the Dirichlet problem has a unique solution). This is difficult even for the Laplacian since the normal is merely a measurable function on Γ. For a rich class of potentials we then show that the restriction of S to $C(\Gamma)$ is a C_0-semigroup on $C(\Gamma)$.

Attention is given to the special case where the semigroup S is positive. Then we deduce that the Dirichlet-to-Neumann operator is resolvent positive on $C(\Gamma)$.

Another main point in this paper is the characterisation of those semigroups in $L_2(K)$ which have a continuous kernel, where K is a compact metric space. This is done in an abstract framework. Moreover, we find criteria for the irreducibility of the semigroup on $C(K)$. Irreducibility is an important property which implies in particular that the first eigenfunction is strictly positive. We apply these results to the Dirichlet-to-Neumann operator but also to elliptic operators with Robin boundary conditions on Ω if Ω is connected. So far, for Robin boundary conditions, strict positivity of the first eigenfunction in $C(\overline{\Omega})$ was not known. There is another reason to consider the Robin operator. Even though Ω is connected, the boundary Γ need not be be connected (an example is an annulus). Still we are able to prove irreducibility for the Dirichlet-to-Neumann semigroup on $C(\Gamma)$ and this is done with the Robin semigroup on $C(\overline{\Omega})$. We should mention that irreducibility on L_2-spaces is much easier to obtain than on $C(K)$ (see [26, Corollary 2.11] for elliptic operators and [10, Theorem 4.2] for the Dirichlet-to-Neumann operator). The difference can be seen by the consequences for the first eigenfunction. The irreducibility on L_2 merely implies that the first eigenfunction is positive almost everywhere, whilst irreducibility on $C(K)$ implies pointwise positive. It is remarkable that our proof of this strict positivity (which is a purely elliptic property) involves considering the parabolic problem.

The paper is organised as follows. In Section 2 we study in an abstract setting when a semigroup S on $L_2(K)$ has a continuous kernel, where K is a compact metric space. If S is positive and has a self-adjoint generator, then we characterise when the restriction of S to $C(K)$ is irreducible. In Section 3 we consider the semigroup S^V generated by $-D_V$, where D_V is the Dirichlet-to-Neumann operator with respect to a symmetric divergence form operator with coefficients $a_{kl} \in L_\infty(\Omega, \mathbb{R})$ and potential $V \in L_\infty(\Omega, \mathbb{R})$. We show that S^V has a continuous kernel and that the resolvent of D_V leaves $C(\Gamma)$ invariant. In Section 4 we prove that the domain of the part of D_V in $C(\Gamma)$ is dense in $C(\Gamma)$ if the coefficients a_{kl} are Lipschitz continuous. In Section 5 we prove an extension of Theorem 1.1 if $a_{kl} \in W^{1,\infty}(\Omega)$ and the potential V is positive or slightly negative. In Section 6 we study the Robin semigroup with boundary condition $\partial_\nu u + \beta \operatorname{Tr} u = 0$ without any sign condition on $\beta \in L_\infty(\Gamma, \mathbb{R})$ and with coefficients of the divergence form operator in $L_\infty(\Omega, \mathbb{R})$. In the last section we show that S^V is irreducible if merely Ω is connected and a positivity condition is satisfied. Again the coefficients a_{kl} are allowed to be measurable.

Using Poisson kernel bounds for the semigroup S^V, it is proved in [18] that the semigroup T is a holomorphic C_0-semigroup on $C(\Gamma)$ if Ω has a $C^{1+\kappa}$-boundary for some $\kappa > 0$ and the coefficients a_{kl} are merely Hölder continuous. Thus more boundary smoothness of Ω is required in [18]. We do not know whether the semigroup on $C(\Gamma)$ in Theorem 1.1 is holomorphic if Ω has merely a Lipschitz boundary.

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2. Continuous kernel and irreducibility

In this section we consider a semigroup S on the space $L_2(K, \mu)$, where K is compact and μ is a finite Borel measure. Our first aim is to investigate when S has a continuous kernel. Subsequently we asume that S is positive (in the lattice sense) and self-adjoint. We will find criteria which imply that the first eigenfunction is continuous and strictly positive. In the sequel of this paper these two results will be applied to both the Dirichlet-to-Neumann operator and an elliptic operator with Robin boundary condition.

In general, by a *semigroup* on a Banach space X we understand simply a map $S\colon (0, \infty) \to \mathcal{L}(X)$ satisfying $S_{t+s} = S_t S_s$ for all $t, s \in (0, \infty)$, without any further continuity assumption. If S is a semigroup on $L_2(K, \mu)$ we say that S has a *continuous kernel* if for all $t > 0$ there exists a continuous function $k_t : K \times K \to \mathbb{C}$ such that for all $u \in L_2(K)$ the function $S_t u$ is given by

$$(S_t u)(x) = \int_K k_t(x, y)\, u(y)\, dy$$

for almost every $x \in K$. In many concrete situations regularity properties of kernels have been investigated, but so far no characterisation for continuity of the kernel seems to be known. The following theorem is such a charcaterisation in terms of a natural property, Condition (ii) in Theorem 2.1, which is frequently easy to verify. Note that the semigroup does not have to be continuous in this theorem.

Theorem 2.1. *Let K be a compact metric space and μ a finite Borel measure on K with* $\operatorname{supp} \mu = K$. *Let S be a semigroup on $L_2(K, \mu)$. Then the following are equivalent:*

(i) *The operator S_t has a continuous kernel for all $t > 0$;*
(ii) *There exists a $p_0 \in [2, \infty)$ such that $S_t L_{p_0}(K) \subset C(K)$ and $S_t^* L_{p_0}(K) \subset C(K)$ for all $t > 0$;*
(iii) *$S_t L_2(K) \subset C(K)$ and $S_t^* L_2(K) \subset C(K)$ for all $t > 0$.*

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). We may assume that $p_0 \in \mathbb{N}$. Let $t > 0$. Then S_t^* is bounded from $L_{p_0}(K)$ into $L_\infty(K)$, so by duality S_t extends to a bounded operator from $L_1(K)$ into $L_{q_0}(K)$, where $\frac{1}{q_0} = 1 - \frac{1}{p_0}$. Also S_t is bounded from $L_{p_0}(K)$ into $L_\infty(K)$. So by interpolation, given $p \in [1, p_0]$, the operator S_t extends to a bounded operator from $L_p(K)$ into $L_q(K)$, where $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0}$. Starting with $p = 1$ and using the semigroup property, iteration gives that for all $t > 0$ and $k \in \{1, \ldots, p_0\}$ the operator S_t extends to a bounded operator from $L_1(K)$ into $L_q(K)$, where $\frac{1}{q} = 1 - \frac{k}{p_0}$. Therefore condition (iii) is valid.

(iii) $\Rightarrow$ (i). Let $t > 0$. Then $S_t^* L_2(K) \subset C(K) \subset L_\infty(K)$, so by duality S_t extends to a bounded operator from $L_1(K)$ into $L_2(K)$, also denoted by S_t. Then by the semigroup property $S_{2t} L_1(K) \subset L_\infty(K)$. Hence by the Dunford–Pettis theorem, for all $t > 0$ there exists a bounded measurable function $\tilde{k}_t \colon K \times K \to \mathbb{C}$ such that

$$(S_t u, v)_{L_2(K)} = \int_{K \times K} (\overline{v} \otimes u)\, \tilde{k}_t$$

for all $u, v \in L_2(K)$. Hence if $u \in L_2(K)$, then

$$(S_t u)(x) = \int_K \tilde{k}_t(x, y)\, u(y)\, dy \tag{2.1}$$

for almost every $x \in K$ and by duality

$$(S_t^* u)(x) = \int_K \tilde{k}_t^*(x, y)\, u(y)\, dy$$

for almost every $x \in K$, where $\tilde{k}_t^*(x, y) = \overline{\tilde{k}_t(y, x)}$ for all $(x, y) \in K \times K$ and $t > 0$. If $t > 0$, then the semigroup property gives that

$$\tilde{k}_{2t}(x, y) = \int_K \tilde{k}_t(x, z)\, \tilde{k}_t(z, y)\, dz \tag{2.2}$$

for almost every $(x, y) \in K \times K$. In particular, for almost all $x \in K$ it follows that (2.2) is valid for almost every $y \in K$.

Fix $t > 0$. Since $S_t L_2(K) \subset C(K)$ it follows from the Riesz representation theorem that for all $x \in K$ there exists a $k_x^t \in \mathcal{L}_2(K)$ such that

$$(S_t u)(x) = (u, k_x^t)_{L_2(K)}$$

for all $u \in L_2(K)$ and $\|k_x^t\|_2 \le \|S_t\|_{2\to\infty}$. Similarly, for all $y \in K$ there exists a $k_y^{*t} \in \mathcal{L}_2(K)$ such that

$$(S_t^* u)(y) = (u, k_y^{*t})_{L_2(K)}$$

for all $u \in L_2(K)$. Then $\|k_x^{*t}\|_2 \le \|S_t^*\|_{2\to\infty}$. Next we use (2.1). Let $u \in L_2(K)$. Then

$$\int_K \tilde{k}_t(x, y)\, u(y)\, dy = (S_t u)(x) = (u, k_x^t)_{L_2(K)} \tag{2.3}$$

for almost every $x \in K$. Since $C(K)$ is separable and $C(K)$ is dense in $L_2(K)$, also the space $L_2(K)$ is separable. Then by continuity and density it follows that (2.3) is valid for all $u \in L_2(K)$ for almost every $x \in K$. Therefore $\overline{k_x^t} = \tilde{k}_t(x, \cdot)$ almost everywhere for almost every $x \in K$. Similarly, $\overline{k_y^{*t}} = \tilde{k}_t^*(y, \cdot)$ almost everywhere for almost every $y \in K$. Hence $k_y^{*t} = \tilde{k}_t(\cdot, y)$ almost everywhere for almost every $y \in K$.

The semigroup property (2.2) and Fubini's theorem give that for almost every $x \in K$ it follows that

$$\tilde{k}_{2t}(x, y) = \int_K \tilde{k}_t(x, z)\, \tilde{k}_t(z, y)\, dz$$

for almost every $y \in K$. Hence for almost every $x \in K$ it follows that

$$\tilde{k}_{2t}(x, y) = \int_K \overline{k_x^t}(z)\, k_y^{*t}(z)\, dz = (k_y^{*t}, k_x^t)_{L_2(K)}$$

for almost every $y \in K$. Define $\hat{k}_{2t} \colon K \times K \to \mathbb{C}$ by

$$\hat{k}_{2t}(x, y) = (k_y^{*t}, k_x^t)_{L_2(K)}.$$

We proved that $\tilde{k}_{2t}(x, \cdot) = \hat{k}_{2t}(x, \cdot)$ almost everywhere for almost every $x \in K$. Clearly $|\hat{k}_{2t}(x, y)| \le \|S_t\|_{2\to\infty}\, \|S_t^*\|_{2\to\infty}$ for all $x, y \in K$.

Since $S_t u \in C(K)$ obviously $x \mapsto (S_t u)(x) = (u, k_x^t)_{L_2(K)}$ is continuous for all $u \in L_2(K)$. Hence if $y \in K$, then the function $x \mapsto \hat{k}_{2t}(x, y)$ is continuous from K into $\mathbb{C}$. Similarly, for all $x \in K$ the function $y \mapsto \hat{k}_{2t}(x, y)$ is continuous from K into $\mathbb{C}$. In particular, $\hat{k}_{2t}$ is a Carathéodory function and therefore measurable (see [1, Lemma 4.51]). Because $\tilde{k}_{2t}(x, \cdot) = \hat{k}_{2t}(x, \cdot)$ almost everywhere for almost every $x \in K$, one deduces from Fubini's theorem that $\tilde{k}_{2t} = \hat{k}_{2t}$ almost everywhere.

Define $k_{4t} \colon K \times K \to \mathbb{C}$ by

$$k_{4t}(x, y) = \int_K \hat{k}_{2t}(x, z)\, \hat{k}_{2t}(z, y)\, dz.$$

Then the semigroup poperty (2.2) gives

$$\tilde{k}_{4t}(x, y) = \int_K \tilde{k}_{2t}(x, z)\, \tilde{k}_{2t}(z, y)\, dz = \int_K \hat{k}_{2t}(x, z)\, \hat{k}_{2t}(z, y)\, dz = k_{4t}(x, y)$$

for almost every $(x, y) \in K \times K$. So $\tilde{k}_{4t} = k_{4t}$ almost everywhere.

Finally, for all $z \in K$ the function $(x, y) \mapsto \hat{k}_{2t}(x, z)\, \hat{k}_{2t}(z, y)$ is continuous from $K \times K$ into $\mathbb{C}$ and bounded by $\|S_t\|_{2\to\infty}^2 \, \|S_t^*\|_{2\to\infty}^2$. Moreover, the measure is finite. Hence by the Lebesgue dominated convergence theorem one deduces that k_{4t} is continuous. Therefore $\tilde{k}_{4t}$ has a continuous representative. □

Remark 2.2. Theorem 2.1 is also valid if K is replaced by a locally compact metric space X and $C(K)$ is replaced by $C_{\mathrm{b}}(X)$. We do not know whether the condition that μ is a finite Borel measure can be relaxed to μ being a regular measure.

In the situation of Theorem 2.1 it follows immediately that S_t leaves $C(K)$ invariant for all $t > 0$. Since kernel operators are compact, it follows that $(S_t|_{C(K)})_{t>0}$ is a semigroup of compact operators in $C(K)$. It is not clear, however, whether it is a C_0-semigroup, even if S is a C_0-semigroup on $L_2(K)$.

A subspace I of a (general) Banach lattice E is called an *ideal* if

$$\left[\begin{array}{l} u \in I \text{ implies } |u| \in I \text{ and} \\ u \in I,\ v \in E \text{ and } 0 \le v \le u \text{ implies } v \in I. \end{array}\right.$$

A semigroup on E is called *irreducible* if the only invariant closed ideals are $\{0\}$ and E. If (X, Σ, μ) is a measure space, $p \in [1, \infty)$ and $I \subset L_p(X)$, then I is a closed ideal if and only if there exists a measurable subset $Y \subset X$ such that $I = \{f \in L_p(X) : f|_Y = 0 \text{ a.e.}\}$ (see [27, Section III.1 Example 1]). A subspace I of $C(K)$ is a closed ideal of $C(K)$ if and only if there exists a closed set $B \subset K$ such that $I = \{f \in C(K) : f|_B = 0\}$ (see [27, Section III.1 Example 2]). We refer to [23] for much more information on irreducible semigroups. An operator $B \colon E \to E$ is called *positive* if $Bf \ge 0$ for all $f \in E$ with $f \ge 0$. A semigroup S on E is called *positive* if S_t is positive for all $t > 0$.

In this paper we need a number of known properties of positive and irreducible semigroups when $E = L_2(K)$, where K is a compact metric space. For convenience and future reference we collect them in the next lemma.

Lemma 2.3. *Let S be a C_0-semigroup on $L_2(K, \mu)$, where K is a compact metric space and μ is a finite Borel measure on K. Suppose the generator $-A$ of S is self-adjoint and that S_t has a bounded kernel for all $t > 0$. Then one has the following:*

(a) *For all $t > 0$ the operator S_t is a Hilbert–Schmidt operator;*
(b) *The operator A has compact resolvent and* $\min \sigma(A)$ *is an eigenvalue;*
(c) *If S is positive, then there exists an eigenfunction u_1 with eigenvalue* $\min \sigma(A)$ *such that $u_1 \geq 0$ almost everywhere;*
(d) *If S is positive and irreducible, then the eigenvalue* $\min \sigma(A)$ *is simple. Moreover, there exists an eigenfunction u_1 with eigenvalue* $\min \sigma(A)$ *such that $u_1(x) > 0$ for almost every $x \in K$.*

Proof. (a) and (b) are easy.

(c). This follows from the Krein–Rutman theorem, see for example [11, Theorem 12.15].

(d). See [11, Proposition 14.12(c) and Example 14.11(a)]. □

We emphasise that the eigenfunction u_1 in Statement (c) is in general not unique, even not up to a positive constant. If moreover $S_t L_2(K) \subset C(K)$ for all $t > 0$, then u_1 is continuous. If S is irreducible (on $L_2(K)$), then $u_1(x) > 0$ for almost all $x \in K$ by Lemma 2.3(c). Of course this does not imply that $u_1(x) > 0$ for all $x \in K$. We will relate this strict positivity with the irreducibility of the semigroup on $C(K)$. The main point of the following proposition is that the very weak nondegeneracy condition (ii) implies that the first eigenfunction is strictly positive.

Proposition 2.4. *Let K be a compact connected metric space and μ a finite Borel measure on K with* $\operatorname{supp} \mu = K$. *Let S be a positive C_0-semigroup on $L_2(K, \mu)$ with self-adjoint generator $-A$. Suppose that $S_t L_2(K) \subset C(K)$ for all $t > 0$. Define*

$$S_t^c = S_t|_{C(K)} \colon C(K) \to C(K)$$

for all $t > 0$. Then the following are equivalent:

(i) *The semigroup $S^c = (S_t^c)_{t>0}$ is irreducible;*
(ii) *For all $x \in K$ there exist $t > 0$ and $f \in C(K)$ such that $(S_t^c f)(x) \neq 0$;*
(iii) *There exists a $\delta > 0$ such that $u_1(x) \geq \delta$ for all $x \in K$, where $u_1 \in L_2(K)$ is an eigenfunction with eigenvalue* $\min \sigma(A)$ *such that $u_1 \geq 0$ almost everywhere;*
(iv) *For all $f \in C(K)$ with $f \geq 0$ and $f \neq 0$ it follows that $(S_t f)(x) > 0$ for all $t > 0$ and $x \in K$.*

Proof. (i) ⇒ (iv). This is a variation of a theorem of Majewski and Robinson [22]. Let $x \in K$. It follows from irreducibility that there exists a $t_1 > 0$ such that $(S_{t_1}^c f)(x) > 0$ (see [23, Section C-III Definition 3.1]). Let $\delta \in (0, t_1)$. We shall show that $(S_t^c f)(x) = 0$ for all $t \in (\delta, \infty)$. Set $t_0 = t_1 - \delta$ and $g = S_\delta^c f$. Then $(S_{t_0}^c g)(x) > 0$. Since S^c has a holomorphic extension to a sector with values in $\mathcal{L}(C(K))$, it follows from the proof in [23, Theorem C-III.3.2(b)] that $(S_t^c g)(x) > 0$ for all $t > 0$.

(iv) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (ii). Take $f = u_1$.

(ii) $\Rightarrow$ (i). By Theorem 2.1 the operator S_t has a continuous kernel k_t for all $t > 0$. Let $B \subset K$ be a closed set with $\emptyset \neq B \neq K$. Define

$$I = \{f \in C(K) : f|_B = 0\}.$$

Suppose that the closed ideal I is invariant under S. Define $g \in C(K)$ by $g(x) = d(x, B)$. Then $g \in I$. Since K is connected there exists an $x_0 \in \partial B$. Let $t > 0$. Because $S_t g \in I$, one deduces that

$$\int_K k_t(x_0, y)\, d(y, B)\, d\mu(y) = (S_t g)(x_0) = 0.$$

Hence $k_t(x_0, y) = 0$ for a.e. $y \in K \setminus B$. Since k_t is continuous and μ is strictly positive on open sets it follows that $k_t(x_0, y) = 0$ for all $y \in K \setminus B$. Because $x_0 \in \partial B$ one establishes that $k_t(x_0, x_0) = 0$. The semigroup property and symmetry then imply that

$$0 = k_t(x_0, x_0) = \int_K k_{t/2}(x_0, y)\, k_{t/2}(y, x_0)\, d\mu(y) = \int_K |k_{t/2}(x_0, y)|^2\, d\mu(y).$$

Hence $k_{t/2}(x_0, y) = 0$ for almost every $y \in K$. Consequently $(S_{t/2} f)(x_0) = 0$ for all $f \in C(K)$. This is for all $t > 0$, which is a contradiction. $\square$

Condition (ii) is automatically satisfied if the semigroup S^c is a C_0-semigroup, because then $\lim_{t \downarrow 0} S_t^c \mathbb{1} = \mathbb{1}$ in $C(K)$. As a consequence the semigroup is irreducible and $u_1(x) > 0$ for all $x \in K$. This is surprising, since only the connectedness of K is responsible for this property. We state this as a corollary.

Corollary 2.5. *Let K be a compact connected metric space and μ a finite Borel measure on K with* $\operatorname{supp} \mu = K$. *Let S be a positive C_0-semigroup on $L_2(K, \mu)$ with self-adjoint generator. Suppose that $S_t L_2(K) \subset C(K)$ for all $t > 0$. Define*

$$S_t^c = S_t|_{C(K)} \colon C(K) \to C(K)$$

for all $t > 0$. If S^c is a C_0-semigroup, then it is irreducible and $\min_{x \in K} u_1(x) > 0$.

There is a remarkable consequence of irreducibility: the semigroup S extends to a consistent C_0-semigroup on $L_p(K)$ for all $p \in [1, \infty)$.

Proposition 2.6. *Let K be a compact connected metric space and μ a finite Borel measure on K with* $\operatorname{supp} \mu = K$. *Let S be a positive C_0-semigroup on $L_2(K, \mu)$ with self-adjoint generator $-A$. Suppose that $S_t L_2(K) \subset C(K)$ for all $t > 0$ and that S^c is irreducible. Then for all $p \in [1, \infty)$ there exists a C_0-semigroup $S^{(p)}$ on $L_p(K)$ which is consistent to S. Moreover, there exists an $M \geq 1$ such that $\|S_t^{(p)}\|_{p \to p} \leq M\, e^{-\lambda_1 t}$ for all $t > 0$ and $p \in [1, \infty)$, where* $\lambda_1 = \min \sigma(A)$.

Proof. Let $\delta > 0$ be as in Proposition 2.4(iii). Let $0 \le f \in L_\infty(K)$. Then

$$0 \le f \le \frac{\|f\|_\infty}{\delta}\,\delta \le \frac{\|f\|_\infty}{\delta}\,u_1.$$

Hence $S_t f \le \frac{\|f\|_\infty}{\delta} S_t u_1 \le \frac{\|f\|_\infty}{\delta} e^{-\lambda_1 t} u_1$ and $\|S_t f\|_\infty \le M e^{-\lambda_1 t} \|f\|_\infty$ for all $t > 0$, where $M = \delta^{-1} \|u_1\|_\infty$. Since S is a self-adjoint semigroup, it follows by duality that $\|S_t f\|_1 \le M e^{-\lambda_1 t} \|f\|_1$ for all $f \in L_2(K)$. Then by interpolation $\|S_t f\|_p \le M e^{-\lambda_1 t} \|f\|_p$ for all $f \in L_2(K) \cap L_p(K)$. Since the measure is finite the semigroup is a C_0-semigroup, see [28]. □

We emphasise that we do not assume in Proposition 2.6 that S^c is a C_0-semigroup on $C(K)$.

3. The Dirichlet-to-Neumann semigroup: invariance of $C(\Gamma)$

In this section we introduce the main setting of this paper and recall some known results for the Dirichlet-to-Neumann operator and the associated semigroup.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Suppose that

$$a_{kl} = a_{lk} \tag{3.1}$$

for all $k, l \in \{1, \ldots, d\}$ and that there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} a_{kl}(x)\, \xi_k\, \overline{\xi_l} \ge \mu\, |\xi|^2 \tag{3.2}$$

for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Define the forms $\mathfrak{a}, \mathfrak{a}_V \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_\Omega a_{kl}\, (\partial_k u)\, \overline{\partial_l v} \quad \text{and} \quad \mathfrak{a}_V(u, v) = \mathfrak{a}(u, v) + \int_\Omega V\, u\, \overline{v}.$$

Let A^N be the operator in $L_2(\Omega)$ associated with the form $\mathfrak{a}$ and let A^D be the operator in $L_2(\Omega)$ associated with the form $\mathfrak{a}|_{H_0^1(\Omega) \times H_0^1(\Omega)}$. Then $A^N + V$ is the operator associated with $\mathfrak{a}_V$ and $A^D + V$ is the operator associated with the form $\mathfrak{a}_V|_{H_0^1(\Omega) \times H_0^1(\Omega)}$. We assume throughout this paper that

$$0 \notin \sigma(A^D + V). \tag{3.3}$$

Let Γ be the boundary of Ω. We provide Γ with the $(d-1)$-dimensional Hausdorff measure. Let D_V be the *Dirichlet-to-Neumann operator* in $L_2(\Gamma)$ associated with

$(\mathfrak{a}_V, \mathrm{Tr})$. This means the following. If $\varphi, \psi \in L_2(\Gamma)$, then $\varphi \in D_V$ and $D_V \varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$ and

$$\mathfrak{a}_V(u, v) = (\psi, \mathrm{Tr}\, v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. It follows from [8, Theorem 4.5], or [12, Theorem 5.10], that D_V is a self-adjoint graph, which is indeed a self-adjoint operator because of the condition (3.3). Moreover, D_V is lower bounded by [8, Theorem 4.15].

We can give another description of the operator D_V, for which we need the notion of a weak conormal derivative. Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$. We define the operators $\mathcal{A}, \mathcal{A} + V : H^1(\Omega) \to H^{-1}(\Omega)$ by

$$\langle \mathcal{A}u, v \rangle = \mathfrak{a}(u, v) \quad \text{and} \quad \langle (\mathcal{A} + V)u, v \rangle = \mathfrak{a}_V(u, v).$$

Let $u \in H^1(\Omega)$ and suppose that $\mathcal{A}u \in L_2(\Omega)$. Then we say that u has a *weak conormal derivative* if there exists a $\psi \in L_2(\Gamma)$ such that

$$\mathfrak{a}(u, v) - \int_\Omega (\mathcal{A}u)\, \overline{v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$. By the Stone–Weierstrass it follows that the function ψ is unique and we write $\partial_\nu u = \psi$. Note that the conormal derivative depends on the coefficients a_{kl}, which is suppressed in the notation.

With this notation the operator A^N can be seen as the realization of $\mathcal{A}$ in $L_2(\Omega)$ with Neumann boundary conditions, since

$$\mathrm{dom}(A^N) = \{u \in H^1(\Omega) : \mathcal{A}u \in L_2(\Omega) \text{ and } \partial_\nu u = 0\}$$

and $A^N u = \mathcal{A}u$ for all $u \in \mathrm{dom}(A^N)$.

The alluded characterisation of D_V is as follows.

Lemma 3.1. *Let $\varphi, \psi \in L_2(\Gamma)$. Then the following are equivalent.*

(i) $\varphi \in \mathrm{dom}(D_V)$ *and* $D_V \varphi = \psi$.
(ii) *There exists a $u \in H^1(\Omega)$ such that $(\mathcal{A} + V)u = 0$, $\mathrm{Tr}\, u = \varphi$ and $\partial_\nu u = \psi$.*

We leave the easy proof to the reader.

Let S^V be the semigroup generated by $-D_V$. In the next proposition we use elliptic regularity to show that the resolvent of D_V leaves $C(\Gamma)$ invariant.

Lemma 3.2. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Let $\omega \in \mathbb{R}$ be such that $\|S_t^V\|_{2\to 2} \le e^{\omega t}$ for all $t > 0$. Let $\lambda \in (\omega, \infty)$ and $\psi \in C(\Gamma)$. Then $(\lambda I + D_V)^{-1} \psi \in C(\Gamma)$.*

Proof. Write $\varphi = (\lambda I + D_V)^{-1}\psi \in L_2(\Gamma)$. Then $D_V\varphi = \psi - \lambda\varphi$. There exists a unique $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\mathfrak{a}_V(u, v) = \int_\Gamma (\psi - \lambda\varphi)\,\overline{\operatorname{Tr} v}$ for all $v \in H^1(\Omega)$. Then

$$\mathfrak{a}(u, v) + \int_\Omega V\, u\, \overline{v} + \lambda \int_\Gamma \operatorname{Tr} u\, \overline{\operatorname{Tr} v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}$$

for all $v \in H^1(\Omega)$. Hence by [25, Theorem 3.14(ii)] one deduces that $u \in C(\overline{\Omega})$. So $\varphi \in C(\Gamma)$. □

Also the semigroup S^V leaves $C(\Gamma)$ invariant. Even stronger, it maps $L_1(\Gamma)$ into $C(\Gamma)$.

Proposition 3.3. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Then $S^V_t L_2(\Gamma) \subset C(\Gamma)$ for all $t > 0$.*

For the proof we need the following lemma:

Lemma 3.4. *Adopt the notation and assumptions of Proposition* 3.3. *Suppose $d \geq 3$. Let $q \in [\frac{2d}{d+2}, \frac{d}{2})$ and $\varepsilon > 0$. Let $u \in H^1(\Omega)$ and $\psi \in L_2(\Gamma) \cap L_{\frac{(d-1)q}{d-q}+\varepsilon}(\Gamma)$. Suppose that*

$$\mathfrak{a}_V(u, v) = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}$$

for all $v \in H^1(\Omega)$. Then $\operatorname{Tr} u \in L_{\frac{(d-1)q}{d-2q}}(\Gamma)$.

Proof. This is a special case of [25, Lemma 3.11]. □

Proof of Proposition 3.3 . First we show that for all $t > 0$ and $\varphi \in L_2(\Gamma)$ there exists an $\varepsilon > 0$ such that $S^V_t \varphi \in L_{d-1+\varepsilon}(\Gamma)$. For this we may assume that $d \geq 3$, since the case $d = 2$ is trivial. For all $n \in \{1, \ldots, d-1\}$ define

$$q_n = \frac{2d}{d+3-n}.$$

Then $q_1 = \frac{2d}{d+2}$, $q_{d-2} = \frac{2d}{5}$ and $q_{d-1} = \frac{d}{2}$. Moreover, $q_{n+1} = \frac{q_n d}{d - \frac{1}{2} q_n}$ for all $n \in \{1, \ldots, d-2\}$. We shall show that for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $n \in \{1, \ldots, d-1\}$ there exists an $\varepsilon > 0$ such that $S^V_t \varphi \in L_{\frac{(d-1)q_n}{d-q_n}+\varepsilon}(\Gamma)$. The proof is by induction on n.

Since $\frac{(d-1)q_1}{d-q_1} = 2\frac{d-1}{d} < 2$, the case $n = 1$ is trivial. Let $n \in \{1, \ldots, d-2\}$ and suppose that for all $t > 0$ and $\varphi \in L_2(\Gamma)$ there exists an $\varepsilon > 0$ such that $S^V_t \varphi \in L_{\frac{(d-1)q_n}{d-q_n}+\varepsilon}(\Gamma)$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Set $\psi = S^V_t\, D_V\, S^V_t \varphi$. Then there exists an $\varepsilon > 0$ such that $\psi \in L_{\frac{(d-1)q_n}{d-q_n}+\varepsilon}(\Gamma)$ by the induction hypothesis. Note that $D_V\, S^V_{2t}\varphi = \psi$. So by definition there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = S^V_{2t}\varphi$ and $\mathfrak{a}_V(u, v) = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}$ for all $v \in H^1(\Omega)$. Because $q_n \leq q_{d-2} < \frac{d}{2}$ one deduces from Lemma 3.4 that $\operatorname{Tr} u \in L_{\frac{(d-1)q_n}{d-2q_n}}(\Gamma)$. Since $\frac{(d-1)q_{n+1}}{d-q_{n+1}} = \frac{(d-1)q_n}{d-q_n} < \frac{(d-1)q_n}{d-2q_n}$, there exists an $\varepsilon' > 0$ such that $S^V_{2t}\varphi = \operatorname{Tr} u \in L_{\frac{(d-1)q_{n+1}}{d-q_{n+1}}+\varepsilon'}(\Gamma)$, which completes the induction step. So by induction for all $t > 0$ and $\varphi \in L_2(\Gamma)$ there exists an $\varepsilon > 0$ such that $S^V_t\varphi \in L_{\frac{(d-1)q_{d-1}}{d-q_{d-1}}+\varepsilon}(\Gamma)$. But $\frac{(d-1)q_{d-1}}{d-q_{d-1}} = d-1$.

Thus we proved for all $d \geq 2$, $t > 0$ and $\varphi \in L_2(\Gamma)$ that there exists an $\varepsilon > 0$ such that $S^V_t\varphi \in L_{d-1+\varepsilon}(\Gamma)$. Now one can argue once again as above and use this time [25, Lemma 3.10] to deduce that $S^V_{2t}\varphi \in C(\Gamma)$ for all $t > 0$ and $\varphi \in L_2(\Gamma)$. □

Corollary 3.5. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Then S^V has a continuous kernel.*

Proof. This follows from Proposition 3.3 and Theorem 2.1. □

For all $t > 0$ define $T^V_t : C(\Gamma) \to C(\Gamma)$ by

$$T^V_t = S^V_t|_{C(\Gamma)}.$$

Obviously $T^V = (T^V_t)_{t>0}$ is a semigroup, but it is unclear whether it is a C_0-semigroup. Define the part $D_{V,c}$ of D_V in $C(\Gamma)$ by

$$\operatorname{dom}(D_{V,c}) = \{\varphi \in C(\Gamma) \cap \operatorname{dom}(D_V) : D_V\varphi \in C(\Gamma)\}$$

and $D_{V,c}\varphi = D_V\varphi$ for all $\varphi \in \operatorname{dom}(D_{V,c})$. If T^V is a C_0-semigroup, then $-D_{V,c}$ is the generator of T^V and consequently $\operatorname{dom}(D_{V,c})$ is dense in $C(\Gamma)$.

4. Density of the domain in $C(\Gamma)$

In this section we shall prove that the operator $D_{V,c}$ has dense domain if the coefficients a_{kl} are Lipschitz continuous.

Theorem 4.1. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Then the domain* $\operatorname{dom}(D_{V,c})$ *of the operator $D_{V,c}$ is dense in $C(\Gamma)$.*

For the proof we need a lot of preparation. Throughout this section we adopt the assumptions of Theorem 4.1.

We aim to prove that $D_{V,c}$ has a dense domain, that is that there are sufficiently many $u \in H^1(\Omega)$ such that $(\mathcal{A} + V)u = 0$, $\operatorname{Tr} u$ is continuous, the function u has a weak conormal derivative and $\partial_\nu u$ is continuous. The next lemma gives existence of a class of functions on Ω with continuous trace, which have a weak conormal derivative and the conormal derivative is bounded (but not necessarily continuous).

Lemma 4.2. *Let $u \in C^1(\overline{\Omega}) \cap H^2(\Omega)$. Then u has a weak conormal derivative and $\partial_\nu u \in L_\infty(\Gamma)$.*

Proof. Since the $a_{kl} \in W^{1,\infty}(\Omega)$ it follows that $\mathcal{A}u \in L_2(\Omega)$. Let $v \in C^\infty(\overline{\Omega})$. For all $k \in \{1, \ldots, d\}$ define $F_k \colon \overline{\Omega} \to \mathbb{C}$ by $F_k = \overline{v} \sum_{l=1}^d a_{kl}\,(\partial_l u)$. Then $F_k \in C(\overline{\Omega}) \cap H^1(\Omega)$. Moreover, $\operatorname{div} F = \sum_{k,l=1}^d a_{kl}\,(\partial_k u)\,\overline{\partial_l v} - (\mathcal{A}u)\,\overline{v} \in L_1(\Omega)$. Hence the divergence theorem gives

$$\sum_{k,l=1}^d \int_\Omega a_{kl}\,(\partial_k u)\,\overline{\partial_l v} - \int_\Omega (\mathcal{A}u)\,\overline{v} = \int_\Omega \operatorname{div} F = \sum_{k=1}^d \int_\Gamma \nu_k \operatorname{Tr} F_k$$
$$= \sum_{k,l=1}^d \int_\Gamma (\nu_k \operatorname{Tr}(a_{kl}\,\partial_l u))\,\overline{\operatorname{Tr} v},$$

where ν is the normal vector. Then by density

$$\sum_{k,l=1}^d \int_\Omega a_{kl}\,(\partial_k u)\,\overline{\partial_l v} - \int_\Omega (\mathcal{A}u)\,\overline{v} = \sum_{k,l=1}^d \int_\Gamma \Big(\nu_k \operatorname{Tr}(a_{kl}\,\partial_l u)\Big)\,\overline{\operatorname{Tr} v}$$

for all $v \in H^1(\Omega)$. So u has a weak conormal derivative and $\partial_\nu u = \sum_{k,l=1}^d \nu_k \operatorname{Tr}(a_{kl}\partial_l u) \in L_\infty(\Gamma)$. ☐

Our next aim is to show that one can approximate an element of $C(\Gamma)$ by functions $u|_\Gamma$, where $u \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ and $(\mathcal{A} + V)u = 0$. We will show this in Lemma 4.7. For such u one deduces from the previous lemma that $u|_\Gamma \in \operatorname{dom}(D_V) \cap C(\Gamma)$ and $D_V(u|_\Gamma) = \partial_\nu u \in L_\infty(\Gamma)$.

The first ingredient is that the Lipschitz domain Ω can be approximated from outside by smooth domains.

Lemma 4.3. *There exist $c_1, c_2 > 0$ and $\Omega_1, \Omega_2, \ldots \subset \mathbb{R}^d$ such that the following is valid:*

(a) *For all $n \in \mathbb{N}$ the set Ω_n is open bounded with C^∞-boundary. Moreover, $\overline{\Omega} \subset \Omega_{n+1} \subset \Omega_n \subset \Omega + B(0, \frac{1}{n})$;*
(b) $\bigcap_{n=1}^\infty \overline{\Omega_n} = \overline{\Omega}$;
(c) *For all $n \in \mathbb{N}$ and $z \in \Gamma$ there exists a $z' \in \Omega_n^c$ such that $|z - z'| \le \frac{c_1}{n}$;*
(d) *If $n \in \mathbb{N}$, $z \in \partial\Omega_n$ and $r \in (0, 1]$, then $|B(z, r) \setminus \Omega_n| \ge c_2\, r^d$.*

Proof. This is a straightforward consequence of [15, Theorem 5.1]. ☐

Since Ω has a Lipschitz boundary, one can extend the coefficients a_{kl} to bounded real valued Lipschitz continuous functions on $\mathbb{R}^d$, which by abuse of notation we continue to denote by a_{kl}. Reducing μ if necessary, we may assume without loss of generality that (3.2) is valid for all $\xi \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$. Similarly we extend V to a bounded real valued measurable function on $\mathbb{R}^d$, still denoted by V. If $\Omega' \subset \mathbb{R}^d$ is open, then we define similarly to $\mathfrak{a}$ the form $\mathfrak{a}_{\Omega'} \colon H^1(\Omega') \times H^1(\Omega') \to \mathbb{C}$ and define similarly the operators $A^D_{\Omega'}$ and $A^N_{\Omega'}$. Moreover, define similarly the operator $\mathcal{A}_{\Omega'} \colon H^1(\Omega') \to H^{-1}(\Omega')$.

If $\Omega', \Omega'' \subset \mathbb{R}^d$ are open with $\Omega' \subset \Omega''$, then we will identify a self-adjoint operator in $L_2(\Omega')$ with a self-adjoint graph in $L_2(\Omega'')$ in a natural way, see [8, Section 3, in particular Proposition 3.3]. Moreover, we identify an element of $H^1_0(\Omega')$ with an element in $H^1_0(\Omega'')$ by extending the function with zero. Then $H^{-1}(\Omega'') \subset H^{-1}(\Omega')$.

If $\Omega_1, \Omega_2, \ldots \subset \mathbb{R}^d$ are as in Lemma 4.3, then the operators $A^D_{\Omega_n} + V$ in $L_2(\Omega_n)$ are a good approximation for the operator $A^D + V$ in $L_2(\Omega)$. This is the content of the next two lemmas. The first lemma is not new. We include the proof for completeness and refer to Daners [13] for a systematic investigation of domain approximation.

Lemma 4.4. *Let $\Omega_1, \Omega_2, \ldots \subset \mathbb{R}^d$ be open bounded sets with $\overline{\Omega} \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^\infty \overline{\Omega_n} = \overline{\Omega}$. Let $\omega \in \mathbb{R}$ and suppose that $V + \omega\, \mathbb{1}_{\Omega_1} \geq \mathbb{1}_{\Omega_1}$. Then*

$$\lim_{n\to\infty} (A^D_{\Omega_n} + V + \omega I)^{-1} = (A^D_{\Omega} + V + \omega I)^{-1}$$

in $\mathcal{L}(L_2(\Omega_1))$.

Proof. Without loss of generality we may assume that $V \geq \mathbb{1}_{\Omega_1}$ and $\omega = 0$. Let $f, f_1, f_2, \ldots \in L_2(\Omega_1)$ and suppose that $\lim f_n = f$ weakly in $L_2(\Omega_1)$. Let $n \in \mathbb{N}$. Set $u_n = (A^D_{\Omega_n} + V)^{-1} f_n$. Then $u_n \in H^1_0(\Omega_n) \subset H^1_0(\Omega_1)$. Moreover,

$$\mathfrak{a}_{\Omega_1}(u_n, v) + (V\, u_n, v)_{L_2(\Omega_1)} = (f_n, v)_{L_2(\Omega_1)} \tag{4.1}$$

for all $v \in H^1_0(\Omega_n)$. Choose $v = u_n$. Then

$$\begin{aligned}\mu \int_{\Omega_1} |\nabla u_n|^2 + \int_{\Omega_1} |u_n|^2 &\leq \mathfrak{a}_{\Omega_1}(u_n) + (V\, u_n, u_n)_{L_2(\Omega_1)} \\ &= (f_n, u_n)_{L_2(\Omega_1)} \leq \|f_n\|_{L_2(\Omega_1)} \, \|u_n\|_{L_2(\Omega_1)}.\end{aligned}$$

Hence $\|u_n\|_{L_2(\Omega_1)} \leq \|f_n\|_{L_2(\Omega_1)}$ and $\mu \int_{\Omega_1} |\nabla u_n|^2 \leq \|f_n\|^2_{L_2(\Omega_1)}$. Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $L_2(\Omega_1)$, it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega_1)$. Passing to a subsequence, if necessary, we may assume that there exists a $u \in H^1_0(\Omega_1)$ such that $\lim u_n = u$ weakly in $H^1_0(\Omega_1)$. Because Ω_1 is bounded, one then obtains that $\lim u_n = u$ (strongly) in $L_2(\Omega_1)$. Since $\operatorname{supp} u_n \subset \overline{\Omega_m}$ for all $n, m \in \mathbb{N}$ with $n \geq m$, it follows that $\operatorname{supp} u \subset \overline{\Omega_m}$ for all $m \in \mathbb{N}$. So $\operatorname{supp} u \subset \bigcap_{m=1}^\infty \overline{\Omega_m} = \overline{\Omega}$. Hence $u \in H^1_0(\Omega)$ since Ω has a Lipschitz boundary. Let $v \in H^1_0(\Omega)$. Then $v \in H^1_0(\Omega_n)$ for all $n \in \mathbb{N}$. Use (4.1) and take the limit $n \to \infty$. Then

$$\mathfrak{a}(u, v) + (V\, u, v)_{L_2(\Omega)} = (f, v)_{L_2(\Omega)}.$$

So $u \in \mathrm{dom}(A^D + V)$ and $(A^D + V)u = f|_\Omega$. Therefore $u = (A^D + V)^{-1} f$.

Choosing $f_n = f$ for all $n \in \mathbb{N}$ we proved that $\lim_{n\to\infty}(A^D_{\Omega_n} + V)^{-1} f = (A^D + V)^{-1} f$ in $L_2(\Omega_1)$ for all $f \in L_2(\Omega_1)$.

Finally, suppose that not $\lim_{n\to\infty}(A^D_{\Omega_n} + V)^{-1} = (A^D_\Omega + V)^{-1}$ in $\mathcal{L}(L_2(\Omega_1))$. Then there are $\varepsilon > 0$ and $f_1, f_2, \ldots \in L_2(\Omega_1)$ such that $\|f_n\|_{L_2(\Omega_1)} = 1$ and

$$\|(A^D_{\Omega_n} + V)^{-1} f_n - (A^D_\Omega + V)^{-1} f_n\|_{L_2(\Omega_1)} \ge \varepsilon$$

for all $n \in \mathbb{N}$. Passing to a subsequence, if necessary, we may assume that there exists an $f \in L_2(\Omega_1)$ such that $\lim f_n = f$ weakly in $L_2(\Omega_1)$. Then $\lim_{n\to\infty}(A^D_{\Omega_n} + V)^{-1} f_n = (A^D + V)^{-1} f$ in $L_2(\Omega_1)$ by the above. Since $(A^D + V)^{-1}$ is compact, also $\lim_{n\to\infty}(A^D_\Omega + V)^{-1} f_n = (A^D + V)^{-1} f$ in $L_2(\Omega_1)$. So $\lim_{n\to\infty} \|(A^D_{\Omega_n} + V)^{-1} f_n - (A^D_\Omega + V)^{-1} f_n\|_{L_2(\Omega_1)} = 0$. This is a contradiction. □

Lemma 4.5. *Let $\Omega_1, \Omega_2, \ldots \subset \mathbb{R}^d$ be open bounded sets with $\overline{\Omega} \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^\infty \overline{\Omega_n} = \overline{\Omega}$. Then there exists a $\delta > 0$ such that*

$$\sigma(A^D_{\Omega_n} + V) \cap (-\delta, \delta) = \emptyset$$

for all large $n \in \mathbb{N}$.

Proof. For all $n \in \mathbb{N}$ the self-adjoint operators $A^D_{\Omega_n} + V$ and $A^D + V$ are lower bounded by $-\|V\|_{L_\infty(\Omega_1)}$ and have compact resolvent. Hence they have a discrete spectrum. Let $n \in \mathbb{N}$. For all $m \in \mathbb{N}$ let $\lambda^{(n)}_m$ be the m-th eigenvalue of $A^D_{\Omega_n} + V$, counted with multiplicity. Define similarly λ_m with respect to $A^D + V$. Since $\lim_{n\to\infty}(A^D_{\Omega_n} + V + \omega I)^{-1} = (A^D_\Omega + V + \omega I)^{-1}$ in $\mathcal{L}(L_2(\Omega_1))$ with $\omega = \|V\|_{L_\infty(\Omega_1)} + 1$ by Lemma 4.4, it follows that $\lim_{n\to\infty} \lambda^{(n)}_m = \lambda_m$ for all $m \in \mathbb{N}$. For a short proof of this well known fact see [16].

By assumption $0 \notin \sigma(A^D + V)$. Hence there exists a $\delta > 0$ such that $\sigma(A^D_\Omega + V) \cap (-\delta, \delta) = \emptyset$. Since the eigenvalues converge, then also $\sigma(A^D_{\Omega_n} + V) \cap (-\delta, \delta) = \emptyset$ for all large $n \in \mathbb{N}$. □

The next lemma is a small extension of a special case of [19, Theorem 1.2].

Lemma 4.6. *For all $c, d, \mu, M > 0$ and $p \in (\frac{d}{2} \vee 2, \infty)$ there exist $\alpha \in (0,1)$ and $c_1 > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be open non-empty and suppose that $|B(z,r) \setminus \Omega| \ge c\, r^d$ for all $z \in \partial\Omega$ and $r \in (0,1]$. Let $V \in L_\infty(\Omega)$ with $\|V\|_{L_\infty(\Omega)} \le M$. For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$ with $\|a_{kl}\|_{L_\infty(\Omega)} \le M$ and suppose that $\mathrm{Re} \sum_{k,l=1}^d a_{kl}(x)\, \xi_k\, \overline{\xi_l} \ge \mu\, |\xi|^2$ for almost all $x \in \Omega$ and all $\xi \in \mathbb{C}^d$. Let $f \in L_p(\Omega) \cap L_2(\Omega)$ and $u \in H^1_0(\Omega)$. Suppose that

$$\sum_{k,l=1}^d \int_\Omega a_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \int_\Omega V\, u\, \overline{v} = \int_\Omega f\, \overline{v}$$

for all $v \in H_0^1(\Omega)$. Then $u \in C^\alpha(\Omega)$ and

$$|||u|||_{C^\alpha(\Omega)} \leq c_1\Big(\|u\|_{H^1(\Omega)} + \|f\|_{L_p(\Omega)}\Big),$$

where

$$|||u|||_{C^\alpha(\Omega)} = \sup\Big\{\frac{|u(x) - u(y)|}{|x-y|^\alpha} : x, y \in \Omega \text{ and } 0 < |x-y| \leq 1\Big\}. \tag{4.2}$$

Proof. If $V = 0$, then this is a special case of [19, Theorem 1.2] with the choice $\Gamma = \emptyset$, $\Upsilon = \Omega$ and $\zeta = 2$. If $V \neq 0$, then one has to replace f by $f - V\,u$ and iterate, using of [19, Proposition 3.2]. $\square$

Now we are able to prove that one can approximate elements in $C(\Gamma)$ by elements $\varphi \in \mathrm{dom}(D_V) \cap C(\Gamma)$ with $D_V\varphi \in L_\infty(\Gamma)$.

Lemma 4.7. *Let $\varphi \in C(\Gamma)$ and $\varepsilon > 0$. Then there exists a $u \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ such that $(\mathcal{A} + V)u = 0$ and $\|u|_\Gamma - \varphi\|_{C(\Gamma)} < \varepsilon$.*

Proof. Since $\{F|_\Gamma : F \in C^2(\mathbb{R}^d)\}$ is dense in $C(\Gamma)$ by the Stone–Weierstraß theorem, we may assume that there exists an $F \in C^2(\mathbb{R}^d)$ such that $\varphi = F|_\Gamma$.

Let $c_1, c_2 > 0$ and $\Omega_1, \Omega_2, \ldots \subset \mathbb{R}^d$ be as in Lemma 4.3. By Lemma 4.5 there exists a $\delta > 0$ such that $\sigma(A^D_{\Omega_n} + V) \cap (-\delta, \delta) = \emptyset$ for all large $n \in \mathbb{N}$. Without loss of generality we may assume that $\sigma(A^D_{\Omega_n} + V) \cap (-\delta, \delta) = \emptyset$ for all $n \in \mathbb{N}$. Then in particular $A^D_{\Omega_n} + V$ is invertible for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Define $G_n \in L_2(\Omega_n)$ by

$$G_n = -\sum_{k,l=1}^d \partial_l\, a_{kl}\, \partial_k (F|_{\Omega_n}).$$

Since $F \in C^2(\mathbb{R}^d)$ and $a_{kl} \in W^{1,\infty}(\mathbb{R}^d)$ one indeed obtains that $G_n \in L_2(\Omega_n)$. Even stronger, $G_n \in L_{d+1}(\Omega_n)$. Since $A^D_{\Omega_n} + V$ is invertible, we can define

$$w_n = (A^D_{\Omega_n} + V)^{-1}(G_n + V\,F).$$

Then $w_n \in H_0^1(\Omega_n) \cap C_0(\Omega_n)$, where the continuity follows for example from Lemma 4.6. Moreover,

$$\mathfrak{a}_{\Omega_n}(w_n, v) + \int_{\Omega_n} V\,w_n\,\overline{v} = \int_{\Omega_n} (G_n + V\,F)\,\overline{v} = \mathfrak{a}_{\Omega_n}(F|_{\Omega_n}, v) + \int_{\Omega_n} V\,F\,\overline{v}$$

for all $v \in H_0^1(\Omega_n)$. Let $u_n = F|_{\Omega_n} - w_n$. Then $(\mathcal{A}_{\Omega_n} + V)u_n = 0$. So $\mathcal{A}_{\Omega_n}u_n = -V\,u_n$ and hence $u_n \in W^{2,p}_{\mathrm{loc}}(\Omega_n)$ for all $p \in (1,\infty)$ by elliptic regularity. In particular $u_n|_\Omega \in C^1(\overline{\Omega}) \cap H^2(\Omega)$. Note that $u_n - F|_{\Omega_n} = -w_n$. By Lemmas 4.6 and 4.3(d) there exist $\alpha \in (0,1)$ and $c_3 > 0$, *independent* of n, such that

$$|||w_n|||_{C^\alpha(\Omega_n)} \le c_3\Big(\|G_n + V\,F\|_{L_{d+1}(\Omega_n)} + \|w_n\|_{H^1(\Omega_n)}\Big) \tag{4.3}$$

for all $n \in \mathbb{N}$, where $|||w_n|||_{C^\alpha(\Omega_n)}$ is defined as in (4.2). Clearly $\|G_n + V\,F\|_{L_{d+1}(\Omega_n)} \le \|G_1 + V\,F\|_{L_{d+1}(\Omega_1)}$ for all $n \in \mathbb{N}$. We next show that $(\|w_n\|_{H^1(\Omega_n)})_{n\in\mathbb{N}}$ is bounded.

Let $n \in \mathbb{N}$. Since $\sigma(A^D_{\Omega_n} + V) \cap (-\delta, \delta) = \emptyset$ it follows that $\|(A^D_{\Omega_n} + V)^{-1}\| \le \delta^{-1}$. Therefore

$$\|w_n\|_{L_2(\Omega_n)} \le \|(A^D_{\Omega_n} + V)^{-1}\|\,\|G_n + V\,F\|_{L_2(\Omega_n)} \le \tfrac{1}{\delta}\,\|G_1 + V\,F\|_{L_2(\Omega_1)}. \tag{4.4}$$

Set $\omega = \|V\|_{L_\infty(G_1)}$. Then

$$\begin{aligned}
\mu \int_{\Omega_n} |\nabla w_n|^2 &\le \mathfrak{a}_{\Omega_n}(w_n) \le \mathfrak{a}_{\Omega_n}(w_n) + \int_{\Omega_n} (V + \omega\,\mathbb{1}_{\Omega_n})\,|w_n|^2 \\
&= \int_{\Omega_n} (G_n + V\,F + \omega\,w_n)\,\overline{w_n} \\
&\le \Big(\|G_1 + V\,F\|_{L_2(\Omega_1)} + \omega\,\|w_n\|_{L_2(G_n)}\Big)\,\|w_n\|_{L_2(G_n)}.
\end{aligned}$$

Together with (4.4) one concludes the sequence $(\|w_n\|_{H^1(\Omega_n)})_{n\in\mathbb{N}}$ is bounded.

Using (4.3) there exists a $c_4 > 0$ such that $|||w_n|||_{C^\alpha(\Omega_n)} \le c_4$ uniformly for all $n \in \mathbb{N}$. Now let $z \in \Gamma$. By Lemma 4.3(c) there exists a $z' \in \Omega_n^c$ such that $|z - z'| \le \frac{c_1}{n}$. Hence if $n \ge c_1$, then $|w_n(z)| = |w_n(z) - w_n(z')| \le |||w_n|||_{C^\alpha(\Omega_n)}\,|z - z'|^\alpha \le c_4\,c_1^\alpha\,n^{-\alpha}$. Therefore $\lim_{n\to\infty} \|w_n|_\Gamma\|_{C(\Gamma)} = 0$. Hence $\lim_{n\to\infty} \|u_n|_\Gamma - F|_\Gamma\|_{C(\Gamma)} = 0$. So choose $u = u_n|_{\overline{\Omega}}$ with n large enough. □

We need one more lemma before we can prove density of $\mathrm{dom}(D_{V,c})$ in $C(\Gamma)$. The main aim in the lemma is to solve the Neumann problem with respect to $A^N + V$ for functions $\psi \in L_p(\Gamma)$. If p is large enough then solutions are continuous on $\overline{\Omega}$. We choose $p = d$. As expected, the kernel of of $A^N + V$ gives problems, so we take orthogonal complements.

Lemma 4.8. *Define*

$$H^1_{V\perp}(\Omega) = \{u \in H^1(\Omega) : (u, v)_{H^1(\Omega)} = 0 \textit{ for all } v \in \ker(A^N + V)\}$$

and

$$L_{d,V\perp}(\Gamma) = \{\tau \in L_d(\Gamma) : (\tau, \mathrm{Tr}\,v)_{L_2(\Gamma)} = 0 \textit{ for all } v \in \ker(A^N + V)\}.$$

Then one has the following:

(a) $\ker(A^N + V) \subset C(\overline{\Omega})$ *is finite dimensional;*
(b) *If* $u \in \ker(A^N + V)$, *then* $\mathrm{Tr}\,u \in \mathrm{dom}(D_{V,c})$*;*

(c) *If* $\tau \in L_{d,V\perp}(\Gamma)$ *and* $\varepsilon > 0$, *then there exists a* $\tau' \in C(\Gamma) \cap L_{d,V\perp}(\Gamma)$ *such that* $\|\tau - \tau'\|_{L_d(\Gamma)} < \varepsilon$;

(d) *For all* $\tau \in L_{d,V\perp}(\Gamma)$ *there exists a unique* $u \in H^1_{V\perp}(\Omega)$ *such that* $\mathfrak{a}_V(u, v) = \int_\Gamma \tau \, \overline{\mathrm{Tr}\, v}$ *for all* $v \in H^1(\Omega)$.

Define $E \colon L_{d,V\perp}(\Gamma) \to H^1_{V\perp}(\Omega)$ *such that*

$$\mathfrak{a}_V(E\tau, v) = \int_\Gamma \tau \, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$.

(e) *The map* E *is continuous;*

(f) *If* $\tau \in L_{d,V\perp}(\Gamma)$, *then* $E\tau \in C(\overline{\Omega})$;

(g) *The map* E *is continuous from* $L_{d,V\perp}(\Gamma)$ *into* $C(\overline{\Omega})$;

(h) *If* $\tau \in L_{d,V\perp}(\Gamma)$, *then* $\mathrm{Tr}\, E\tau \in \mathrm{dom}(D_V)$ *and* $D_V \mathrm{Tr}\, E\tau = \tau$.

Proof. (a). The operator $A^N + V$ has compact resolvent. Hence its kernel is finite dimensional. The inclusion follows from [25, Theorem 3.14(ii)].

(b). If $v \in H^1(\Omega)$, then $\mathfrak{a}_V(u, v) = ((A^N + V)u, v)_{L_2(\Omega)} = 0$. Therefore $\mathrm{Tr}\, u \in \mathrm{dom}(D_V)$ and $D_V \mathrm{Tr}\, u = 0$. Since $\mathrm{Tr}\, u \in C(\Gamma)$ by Statement (a) and obviously the zero function is continuous, one deduces that $\mathrm{Tr}\, u \in \mathrm{dom}(D_{V,c})$.

(c). By Statement (a) there exist $N \in \mathbb{N}_0$ and $\varphi_1, \ldots, \varphi_N \in \mathrm{Tr}\, \ker(A^N + V)$ such that $\varphi_1, \ldots, \varphi_N$ is a basis for $\mathrm{Tr}\, \ker(A^N + V)$. We may assume without loss of generality that $\varphi_1, \ldots, \varphi_N$ is orthonormal in $L_2(\Gamma)$. Since $C(\Gamma)$ is dense in $L_d(\Gamma)$ there exists a $\tau'' \in C(\Gamma)$ such that $\|\tau - \tau''\|_{L_d(\Gamma)} < \varepsilon$. For all $k \in \{1, \ldots, N\}$ set $c_k = (\tau'', \varphi_k)_{L_2(\Gamma)}$. Then $|c_k| = |(\tau'' - \tau, \varphi_k)_{L_2(\Gamma)}| \le \varepsilon \, \|\varphi_k\|_{L_p(\Gamma)}$, where $p \in (1, \infty)$ is the dual exponent of d. Set $\tau' = \tau'' - \sum_{k=1}^N c_k \, \varphi_k$. Then

$$\|\tau - \tau'\|_{L_d(\Gamma)} \le \|\tau - \tau''\|_{L_d(\Gamma)} + \sum_{k=1}^N |c_k| \|\varphi_k\|_{L_d(\Gamma)} \le \Big(1 + \sum_{k=1}^N \|\varphi_k\|_{L_d(\Gamma)} \|\varphi_k\|_{L_p(\Gamma)}\Big)\varepsilon$$

and $\tau' \in C(\Gamma) \cap L_{d,V\perp}(\Gamma)$.

(d). Define the form $\mathfrak{b} \colon H^1_{V\perp}(\Omega) \times H^1_{V\perp}(\Omega) \to \mathbb{C}$ by $\mathfrak{b} = \mathfrak{a}_V|_{H^1_{V\perp}(\Omega) \times H^1_{V\perp}(\Omega)}$. Then $\mathfrak{b}$ is a continuous symmetric sesquilinear form. Hence there exists a $T \in \mathcal{L}(H^1_{V\perp}(\Omega))$ such that $\mathfrak{b}(u, v) = (Tu, v)_{H^1_{V\perp}(\Omega)}$ for all $u, v \in H^1_{V\perp}(\Omega)$.

We next show that T is injective. Indeed, if $u \in \ker T$, then $\mathfrak{a}_V(u, v) = 0$ for all $v \in H^1_{V\perp}(\Omega)$. Obviously $\mathfrak{a}_V(u, v) = (u, (A^N + V)v)_{L_2(\Omega)} = 0$ for all $v \in \ker(A^N + V)$. Since $H^1(\Omega) = H^1_{V\perp}(\Omega) \oplus \ker(A^N + V)$, it follows that $\mathfrak{a}_V(u, v) = 0$ for all $v \in H^1(\Omega)$. Hence $u \in \mathrm{dom}(A^N + V)$ and $(A^N + V)u = 0$. So $u \in \ker(A^N + V)$. Also $u \in H^1_{V\perp}(\Omega)$. Therefore $u = 0$ and T is injective.

The inclusion map $H^1_{V\perp}(\Omega) \subset L_2(\Omega)$ is compact and the form $\mathfrak{b}$ is $L_2(\Omega)$-elliptic. Hence by [8, Lemma 4.1] the operator T is invertible.

The Sobolev embedding theorem, [24, Theorems 2.4.2 and 2.4.6], gives $\operatorname{Tr} H^1(\Omega) \subset L_{\frac{2d-2}{d-2+\varepsilon}}(\Gamma)$ for all $\varepsilon \in (0,1]$. Moreover, $L_{\frac{2d-2}{d-2+\varepsilon}}(\Gamma) \subset L_{\frac{d}{d-1}}(\Gamma)$. Hence there exists a $c > 0$ such that

$$\Big| \int_\Gamma \tau \, \overline{\operatorname{Tr} v} \Big| \le c \, \|\tau\|_{L_d(\Gamma)} \, \|v\|_{H^1(\Omega)}$$

for all $\tau \in L_d(\Gamma)$ and $v \in H^1(\Omega)$. Now let $\tau \in L_{d,V\perp}(\Gamma)$. Then the map $\alpha\colon H^1_{V\perp}(\Omega) \to \mathbb{C}$ given by $\alpha(v) = \int_\Gamma \tau \, \overline{\operatorname{Tr} v}$ is continuous and anti-linear. Hence there exists a unique $u \in H^1_{V\perp}(\Omega)$ such that $(Tu, v)_{H^1_{V\perp}(\Omega)} = \alpha(v)$ for all $v \in H^1_{V\perp}(\Omega)$. Moreover, $\|u\|_{H^1(\Omega)} \le c \, \|T^{-1}\| \, \|\tau\|_{L_d(\Gamma)}$. Then

$$\mathfrak{a}_V(u, v) = \mathfrak{b}(u, v) = (Tu, v)_{H^1_{V\perp}(\Omega)} = \alpha(v) = \int_\Gamma \tau \, \overline{\operatorname{Tr} v}$$

for all $v \in H^1_{V\perp}(\Omega)$. Clearly $\mathfrak{a}(u, v) = 0$ and $\int_\Gamma \tau \, \overline{\operatorname{Tr} v} = 0$ for all $v \in \ker(A^N + V)$. Hence $\mathfrak{a}_V(u, v) = \int_\Gamma \tau \, \overline{\operatorname{Tr} v}$ for all $v \in H^1(\Omega)$. Note that $E\tau = u$.

(e). In the proof of Statement (d) we deduced that $\|E\tau\|_{H^1(\Omega)} \le c\|T^{-1}\| \|\tau\|_{L_d(\Gamma)}$ for all $\tau \in L_{d,V\perp}(\Gamma)$. So E is continuous.

(f). This follows from [25, Theorem 3.14(ii)].

(g). By [25, Theorem 3.14(ii)] there exists a $c' > 0$ such that

$$\|E\tau\|_{C(\overline{\Omega})} \le c'(\|E\tau\|_{L_2(\Omega)} + \|\tau\|_{L_d(\Gamma)})$$

for all $\tau \in L_{d,V\perp}(\Gamma)$. But

$$\|E\tau\|_{L_2(\Omega)} \le \|E\tau\|_{H^1(\Omega)} \le c \, \|T^{-1}\| \, \|\tau\|_{L_d(\Gamma)}.$$

So $\|E\tau\|_{C(\overline{\Omega})} \le c'(c \, \|T^{-1}\| + 1)\|\tau\|_{L_d(\Gamma)}$ for all $\tau \in L_{d,V\perp}(\Gamma)$.

(h). This follows from the definitions of E and D_V. □

Now we are able to prove that the operator $D_{V,c}$ is densely defined.

Proof of Theorem 4.1. Let $H^1_{V\perp}(\Omega)$, $L_{d,V\perp}(\Gamma)$ and the map $E\colon L_{d,V\perp}(\Gamma) \to H^1_{V\perp}(\Omega) \cap C(\overline{\Omega})$ be as in Lemma 4.8. Let $M > 0$ be such that

$$\|E\tau\|_{C(\overline{\Omega})} \le M \, \|\tau\|_{L_d(\Gamma)}$$

for all $\tau \in L_{d,V\perp}(\Gamma)$. Let $N \in \mathbb{N}_0$ and $u_1, \ldots, u_N \in \ker(A^N + V)$ be such that $u_1, \ldots, u_N$ is a basis for $\ker(A^N + V)$ and is orthonormal in $H^1(\Omega)$. Note that $u_k \in C(\overline{\Omega})$ for all $k \in \{1, \ldots, N\}$ by Lemma 4.8(a).

Let $\varphi \in C(\Gamma)$ and $\varepsilon > 0$. By Lemma 4.7 there exists a $u \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ such that $(\mathcal{A} + V)u = 0$ and $\|u|_\Gamma - \varphi\|_{C(\Gamma)} < \varepsilon$. Then u has a weak conormal derivative and $\partial_\nu u \in L_\infty(\Gamma)$ by Lemma 4.2. If $v \in \ker(A^N + V)$, then

$$\begin{aligned}\int_\Gamma (\partial_\nu u)\,\overline{\operatorname{Tr} v} &= \mathfrak{a}(u, v) - \int_\Omega (\mathcal{A}u)\,\overline{v} = \mathfrak{a}(u, v) + \int_\Omega V\,u\,\overline{v}\\ &= \mathfrak{a}_V(u, v) = (u, (A^N + V)v)_{L_2(\Omega)} = 0.\end{aligned}$$

So $\partial_\nu u \in L_{d,V\perp}(\Gamma)$. By Lemma 4.8(c) there exists a $\tau \in C(\Gamma) \cap L_{d,V\perp}(\Gamma)$ such that $\|\tau - \partial_\nu u\|_{L_d(\Gamma)} < \varepsilon$. Choose $w = E\tau$. Then $w \in H^1_{V\perp}(\Omega) \cap C(\overline{\Omega})$ and

$$\mathfrak{a}_V(w, v) = \int_\Gamma \tau\,\overline{\operatorname{Tr} v}$$

for all $v \in H^1(\Omega)$. Set $c_k = (u, u_k)_{H^1(\Omega)} \in \mathbb{C}$ for all $k \in \{1, \ldots, N\}$. Then by construction $w - u + \sum_{k=1}^N c_k\,u_k \in H^1_{V\perp}(\Omega)$. Let $v \in H^1(\Omega)$. Then

$$\begin{aligned}\mathfrak{a}_V\Big(w - u + \sum_{k=1}^N c_k\,u_k, v\Big) &= \mathfrak{a}_V(w, v) - \mathfrak{a}_V(u, v)\\ &= \int_\Gamma \tau\,\overline{\operatorname{Tr} v} - \Big(\int_\Gamma (\partial_\nu u)\,\overline{\operatorname{Tr} v} + \int_\Omega ((\mathcal{A} + V)u)\,\overline{v}\Big)\\ &= \int_\Gamma (\tau - \partial_\nu u)\,\overline{\operatorname{Tr} v}.\end{aligned}$$

Note that $\tau - \partial_\nu u \in L_{d,V\perp}(\Gamma)$. So

$$w - u + \sum_{k=1}^N c_k\,u_k = E(\tau - \partial_\nu u).$$

Hence

$$\|w - u + \sum_{k=1}^N c_k\,u_k\|_{C(\overline{\Omega})} \le M\,\|\tau - \partial_\nu u\|_{L_d(\Gamma)} \le M\,\varepsilon.$$

Then $\|w|_\Gamma - \varphi + \sum_{k=1}^N c_k\,u_k|_\Gamma\|_{C(\Gamma)} \le (M + 1)\varepsilon$.

Finally note that $\operatorname{Tr} w \in \operatorname{dom}(D_V)$ and $D_V(\operatorname{Tr} w) = \tau$ by Lemma 4.8(h). Since both $\operatorname{Tr} w$ and τ are continuous, one deduces that $\operatorname{Tr} w \in \operatorname{dom}(D_{V,c})$. Moreover, $\operatorname{Tr} u_k \in \operatorname{dom}(D_{V,c})$ for all $k \in \{1, \ldots, N\}$ by Lemma 4.8(b). So $\varphi \in \overline{\operatorname{dom}(D_{V,c})}$. The proof of Theorem 4.1 is complete. □

5. C_0-semigroup on $C(\Gamma)$

We next consider the problem whether $-D_{V,c}$ generates a C_0-semigroup on $C(\Gamma)$. If $(X, \mathcal{B}, \mu)$ is a measure space, then for operators on the Hilbert space $L_2(X)$ the notation of positivity has two different meanings and in the next lemma we need both of them. We will use the following terminology if confusion is possible. If B is an operator in a Hilbert space H, then we say that B is *positive in the Hilbert space sense* if $(Bu, u)_H \geq 0$ for all $u \in \mathrm{dom}(B)$. If $B\colon L_2(X) \to L_2(X)$ is a linear operator, then we say that B is *positive in the Banach lattice sense* if $Bf \geq 0$ for all $f \in L_2(X)$ with $f \geq 0$. Here $f \geq 0$ means that $f(x) \geq 0$ for almost all $x \in X$. Below we consider the two cases $X = \Omega$, provided with the Lebesgue measure, and $X = \Gamma$, provided with the $(d-1)$-dimensional Hausdorff measure.

The following proposition is known if $a_{kl} = \delta_{kl}$, that is if $A = -\Delta$.

Proposition 5.1. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid.*

(a) *Suppose that $A^D + V$ is positive in the Hilbert space sense and $0 \notin \sigma(A^D + V)$. Then the semigroup S^V is positive in the Banach lattice sense;*
(b) *Suppose that $V \geq 0$. Then the semigroup S^V is submarkovian.*

Proof. Statement (a) can be proved as in [9, Theorem 5.1] or [17, Theorem 2.3(a)], with obvious modifications. Statement (b) is similar to [17, Theorem 2.3(b)]. □

It turns out that the resolvent of $D_{V,c}$ behaves well. Recall that D_V is a lower-bounded self-adjoint operator.

Lemma 5.2. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Let $\omega \in \mathbb{R}$ be such that $\|S_t^V\|_{2\to 2} \leq e^{\omega t}$ for all $t > 0$. Let $\lambda \in (\omega, \infty)$. Then one has the following:*

(a) *$\lambda I + D_{V,c}$ is invertible;*
(b) *$(\lambda I + D_{V,c})^{-1}\psi = (\lambda I + D_V)^{-1}\psi$ for all $\psi \in C(\Gamma)$;*
(c) *If $A^D + V$ is positive in the Hilbert space sense, then $(\lambda I + D_{V,c})^{-1}$ is positive in the Banach lattice sense.*

Proof. (a). Let $\psi \in C(\Gamma)$. Write $\varphi = (\lambda I + D_V)^{-1}\psi \in L_2(\Gamma)$. Then $D_V\varphi = \psi - \lambda\varphi$ and $\varphi \in C(\Gamma)$ by Lemma 3.2. So $\psi - \lambda\varphi \in C(\Gamma)$ and $\varphi \in \mathrm{dom}(D_{V,c})$. Obviously $(\lambda I + D_{V,c})(\lambda I + D_V)^{-1}\psi = \psi$. So the operator $\lambda I + D_{V,c}$ is surjective. Since $\lambda I + D_V$ is injective, also the operator $\lambda I + D_{V,c}$ is injective. Therefore $\lambda I + D_{V,c}$ is bijective, that is invertible.

Statement (b) is now clear.

(c). It follows from Proposition 5.1(a) that the operator $(\lambda I + D_V)^{-1}$ is positive in the Banach lattice sense on $L_2(\Gamma)$. Then the statement is a consequence of Statement (b). □

We now prove the main theorem of this paper. In view of our general assumption (3.3), Condition (a) can be reformulated by saying that the first eigenvalue of $A^D + V$ is strictly positive. In contrast to this, Condition (b) does not include any spectral condition (except that $0 \notin \sigma(A^D + V)$). As a matter of fact, in fact the potential can be very negative.

Theorem 5.3. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Moreover, suppose that at least one of the following conditions is valid:*

(a) *$A^D + V$ is positive in the Hilbert space sense;*
(b) *One has $a_{kl} = \delta_{kl}$ for all $k, l \in \{1, \ldots, d\}$ and the set Ω has a $C^{1,1}$-boundary.*

Then $S_t^V C(\Gamma) \subset C(\Gamma)$ for all $t > 0$ and $(S_t^V|_{C(\Gamma)})_{t>0}$ is a C_0-semigroup whose generator is $-D_{V,c}$.

Proof. (a). The operator $-D_{V,c}$ is a densely defined resolvent positive operator by Theorem 4.1 and Lemma 5.2(c). Moreover, the positive cone in $C(\Gamma)$ has a non-empty interior. Hence $-D_{V,c}$ is the generator of a C_0-semigroup by [2, Corollary 2.3].

(b). By [7, Proposition 6.10] the semigroup S^V leaves $L_\infty(\Gamma)$ invariant and there exists an $M \geq 1$ such that $\|S_t^V \varphi\|_\infty \leq M \|\varphi\|_\infty$ for all $t \in (0, 1]$ and $\varphi \in L_\infty(\Gamma)$. Then $\|T_t^V\|_{C(\Gamma)\to C(\Gamma)} \leq M$ for all $t \in (0, 1]$. If $\varphi \in \operatorname{dom}(D_{V,c})$, then

$$\|(I - T_t^V)\varphi\|_{C(\Gamma)} \leq \int_0^t \|S_s^V D_V \varphi\|_\infty \, ds \leq M\, t\, \|D_V \varphi\|_\infty$$

for all $t \in (0, 1]$. Hence $\lim_{t\downarrow 0} T_t^V \varphi = \varphi$ in $C(\Gamma)$. Since $\operatorname{dom}(D_{V,c})$ is dense in $C(\Gamma)$ by Theorem 4.1, one deduces that T^V is a C_0-semigroup on $C(\Gamma)$. It is easy to verify that $-D_{V,c}$ is the generator. (This argument also works if $V \geq 0$ and merely the $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$, by using Proposition 5.1(b) instead of [7, Proposition 6.10].) □

Whereas under Condition (a) the semigroup T^V is positive (in the Banach lattice sense), this is in general not the case under Condition (b), see [14].

Corollary 5.4. *For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose* (3.1), (3.2) *and* (3.3) *are valid. Suppose $A^D + V$ is positive in the Hilbert space sense. Then for all $p \in [1, \infty)$ the semigroup S^V extends to a C_0-semigroup on $L_p(\Gamma)$.*

Proof. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Then

$$\begin{aligned}\|S_t^V \varphi\|_1 &= \sup_{\substack{\psi \in C(\Gamma) \\ \|\psi\|_\infty \leq 1}} |(S_t^V \varphi, \psi)_{L_2(\Gamma)}| \\ &= \sup_{\substack{\psi \in C(\Gamma) \\ \|\psi\|_\infty \leq 1}} |(\varphi, T_t^V \psi)_{L_2(\Gamma)}| \leq \sup_{\substack{\psi \in C(\Gamma) \\ \|\psi\|_\infty \leq 1}} \|\varphi\|_1 \, \|T_t^V \psi\|_\infty \leq \|T_t^V\| \, \|\varphi\|_1.\end{aligned}$$

Hence S_t^V extends to a bounded operator $S_t^{V(1)} \colon L_1 \to L_1$ and $\|S_t^{V(1)}\| \leq \|T_t^V\|$. It is easy to verify that $S^{V(1)}$ is a semigroup on L_1. Moreover, $\sup_{t\in(0,1]} \|S_t^{V(1)}\| \leq \sup_{t\in(0,1]} \|T_t^V\| < \infty$. Since Γ has finite measure, the semigroup $S^{V(1)}$ is a C_0-semigroup. Then by duality and interpolation the corollary follows. □

6. The Robin semigroup on $C(\overline{\Omega})$

In order to prove irreducibility of T^V in case $A^D + V$ is positive in the Hilbert space sense, we make a detour and prove irreducibility for the Robin Laplacian.

Throughout this section we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected set with Lipschitz boundary, $a_{kl} = a_{lk} \in L_\infty(\Omega, \mathbb{R})$, the ellipticity condition (3.2) is valid and $V \in L_\infty(\Omega, \mathbb{R})$. Moreover, let $\beta \in L_\infty(\Gamma, \mathbb{R})$. We do not assume that $0 \notin \sigma(A^D + V)$. Define the sesquilinear form $\mathfrak{a}_{V,\beta} \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_{V,\beta}(u, v) = \mathfrak{a}_V(u, v) + \int_\Gamma \beta \operatorname{Tr} u \, \overline{\operatorname{Tr} v}.$$

Then $\mathfrak{a}_{V,\beta}$ is an $L_2(\Omega)$-elliptic sesquilinear form. Let $A_{V,\beta}$ be the associated operator. Then $A_{V,\beta}$ is self-adjoint and bounded below. It is easy to see that

$$\operatorname{dom}(A_{V,\beta}) = \{u \in H^1(\Omega) : \mathcal{A}u \in L_2(\Omega) \text{ and } \partial_\nu u + \beta \operatorname{Tr} u = 0\}$$

and $A_{V,\beta} u = \mathcal{A}u + V\,u$ for all $u \in \operatorname{dom}(A_{V,\beta})$. So $A_{V,\beta}$ is the realisation of $\mathcal{A} + V$ with Robin boundary conditions. The operator $-A_{V,\beta}$ generates a C_0-semigroup $S^{V,\beta}$ on $L_2(\Omega)$, which is called the Robin semigroup. If $\beta \geq 0$ then it is well known that the semigroup $S^{V,\beta}$ has Gaussian kernel bounds (see [4, Theorem 4.9]) and therefore the semigroup $S^{V,\beta}$ on $L_2(\Omega)$ extrapolates to a C_0-semigroup on $L_p(\Omega)$ for all $p \in [1, \infty)$. It is an open problem whether the same is valid without the condition $\beta \geq 0$. Added in proof: this also remains true without the condition $\beta \geq 0$.[1]

The main theorem of this section is as follows:

Theorem 6.1. *Adopt the above notation and assumptions:*

(a) *The semigroup $S^{V,\beta}$ is positive (in the Banach lattice sense);*
(b) *If $t > 0$ then $S_t^{V,\beta} L_2(\Omega) \subset C(\overline{\Omega})$;*
(c) *If $\lambda > \omega$, then $(\lambda I + A_{V,\beta})^{-d} L_2(\Omega) \subset C(\overline{\Omega})$. Here $\omega \in \mathbb{R}$ is chosen large enough such that $\sup_{t>0} e^{-\omega t} \|S_t^{V,\beta}\|_{2\to 2} < \infty$;*
(d) *For all $t > 0$ the operator $S_t^{V,\beta}$ has a continuous kernel $k_t \colon \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$;*
(e) *The operator $A_{V,\beta}$ has compact resolvent;*
(f) *The semigroup $S^{V,\beta}$ is irreducible (on $L_2(\Omega)$);*
(g) *The eigenvalue $\min \sigma(A_{V,\beta})$ is simple;*
(h) *The semigroup $(S_t^{V,\beta}|_{C(\overline{\Omega})})_{t>0}$ is a C_0-semigroup on $C(\overline{\Omega})$;*
(i) *The semigroup $(S_t^{V,\beta}|_{C(\overline{\Omega})})_{t>0}$ is irreducible (on $C(\overline{\Omega})$);*
(j) *There exists a $\delta > 0$ such that $u_1(x) \geq \delta$ for all $x \in \overline{\Omega}$, where $u_1 \in L_2(\overline{\Omega})$ is an eigenfunction of $A_{V,\beta}$ with eigenvalue $\min \sigma(A_{V,\beta})$ such that $u_1 \geq 0$ almost everywhere;*

[1] See D. Daners, *Inverse positivity for general Robin problems on Lipschitz domain*, Archiv. Math. (Basel) **29** (2009), 57–69, Theorem 2.2 and Lemma 3.2.

(k) *For all $p \in [1, \infty)$ the semigroup $S^{V,\beta}$ extends consistently to a C_0-semigroup on $L_p(\Omega)$.*

Proof.

(a). This follows as in the proof of [4, Theorem 4.9]. The positivity of β is not needed in that proof;

(b). This follows from [25, Theorem 3.14(ii)] and Theorem 2.1(i)⇒ (ii);

(c). This follows from [25, Lemmas 3.11 and 3.10];

(d). This is a consequence of Statement (b) and Theorem 2.1;

(e). Easy;

(f). This is a consequence of [26, Corollary 2.11];

(g). See Lemma 2.3(d);

(h). Define the part $A_{V,\beta,c}$ of $A_{V,\beta}$ in $C(\overline{\Omega})$ by

$$\mathrm{dom}(A_{V,\beta,c}) = \{u \in C(\overline{\Omega}) \cap \mathrm{dom}(A_{V,\beta}) : A_{V,\beta}u \in C(\overline{\Omega})\}$$

and $A_{V,\beta,c}u = A_{V,\beta}u$ for all $u \in \mathrm{dom}(A_{V,\beta,c})$. Then $\mathrm{dom}(A_{V,\beta,c})$ is dense in $C(\overline{\Omega})$ by the arguments in the proof of [25, Theorem 4.3]. (Remark, unfortunately there is a gap in the proof of [25, Theorem 4.3] for the part that the restriction $(S_t^{V,\beta}|_{C(\overline{\Omega})})_{t>0}$ of the Robin semigroup in $C(\overline{\Omega})$ is a C_0-semigroup, since it is unclear whether $\sup_{t\in(0,1]} \|S_t^{V,\beta}\|_{\infty\to\infty} < \infty$. He used that the semigroup $S^{V,\beta}$ has a kernel with Gaussian bounds, which is only known in case $\beta \geq 0$.)
Let $\omega \in \mathbb{R}$ be as in Statement (c). Let $\lambda > \omega$. Then the operator $\lambda I + A_{V,\beta,c}$ is invertible by the same argument as in the proof of Lemma 5.2(a). Since the resolvent operator $(\lambda I + A_{V,\beta})^{-1}$ is positive on $L_2(\Omega)$, also the resolvent operator $(\lambda I + A_{V,\beta,c})^{-1}$ is positive on $C(\overline{\Omega})$. Moreover, the positive cone in $C(\overline{\Omega})$ has a non-empty interior. Hence $-A_{V,\beta,c}$ is the generator of a C_0-semigroup by [2, Corollary 2.3];

(i). and (j). This follows from Corollary 2.5;

(k). The proof is similar to the proof of Corollary 5.4. □

Remark 6.2. In order to avoid confusion with the assumptions and notation in the rest of this paper we continued to assume in this section that the coefficients are symmetric and that there are no first-order terms. One can, however, consider the full Robin form $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ given by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_\Omega a_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \sum_{k=1}^{d} \int_\Omega b_k\,(\partial_k u)\,\overline{v} + \sum_{k=1}^{d} \int_\Omega c_k\,u\,\overline{\partial_k v} + \int_\Omega c_0\,u\,\overline{v} + \int_\Gamma \beta\,\mathrm{Tr}\,u\,\overline{\mathrm{Tr}\,v},$$

where $a_{kl}, b_k, c_k, c_0 \in L_\infty(\Omega, \mathbb{R})$ and $\beta \in L_\infty(\Gamma, \mathbb{R})$, together with the ellipticity condition (3.2). We do not assume any longer that the a_{kl} are symmetric. Let A be the m-sectorial operator associated with $\mathfrak{a}$ and let S be the semigroup generated by $-A$ on $L_2(\Omega)$. Then Statements (a), (b), (c), (d), (e), (f), (h) and (k) are still valid, with the same proof. Instead of Statement (g) one can consider $\lambda_1 = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$. Then $\lambda_1 \in \sigma(A)$ by [3, Proposition 3.11.2] and it follows as before that λ_1 is a simple eigenvalue. If A is symmetric, then also Statement (j) is valid.

We do not know whether Statement (i) is still valid if A is not symmetric. We also do not know whether Statement (k) is valid if $b_k, c_k, c_0 \in L_\infty(\Omega)$ and $\beta \in L_\infty(\Gamma)$ are complex valued.

7. Strictly positive first eigenfunction and extensions to $L_p(\Gamma)$

In this section we consider the case where the semigroup generated by $-D_V$ is positive (in the Banach lattice sense) and under the condition that Ω is connected we show that the first eigenfunction is strictly positive. We deduce from this that the Dirichlet-to-Neumann semigroup is irreducible on $C(\Gamma)$. This is surprising since we merely assume that Ω is connected. For example, if Ω is an annulus, then Γ is not connected. The result also allows us to extend the semigroup S^V consistently to a C_0-semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty)$.

We adopt the assumptions and notation as in Section 3. In particular, for all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \in L_\infty(\Omega, \mathbb{R})$. Let $V \in L_\infty(\Omega, \mathbb{R})$. We suppose that (3.1), (3.2) and (3.3) are valid. In addition we assume that Ω is connected and that $A^D + V$ is positive (in the Hilbert space sense). Then S^V is a positive semigroup by Proposition 5.1(a). Moreover, $S_t^V L_2(\Gamma) \subset C(\Gamma)$ for all $t > 0$ by Proposition 3.3 and D_V is self-adjoint with compact resolvent. So all eigenfunctions of D_V are elements of $C(\Gamma)$. Let $\lambda_1 = \min \sigma(D_V)$. Let $\varphi_1 \in C(\Gamma)$ be an eigenfunction with eigenvalue λ_1 such that $\varphi_1 \geq 0$.

Theorem 7.1. *Adopt the above notation and assumptions. Then* $\min \varphi_1 > 0$.

The theorem is an immediate consequence of the next proposition.

Proposition 7.2. *Let $\beta \in \mathbb{R}$ and $\varphi \in \operatorname{dom}(D_V)$ be and eigenfunction of D_V with eigenvalue $-\beta$. Suppose that $\varphi \geq 0$. We identify the real number β with the constant function $\beta\, \mathbb{1}_\Gamma$ on Γ. Consider the Robin operator $A_{V,\beta}$ as in Section 6. Then $\min \sigma(A_{V,\beta}) = 0$. Let $u_1 \in C(\overline{\Omega})$ be an eigenfunction of $A_{V,\beta}$ with eigenvalue 0 as in Theorem 6.1(j). Then there exists a $c > 0$ such that $\varphi = c\, u_1|_\Gamma$. In particular,* $\dim \operatorname{span}\{\psi \in \ker(\beta I + D_V) : \psi \geq 0\} = 1$ *and $\varphi(z) > 0$ for all $z \in \Gamma$.*

Proof. By definition of D_V there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and

$$\mathfrak{a}_V(u, v) = -\beta\, (\varphi, \operatorname{Tr} v)_{L_2(\Gamma)} \tag{7.1}$$

for all $v \in H^1(\Omega)$. Since $\varphi \geq 0$ it follows that u is real valued and $u^- \in H_0^1(\Omega)$. Choose $v = u^-$. Then $\mathfrak{a}_V(u, u^-) = 0$. But $\partial_k(u^-) = -(\partial_k u)\, \mathbb{1}_{[u<0]}$ for all

$k \in \{1, \ldots, d\}$. Therefore $\mathfrak{a}_V(u^-, u^-) = \mathfrak{a}_V(u, u^-) = 0$. Since $A^D + V$ is a positive operator in the Hilbert space sense with trivial kernel by assumption (3.3), it follows that $u^- = 0$. Therefore $u \geq 0$ and clearly $u \neq 0$. It follows from (7.1) that $\mathfrak{a}_{V,\beta}(u, v) = 0$ for all $v \in H^1(\Omega)$. Therefore $u \in \operatorname{dom}(A_{V,\beta})$ and $A_{V,\beta} u = 0$. The operator $-A_{V,\beta}$ is self-adjoint, has compact resolvent and generates a positive irreducible semigroup in $L_2(\Omega)$. Hence it follows from the inverse Krein–Rutman theorem [6, Lemma 5.14] that $0 = \min \sigma(A_{V,\beta})$.

Since $\min \sigma(A_{V,\beta})$ is a simple eigenvalue of $A_{V,\beta}$ by Theorem 6.1(g), it follows that there exists a $c \in \mathbb{C} \setminus \{0\}$ such that $u = c\, u_1$. But both $u_1, u \geq 0$. Therefore $c > 0$. Then $\varphi = \operatorname{Tr} u = c \operatorname{Tr} u_1$. Since $u_1(x) > 0$ for all $x \in \overline{\Omega}$ by Theorem 6.1(j), obviously $\varphi(z) > 0$ for all $z \in \Gamma$. □

Recall that T^V is the restriction of the Dirichlet-to-Neumann semigroup S^V to $C(\Gamma)$. We show below that T^V is irreducible. We cannot deduce this in general from the strict positivity of the first eigenfunction via Proposition 2.4, since Γ is not connected in general.

Irreducibility of S^V in $L_2(\Gamma)$ is much easier. We need the following result.

Proposition 7.3. *Let (Y, Σ, μ) be a finite measure space. Let B be a lower bounded self-adjoint operator in $L_2(Y)$ and suppose that $-B$ generates a positive C_0-semigroup S on $L_2(Y)$. Let $\varphi \in L_2(Y)$ and suppose that $\varphi(y) > 0$ for almost every $y \in Y$. Further suppose that $S_t \varphi = \varphi$ for all $t > 0$ and that* $\dim \operatorname{span}\{\psi \in \ker B : \psi \geq 0\} = 1$. *Then S is irreducible.*

Proof. The proof is a variation of the proof of [5, Proposition 2.2]. Let Y_1 be a measurable subset of Y and suppose that $S_t L_2(Y_1) \subset L_2(Y_1)$ for all $t > 0$. Set $Y_2 = Y \setminus Y_1$. Since S_t is self-adjoint one deduces that $S_t L_2(Y_2) \subset L_2(Y_2)$ for all $t > 0$. Let $t > 0$. Then

$$\varphi\, \mathbb{1}_{Y_1} + \varphi\, \mathbb{1}_{Y_2} = \varphi = S_t \varphi = S_t(\varphi\, \mathbb{1}_{Y_1}) + S_t(\varphi\, \mathbb{1}_{Y_2}).$$

Since S_t leaves $L_2(Y_1)$ and $L_2(Y_2)$ invariant, it follows that $S_t(\varphi\, \mathbb{1}_{Y_1}) = \varphi\, \mathbb{1}_{Y_1}$ and $S_t(\varphi\, \mathbb{1}_{Y_2}) = \varphi\, \mathbb{1}_{Y_2}$. So $\varphi\, \mathbb{1}_{Y_1} \in \ker B$ and $\varphi\, \mathbb{1}_{Y_2} \in \ker B$. Since $\dim \operatorname{span}\{\psi \in \ker B : \psi \geq 0\} = 1$ one deduces that $\varphi\, \mathbb{1}_{Y_1} = 0$ or $\varphi\, \mathbb{1}_{Y_2} = 0$. Therefore $\mu(Y_1) = 0$ or $\mu(Y_2) = 0$. □

Proposition 7.4. *The semigroup S^V is irreducible on $L_2(\Gamma)$ and* $\min \sigma(D_V)$ *is a simple eigenvalue.*

Proof. It follows from Proposition 7.2 that $\varphi_1(z) > 0$ for all $z \in \Gamma$ and $\dim \operatorname{span}\{\psi \in \ker(D_V - \lambda_1 I) : \psi \geq 0\} = 1$. Apply Proposition 7.3 to the operator $D_V - \lambda_1 I$. One deduces that S^V is irreducible. Then the eigenvalue $\min \sigma(D_V)$ is simple by Lemma 2.3(d). □

Now we prove the irreducibility in $C(\Gamma)$.

Theorem 7.5. *The semigroup T^V is irreducible on $C(\Gamma)$.*

Proof. Let Γ_1 be a closed subset of Γ with $\emptyset \neq \Gamma_1 \neq \Gamma$. Set $I = \{\varphi \in C(\Gamma) : \varphi|_{\Gamma_1} = 0\}$. Assume that $T_t^V I \subset I$ for all $t > 0$. We consider two cases.

Case I. *Suppose* Γ_1 *is not open*. Then there exists an $x_0 \in \partial\Gamma_1$. Then one can argue as in the proof of the implication (ii) $\Rightarrow$ (i) in the proof of Proposition 2.4 to deduce that $(T_t^V u)(x_0) = 0$ for all $u \in C(\Gamma)$ and $t > 0$. But $(T_t^V \varphi_1)(x_0) = e^{-\lambda_1 t}\, \varphi_1(x_0) > 0$ for all $t > 0$ by Theorem 7.1. This is a contradiction.

Case II. *Suppose* Γ_1 *is open*. Then Γ_1 is a connected component of Γ. Hence $\sigma(\Gamma_1) > 0$ and $\sigma(\Gamma \setminus \Gamma_1) > 0$. Let $J = \{\varphi \in L_2(\Gamma) : \varphi|_{\Gamma_1} = 0\}$. Then J is the closure of I in $L_2(\Gamma)$ and $S_t^V J \subset J$ for all $t > 0$. Since S^V is irreducible one deduces that $\sigma(\Gamma_1) = 0$ or $\sigma(\Gamma \setminus \Gamma_1) = 0$. This is a contradiction. □

Corollary 7.6. *For all* $p \in [1, \infty)$ *the semigroup* S^V *extends consistently to a* C_0-*semigroup on* $L_p(\Gamma)$.

Proof. This follows from Proposition 2.6. □

Corollary 7.7. *Let* $\varphi \in C(\Gamma)$ *with* $\varphi \geq 0$ *and* $\varphi \neq 0$. *Then* $(T_t^V \varphi)(z) > 0$ *for all* $t > 0$ *and* $z \in \Gamma$.

Proof. Apply Proposition 2.4(i) $\Rightarrow$ (iv). □

We do not know whether T^V is a C_0-semigroup (unless the a_{kl} are Lipschitz continuous, see Theorem 5.3(a)).

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