

Convergence of the Yamabe flow on manifolds with minimal boundary

SÉRGIO ALMARAZ AND LIMING SUN

Abstract. We study the Yamabe flow on compact Riemannian manifolds of dimensions greater than two with minimal boundary. Convergence to a metric with constant scalar curvature and minimal boundary is established in dimensions up to seven, and in any dimensions if the manifold is spin.

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1. Introduction

Let M^n be a closed manifold with dimension $n \geq 3$. In order to solve the Yamabe problem (see [33]), R. Hamilton introduced the Yamabe flow, which evolves Riemannian metrics on M according to the equation

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \bar{R}_{g(t)})g(t),$$

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where R_g denotes the scalar curvature of the metric g and $\overline{R}_g$ stands for the average

$$\left(\int_M dv_g \right)^{-1} \int_M R_g dv_g.$$

Here, dv_g is the volume form of (M, g) . Although the Yamabe problem was solved using a different approach in [4, 25, 31], the Yamabe flow is a natural geometric deformation to metrics of constant scalar curvature. The convergence of the Yamabe flow on closed manifolds was studied in [13, 28, 34]. This question was solved in [8, 9] under the hypotheses of the positive mass theorem.

In this work, we study the convergence of the Yamabe flow on compact n -dimensional manifolds with boundary, when $n \geq 3$. For those manifolds, J. Escobar raised the question of existence of conformal metrics with constant scalar curvature which have the boundary as a minimal hypersurface. This problem was studied in [10, 15, 23]; see also [3, 19].

Let (M^n, g_0) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. We consider the following conformal invariant defined in [15]:

$$\begin{aligned} Q(M) &= \inf_{g \in [g_0]} \frac{\int_M R_g dv_g + 2 \int_{\partial M} H_g d\sigma_g}{\left(\int_M dv_g \right)^{\frac{n-2}{n}}} \\ &= \inf_{\{u \in C^1(M), u \neq 0\}} \frac{\int_M \left(\frac{4(n-1)}{n-2} |du|_{g_0}^2 + R_{g_0} u^2 \right) dv_{g_0} + \int_{\partial M} 2H_{g_0} u^2 d\sigma_{g_0}}{\left(\int_M |u|^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}}, \end{aligned}$$

where H_g and $d\sigma_g$ denote respectively the trace of the 2nd fundamental form and the volume form of ∂M , with respect to the metric g , and $[g_0]$ stands for the conformal class of the metric g_0 .

We are interested in a formulation of the Yamabe flow for compact manifolds with minimal boundary proposed by S. Brendle in [7]. This flow evolves a conformal family of metrics $g(t)$, $t \geq 0$, according to the equations

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \overline{R}_{g(t)})g(t) & \text{in } M \\ H_{g(t)} = 0 & \text{on } \partial M. \end{cases} \quad (1.1)$$

Theorem 1.1 ([7]). *Suppose that:*

- (i) $Q(M) \leq 0$, or
- (ii) $Q(M) > 0$ and M is locally conformally flat with umbilic boundary.

Then, for every initial metric $g(0)$ on M with minimal boundary, the flow (1.1) exists for all time $t \geq 0$ and converges to a constant scalar curvature metric with minimal boundary.

Inspired by the ideas in [8,9], we handle the remaining cases of this problem. Define

$$\mathcal{Z}_M = \{x_0 \in M \setminus \partial M; \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)| = 0\},$$

$$\mathcal{Z}_{\partial M} = \{x_0 \in \partial M; \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)| = \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{1-d} |\pi_{g_0}(x)| = 0\},$$

$$\text{and } \mathcal{Z} = \mathcal{Z}_M \cup \mathcal{Z}_{\partial M},$$

where W_{g_0} denotes the Weyl tensor of M , π_{g_0} the trace-free second fundamental form of ∂M , and $d = \left\lceil \frac{n-2}{2} \right\rceil$. Our first result is the following:

Theorem 1.2. *Suppose that (M^n, g_0) is not conformally diffeomorphic to the hemisphere S_+^n and satisfies $Q(M) > 0$. If*

- (a) $\mathcal{Z} = \emptyset$, or
- (b) $n \leq 7$, or
- (c) M is spin,

then, for any initial metric $g(0)$ on M with minimal boundary, the flow (1.1) exists for all time $t \geq 0$ and converges to a metric with constant scalar curvature and minimal boundary.

Since the round sphere S^n minus a point is diffeomorphic to $\mathbb{R}^n$, which is spin, the following is an immediate consequence of Theorems 1.1 and 1.2:

Corollary 1.3. *If $M \subset S^n$ is a compact domain with smooth boundary, then the flow (1.1), starting with any metric with minimal boundary, exists for all time $t \geq 0$ and converges to a metric with constant scalar curvature and minimal boundary.*

Condition (a) in Theorem 1.2 is particularly satisfied if the Weyl tensor and the trace-free second fundamental form are nonzero everywhere on $M \setminus \partial M$ and ∂M respectively. Conditions (b) and (c) allow us to make use of the positive mass theorem in [26, 27, 32] and its corresponding version for manifolds with a non-compact boundary in [2].

Before stating our main result, from which Theorem 1.2 follows, we will briefly discuss those positive mass theorems.

Definition 1.4. Let (N, g) be a Riemannian manifold with a (possibly empty) boundary ∂N . We say that N is *asymptotically flat* with order $p > 0$, if there is a compact set $K \subset N$ and a diffeomorphism $f : N \setminus K \rightarrow \mathbb{R}^n \setminus \overline{B_1(0)}$ or $f : N \setminus K \rightarrow \mathbb{R}_+^n \setminus \overline{B_1^+(0)}$ such that, in the coordinate chart defined by f (which we call the *asymptotic coordinates* of N), we have

$$|g_{ab}(y) - \delta_{ab}| + |y| |g_{ab,c}(y)| + |y|^2 |g_{ab,cd}(y)| = O(|y|^{-p}), \quad \text{as } |y| \rightarrow \infty,$$

where $a, b, c, d = 1, \dots, n$. Here, $\mathbb{R}_+^n = \{(y_1, \dots, y_n) \in \mathbb{R}^n; y_n \geq 0\}$, $\overline{B_1(0)} = \{y \in \mathbb{R}^n; |y| \leq 1\}$ and $\overline{B_1^+(0)} = \overline{B_1(0)} \cap \mathbb{R}_+^n$.

Suppose the manifold N^n , with dimension $n \geq 3$, is asymptotically flat with order $p > \frac{n-2}{2}$, as defined above. Assume also that R_g is integrable on N , and H_g is integrable on ∂N if ∂N is noncompact. Let $(y_1, \dots, y_n)$ be the asymptotic coordinates induced by the diffeomorphism f .

If f takes values in $\mathbb{R}^n \setminus \overline{B_1(0)}$ then ∂N is compact (or empty) and the limit

$$m_{ADM}(g) := \lim_{R \rightarrow \infty} \sum_{a,b=1}^n \int_{y \in \mathbb{R}^n, |y|=R} (g_{ab,b} - g_{bb,a}) \frac{y_a}{|y|} d\sigma_R$$

exists and is called the *ADM mass* of (N, g) . Moreover, $m_{ADM}(g)$ is a geometric invariant in the sense that it does not depend on the asymptotic coordinates; see [6].

Conjecture 1.5 (Positive mass). If $R_g, H_g \geq 0$, then we have $m_{ADM}(g) \geq 0$ and the equality holds if and only if N is isometric to $\mathbb{R}^n$. In particular, $\partial N = \emptyset$ when the equality holds.

As a consequence of [26, 27, 32] we have:

Theorem 1.6. *Conjecture 1.5 holds true if $n \leq 7$ or if N is spin.*

The proof for $n \leq 7$ was obtained by Schoen and Yau in [26, 27], and the one for spin manifolds by Witten in [32] when $M = \emptyset$. The boundary condition used in [2] can be used to extend Witten's result to the case $\partial M \neq \emptyset$.

If f takes values in $\mathbb{R}_+^n \setminus \overline{B_1^+(0)}$ then the limit

$$m(g) := \lim_{R \rightarrow \infty} \left\{ \sum_{a,b=1}^n \int_{y \in \mathbb{R}_+^n, |y|=R} (g_{ab,b} - g_{bb,a}) \frac{y_a}{|y|} d\sigma_R + \sum_{i=1}^{n-1} \int_{y \in \partial \mathbb{R}_+^n, |y|=R} g_{ni} \frac{y_i}{|y|} d\sigma_R \right\} \quad (1.2)$$

exists, and we call it the *mass* of (M, g) . Moreover, $m(g)$ is a geometric invariant in the sense that it does not depend on the asymptotic coordinates; see [2].

Conjecture 1.7 (Positive mass with a noncompact boundary). If $R_g, H_g \geq 0$, then we have $m(g) \geq 0$ and the equality holds if and only if N is isometric to $\mathbb{R}_+^n$.

In [2], this conjecture is reduced to Conjecture 1.5, so we have the following result:

Theorem 1.8. *Conjecture 1.7 holds true if $n \leq 7$ or if N is spin.*

The asymptotically flat manifolds used in this paper are obtained as the generalized stereographic projections of the compact Riemannian manifold (M, g_0) with nonempty boundary. Those stereographic projections are performed around points $x_0 \in M$ by means of Green functions G_{x_0} , with singularity at x_0 . After choosing a new background metric $g_{x_0} \in [g_0]$ with better coordinates expansion around x_0 (see Section 3), we consider the asymptotically flat manifold $(M \setminus \{x_0\}, \bar{g}_{x_0})$, where

$\bar{g}_{x_0} = G_{x_0}^{\frac{4}{n-2}} g_{x_0}$ satisfies $R_{\bar{g}_{x_0}} \equiv 0$ and $H_{\bar{g}_{x_0}} \equiv 0$. If $x_0 \in \mathcal{Z}_{\partial M}$, according to Proposition 3.13 below, this manifold has asymptotic order $p > \frac{n-2}{2}$, so Conjecture 1.7 claims that $m(\bar{g}_{x_0}) > 0$ unless M is conformally equivalent to the unit hemisphere. If $x_0 \in \mathcal{Z}_M$, this manifold has asymptotic order $p > \frac{n-2}{2}$ (see [9, Proposition 19]), so Conjecture 1.5 claims that $m_{ADM}(\bar{g}_{x_0}) > 0$.

Our main result, which implies Theorem 1.2, is the following:

Theorem 1.9. *Suppose that (M^n, g_0) is not conformally diffeomorphic to the unit hemisphere S_+^n and satisfies $Q(M) > 0$. Assume that $m_{ADM}(\bar{g}_{x_0}) > 0$ for all $x_0 \in \mathcal{Z}_M$ and $m(\bar{g}_{x_0}) > 0$ for all $x_0 \in \mathcal{Z}_{\partial M}$. Then, for any initial metric $g(0)$ with minimal boundary, the flow (1.1) exists for all $t \geq 0$ and converges to a constant scalar curvature metric with minimal boundary.*

The proof of Theorem 1.9 follows the arguments in [8]; see also [1]. An essential step is the construction of a family of test functions around each point $x_0 \in M$, whose energies are uniformly bounded by the Yamabe quotient $Y(S^n)$ if $x_0 \in M \setminus \partial M$, and by $Q(S_+^n)$ if $x_0 \in \partial M$. If $x_0 \in M \setminus \partial M$, the test functions used are essentially the ones introduced by S. Brendle in [9] for the case of closed manifolds. If $x_0 \in \partial M$, the functions used here were obtained in [10] in the case of umbilic boundary, where the authors address the existence of solutions to the Yamabe problem for manifolds with boundary. In this paper, however, we estimate their energies without any assumption on the boundary.

An additional difficulty in controlling the energy of interior test functions by $Y(S^n)$ arises when their centers get close to the boundary (see Subsection 3.3). In this case, the techniques in [9] cannot be directly adapted because the standard (and symmetric) bubble in $\mathbb{R}^n$, which represents the sphere metric and is essential in the construction of the test functions, does not satisfy the Neumann boundary condition unless it is centered on $\partial \mathbb{R}_+^n$. However, here we are able to exploit the sign of this Neumann derivative, when centered in $\mathbb{R}_+^n \setminus \partial \mathbb{R}_+^n$, to obtain the necessary estimates.

This paper is organized as follows. In Section 2, we establish some preliminaries and prove the long-time existence of the flow. In Section 3, we construct the necessary test functions and estimate their energy. In Section 4, we make use of the decomposition theorem in [24] to carry out a blow-up analysis using the test functions. In Section 5, first we use the blow-up analysis to prove a result which is analogous to Proposition 3.3 of [8]. Then we use it to prove our main theorem by estimating the solution to the flow uniformly in $t \geq 0$.

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2. Preliminary results and long-time existence

Notation. In the rest of this paper, M^n will denote a compact manifold of dimension $n \geq 3$ with boundary ∂M , and g_0 will denote a background Riemannian metric on M . We will denote by $B_r(x)$ the metric ball in M of radius r with center $x \in M$ (observe that $B_r(x)$ intersects ∂M when $g_{g_0}(x, \partial M) < r$).

For any Riemannian metric g on M , η_g will denote the inward unit normal vector to ∂M with respect to g and Δ_g the Laplace-Beltrami operator.

If $z_0 \in \mathbb{R}_+^n$, we set $B_r^+(z_0) = \{z \in \mathbb{R}_+^n; |z - z_0| < r\}$,

$$D_r(z_0) = B_r^+(z_0) \cap \partial \mathbb{R}_+^n, \quad \text{and} \quad \partial^+ B_r^+(z_0) = \partial B_r^+(z_0) \cap \mathbb{R}_+^n.$$

Finally, for any $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ we set $\bar{z} = (z_1, \dots, z_{n-1}, 0) \in \partial \mathbb{R}_+^n \cong \mathbb{R}^{n-1}$.

Convention. We assume that (M, g_0) satisfies $Q(M) > 0$. According to [15, Lemma 1.1], we can also assume that $R_{g_0} > 0$ and $H_{g_0} \equiv 0$, after a conformal change of the metric. Multiplying g_0 by a positive constant, we can suppose that $\int_M dv_{g_0} = 1$. We will adopt the summation convention whenever confusion is not possible, and use indices $a, b, c, d = 1, \dots, n$, and $i, j, k, l = 1, \dots, n-1$.

If $g = u^{\frac{4}{n-2}} g_0$ for some positive smooth function u on M , we know that

$$\begin{cases} R_g = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u \right) & \text{in } M \\ H_g = u^{-\frac{n}{n-2}} \left(-\frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} u + H_{g_0} u \right) & \text{on } \partial M, \end{cases} \quad (2.1)$$

and the operators $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ and $B_g = \frac{\partial}{\partial \eta_g} - \frac{n-2}{2(n-1)} H_g$ satisfy

$$L_{u^{\frac{4}{n-2}} g_0} (u^{-1} \zeta) = u^{-\frac{n+2}{n-2}} L_{g_0} \zeta, \quad (2.2)$$

$$B_{u^{\frac{4}{n-2}} g_0} (u^{-1} \zeta) = u^{-\frac{n}{n-2}} B_{g_0} \zeta, \quad (2.3)$$

for any smooth function ζ .

If $u(t) = u(\cdot, t)$ is a 1-parameter family of positive smooth functions on M and $g(t) = u(t)^{\frac{4}{n-2}} g_0$ with $H_{g_0} \equiv 0$, then (1.1) can be written as

$$\begin{cases} \frac{\partial}{\partial t} u(t) = -\frac{n-2}{4} (R_{g(t)} - \bar{R}_{g(t)}) u(t), & \text{in } M, \\ \frac{\partial}{\partial \eta_{g_0}} u(t) = 0, & \text{on } \partial M. \end{cases} \quad (2.4)$$

The first equation of (2.4) can also be written as

$$\frac{\partial}{\partial t} u(t)^{\frac{n+2}{n-2}} = \frac{n+2}{4} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + \bar{R}_{g(t)} u^{\frac{n+2}{n-2}} \right).$$

Short-time existence of solutions to the equations (2.4) can be obtained by standard theory for quasilinear parabolic equations. Hence, the equations (2.4) have a solution $u(t)$ defined for all t in the maximal interval $[0, T_{\max})$.

Taking $\partial/\partial\eta_{g_0}$ on both sides of the first equation of (2.4) and using the second one, one gets $\partial R_{g(t)}/\partial\eta_{g_0} = 0$ on ∂M . Hence the scalar curvature has evolution equations

$$\begin{cases} \frac{\partial}{\partial t} R_{g(t)} = (n-1)\Delta_{g(t)} R_{g(t)} + (R_{g(t)} - \bar{R}_{g(t)})R_{g(t)} & \text{in } M \\ \frac{\partial}{\partial \eta_{g(t)}} R_{g(t)} = 0 & \text{on } \partial M, \end{cases} \quad (2.5)$$

where the first equation comes from the well known first variation formula of scalar curvature.

Observe that for all $t \geq 0$ we have

$$\frac{\partial}{\partial t} dv_{g(t)} = -\frac{n}{2}(R_{g(t)} - \bar{R}_{g(t)}) dv_{g(t)} \quad (2.6)$$

and

$$\frac{\partial}{\partial t} \bar{R}_{g(t)} = -\frac{n-2}{2} \int_M (R_{g(t)} - \bar{R}_{g(t)})^2 dv_{g(t)}. \quad (2.7)$$

In particular, $\bar{R}_{g(t)}$ is decreasing and one can easily derive that (1.1) preserves the volume which we can normalize to

$$\int_M dv_{g(t)} = 1, \quad \text{for all } t \in [0, T_{\max}).$$

So, $\bar{R}_{g(t)} \geq Q(M) > 0$ for all $t \geq 0$.

Proposition 2.1. *We have $R_{g(t)} \geq \min\{\inf_M R_{g(0)}, 0\}$, for all $t \in [0, T_{\max})$.*

Proof. Following (2.5), this is an application of maximum principle. $\square$

Proposition 2.2. *For each $T \in (0, T_{\max})$, there exist $C(T), c(T) > 0$ such that*

$$\sup_M u(t) \leq C(T) \quad \text{and} \quad \inf_M u(t) \geq c(T), \quad \text{for all } t \in [0, T]. \quad (2.8)$$

In particular, $T_{\max} = \infty$.

Proof. Set $\sigma = 1 - \min\{\inf_M R_{g(0)}, 0\} = \max\{\sup_M (1 - R_{g(0)}), 1\}$. Then, by Proposition 2.1, $R_{g(t)} + \sigma \geq 1$ for all $t \in [0, T_{\max})$. It follows from (2.4) and (2.7) that

$$\frac{\partial}{\partial t} \log u(t) = \frac{n-2}{4} (\bar{R}_{g(t)} - R_{g(t)}) \leq \frac{n-2}{4} (\bar{R}_{g(0)} + \sigma).$$

Then there exists $C(T) > 0$ such that $\sup_M u(t) \leq C(T)$ for all $t \in [0, T]$.

Defining $P = R_{g_0} + \sigma \left(\sup_{0 \leq t \leq T} \sup_M u(t) \right)^{\frac{4}{n-2}}$ we obtain

$$\begin{aligned} -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + Pu(t) &\geq -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + R_{g_0} u(t) + \sigma u(t)^{\frac{n+2}{n-2}} \\ &= (R_{g(t)} + \sigma) u(t)^{\frac{n+2}{n-2}} \geq 0 \end{aligned}$$

for all $0 \leq t \leq T$. Then it follows from Proposition A.4 in the appendix that

$$\inf_M u(t) \left(\sup_M u(t) \right)^{\frac{n+2}{n-2}} \geq c(T) \int_M u(t)^{\frac{2n}{n-2}} dv_{g_0} = c(T),$$

by our volume normalization. This proves the second equation of (2.8).

Now we can follow [8, Proposition 2.6] to prove that if $0 < \alpha < \min\{4/n, 1\}$ then there is $\tilde{C}(T)$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \tilde{C}(T) \left((t_1 - t_2)^{\alpha/2} + d_{g_0}(x_1, x_2)^\alpha \right)$$

for all $x_1, x_2 \in M$ and $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. Then standard regularity theory for parabolic equations can be used to prove that all higher order derivatives of u are uniformly bounded on every fixed interval $[0, T]$. This implies the long-time existence of u . $\square$

Set

$$\overline{R}_\infty = \lim_{t \rightarrow \infty} \overline{R}_{g(t)} > 0. \quad (2.9)$$

Because $\partial R_{g(t)} / \partial \eta_{g(t)} = 0$ holds on ∂M , we can follow the proof of Corollary 3.2 in [8] line by line, making use of (2.5), (2.6) and (2.7), to obtain

Corollary 2.3. *For any $1 < p < n/2 + 1$ we have*

$$\lim_{t \rightarrow \infty} \int_M |R_{g(t)} - \overline{R}_\infty|^p dv_{g(t)} = 0.$$

3. The test functions

In this section we construct the test functions to be used in the blow-up analysis of Section 4. Those functions are perturbations of the symmetric functions U_ϵ (see (3.1) below), which represent the spherical metric on $\mathbb{R}^n$ and have maximum at the origin.

We will make use of the following coordinate systems:

Definition 3.1. Fix $x_0 \in \partial M$ and geodesic normal coordinates for ∂M centered at x_0 . Let $(y_1, \dots, y_{n-1})$ be the coordinates of $x \in \partial M$ and $\eta(x)$ be the inward unit vector normal to ∂M at x . For small $y_n \geq 0$, the point $\exp_x(y_n \eta(x)) \in M$ is said to have *Fermi coordinates* $(y_1, \dots, y_n)$ (centered at x_0).

Definition 3.2. Let g be any (smooth) Riemannian metric on M . Consider $\tilde{M}$ the double of M along its boundary and extend g to a (smooth) Riemannian metric $\tilde{g}$ on $\tilde{M}$. Fix $x_0 \in M$ and let $\tilde{\psi}_{x_0} : B_r(0) \subset \mathbb{R}^n \rightarrow \tilde{M}$ be normal coordinates (with respect to $\tilde{g}$) centered at x_0 . If $\tilde{B}_{x_0,r} = \tilde{\psi}_{x_0}^{-1}(\tilde{\psi}_{x_0}(B_r(0)) \cap M)$, we define the *extended normal coordinates* (centered at x_0)

$$\psi_{x_0} : \tilde{B}_{x_0,r} \subset \mathbb{R}^n \rightarrow M$$

as the restriction of $\tilde{\psi}_{x_0}$ to $\tilde{B}_{x_0,r}$.

Observe that this definition depends on the metric $\tilde{g}$ chosen, but this does not harm our arguments in this section because we can fix the extension to $\tilde{M}$ of the background metric g_0 .

Convention. We will refer to extended normal coordinates as *normal coordinates* for short.

Notation. We set $\tilde{D}_{x_0,r} = \psi_{x_0}^{-1}(\psi_{x_0}(\tilde{B}_{x_0,r}) \cap \partial M)$ and $\partial^+ \tilde{B}_{x_0,r} = \partial \tilde{B}_{x_0,r} \setminus \tilde{D}_{x_0,r} \subset \partial B_r(0)$.

Set $M_t = \{x \in M ; d_{g_0}(x, \partial M) \leq t\}$ and let $\delta_0 > 0$ be a small constant to be chosen later (see Remark 4.1 below). In the next subsections we will define three types of test functions:

- **Type A** test functions $(\bar{u}_{A;(x_0,\epsilon)})$: defined in Subsection 3.2 using Fermi coordinates centered at any $x_0 \in \partial M$ and with energy to be controlled by $Q(S_+^n)$;
- **Type B** test functions $(\bar{u}_{B;(x_0,\epsilon)})$: defined in Subsection 3.3 using normal coordinates centered at any $x_0 \in M_{2\delta_0} \setminus \partial M$ and with energy to be controlled by $Y(S^n)$;
- **Type C** test functions $(\bar{u}_{C;(x_0,\epsilon)})$: defined in Subsection 3.4 using normal coordinates centered at any $x_0 \in M \setminus M_{\delta_0}$ and with energy to be controlled by $Y(S^n)$.

We fix $P_0 = P_0(M, g_0) > 0$ small such that (extended) normal coordinates with center x_0 are defined in $\tilde{B}_{x_0, 2P_0}$ for all $x_0 \in M \setminus \partial M$, and Fermi coordinates with center at x_0 are defined in $B_{2P_0}^+(0)$ for all $x_0 \in \partial M$.

Convention. In this section we will use the normalization $\bar{R}_\infty = 4n(n-1)$, without loss of generality.

3.1. The auxiliary function ϕ and some algebraic preliminaries

Firstly we fix some notation. If $\epsilon > 0$, we define

$$U_\epsilon(y) = \left(\frac{\epsilon}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}} \quad \text{for } y \in \mathbb{R}^n. \quad (3.1)$$

It is well known that U_ϵ satisfies

$$\begin{cases} \Delta U_\epsilon + n(n-2)U_\epsilon^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}_+^n \\ \partial_n U_\epsilon = 0 & \text{on } \partial \mathbb{R}_+^n, \end{cases} \quad (3.2)$$

and

$$4n(n-1) \left(\int_{\mathbb{R}_+^n} U_\epsilon(y)^{\frac{2n}{n-2}} dy \right)^{\frac{2}{n}} = Q(S_+^n). \quad (3.3)$$

In this subsection $\mathcal{H}$ will denote a symmetric trace-free 2-tensor on $\mathbb{R}_+^n$ with components $\mathcal{H}_{ab}$, $a, b = 1, \dots, n$, satisfying

$$\begin{cases} \mathcal{H}_{ab}(0) = 0 & \text{for } a, b = 1, \dots, n \\ \mathcal{H}_{an}(x) = 0 & \text{for } x \in \mathbb{R}_+^n, a = 1, \dots, n \\ \partial_k \mathcal{H}_{ij}(0) = 0 & \text{for } i, j, k = 1, \dots, n-1 \\ \sum_{j=1}^{n-1} x_j \mathcal{H}_{ij}(x) = 0 & \text{for } x \in \partial\mathbb{R}_+^n, i = 1, \dots, n-1. \end{cases} \quad (3.4)$$

We will also assume that those components are of the form

$$\mathcal{H}_{ab}(x) = \sum_{1 \leq |\alpha| \leq d} h_{ab,\alpha} x^\alpha \quad \text{for } x \in \mathbb{R}_+^n, \quad (3.5)$$

where $d = \left\lceil \frac{n-2}{2} \right\rceil$ and each α stands for a multi-index. Obviously, the constants $h_{ab,\alpha} \in \mathbb{R}$ satisfy $h_{an,\alpha} = 0$ for any α , and $h_{ab,\alpha} = 0$ for any $\alpha \neq (0, \dots, 0, 1)$ with $|\alpha| = 1$, where $a, b = 1, \dots, n$.

Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative smooth function such that $\chi|_{[0,4/3]} \equiv 1$ and $\chi|_{[5/3,\infty)} \equiv 0$. If $\rho > 0$, we define

$$\chi_\rho(x) = \chi\left(\frac{|x|}{\rho}\right) \quad \text{for } x \in \mathbb{R}^n. \quad (3.6)$$

Notice that $\partial_n \chi_\rho = 0$ on $\partial\mathbb{R}_+^n$.

Let $V = V(\epsilon, \rho, \mathcal{H})$ be the smooth vector field on $\mathbb{R}_+^n$ obtained in [10, Theorem A.4], which satisfies

$$\begin{cases} \sum_{b=1}^n \partial_b \left\{ U_\epsilon^{\frac{2n}{n-2}} (\chi_\rho \mathcal{H}_{ab} - \partial_a V_b - \partial_b V_a + \frac{2}{n} (\operatorname{div} V) \delta_{ab}) \right\} = 0 & \text{in } \mathbb{R}_+^n \\ \partial_n V_i = V_n = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases} \quad (3.7)$$

for $a = 1, \dots, n$, and $i = 1, \dots, n-1$, and

$$|\partial^\beta V(x)| \leq C(n, |\beta|) \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| (\epsilon + |x|)^{|\alpha|+1-|\beta|} \quad (3.8)$$

for any multi-index β . Here $\delta_{ab} = 1$ if $a = b$ and $\delta_{ab} = 0$ if $a \neq b$.

We define symmetric trace-free 2-tensors S and T on $\mathbb{R}_+^n$ by

$$S_{ab} = \partial_a V_b + \partial_b V_a - \frac{2}{n} \partial_c V_c \delta_{ab} \quad \text{and} \quad T = \mathcal{H} - S. \quad (3.9)$$

(Recall that we are adopting the summation convention.) Observe that $T_{in} = S_{in} = 0$ on $\partial\mathbb{R}_+^n$ for $i = 1, \dots, n-1$. It follows from (3.7) that T satisfies

$$U_\epsilon \partial_b T_{ab} + \frac{2n}{n-2} \partial_b U_\epsilon T_{ab} = 0, \quad \text{in } B_\rho^+(0), \quad \text{for } a = 1, \dots, n. \quad (3.10)$$

In particular,

$$\frac{n-2}{4(n-1)} U_\epsilon \partial_a \partial_b T_{ab} + \partial_a (\partial_b U_\epsilon T_{ab}) = 0, \quad \text{in } B_\rho^+(0), \quad (3.11)$$

where we have used $U_\epsilon \partial_a \partial_b U_\epsilon - \frac{n-2}{n-2} \partial_a U_\epsilon \partial_b U_\epsilon = \frac{1}{n} (U_\epsilon \Delta U_\epsilon - \frac{n-2}{n-2} |dU_\epsilon|^2) \delta_{ab}$ in $\mathbb{R}_+^n$ for all $a, b = 1, \dots, n$.

Next we define the auxiliary function $\phi = \phi_{\epsilon, \rho, \mathcal{H}}$ by

$$\phi = \partial_a U_\epsilon V_a + \frac{n-2}{2n} U_\epsilon \partial_a V_a. \quad (3.12)$$

By a direct computation, we have

$$\begin{cases} \Delta \phi + n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi = \frac{n-2}{4(n-1)} U_\epsilon \partial_b \partial_a \mathcal{H}_{ab} + \partial_b (\partial_a U_\epsilon \mathcal{H}_{ab}) & \text{in } B_\rho^+(0) \\ \partial_n \phi = 0 & \text{on } \partial\mathbb{R}_+^n. \end{cases} \quad (3.13)$$

By the estimate (3.8), ϕ satisfies

$$|\phi(x)| \leq C \epsilon^{\frac{n-2}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| (\epsilon + |x|)^{|\alpha|+2-n} \quad (3.14)$$

and

$$\left| \Delta \phi(x) + n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi(x) \right| \leq C \epsilon^{\frac{n-2}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| (\epsilon + |x|)^{|\alpha|-n}, \quad (3.15)$$

for all $x \in \mathbb{R}_+^n$.

Observe that if $n = 3$ then $d = 0$, in which case $\mathcal{H} \equiv 0$ and $\phi \equiv 0$.

Convention. In the rest of Subsection 3.1 we will assume that $n \geq 4$.

We define algebraic Schouten tensor and algebraic Weyl tensor by

$$A_{ac} = \partial_c \partial_e \mathcal{H}_{ae} + \partial_a \partial_e \mathcal{H}_{ce} - \partial_e \partial_e \mathcal{H}_{ac} - \frac{1}{n-1} \partial_e \partial_f \mathcal{H}_{ef} \delta_{ac}$$

and

$$\begin{aligned} Z_{abcd} &= \partial_b \partial_d \mathcal{H}_{ac} - \partial_b \partial_c \mathcal{H}_{ad} + \partial_a \partial_c \mathcal{H}_{db} - \partial_a \partial_d \mathcal{H}_{bc} \\ &\quad + \frac{1}{n-2} (A_{ac} \delta_{bd} - A_{ad} \delta_{bc} + A_{bd} \delta_{ac} - A_{bc} \delta_{db}). \end{aligned}$$

We also set

$$\begin{aligned} Q_{ab,c} &= U_\epsilon \partial_c T_{ab} - \frac{2}{n-2} \partial_a U_\epsilon T_{bc} - \frac{2}{n-2} \partial_b U_\epsilon T_{ac} \\ &\quad + \frac{2}{n-2} \partial_d U_\epsilon T_{ad} \delta_{bc} + \frac{2}{n-2} \partial_d U_\epsilon T_{bd} \delta_{ac}. \end{aligned} \quad (3.16)$$

Lemma 3.3. *If the tensor $\mathcal{H}$ satisfies*

$$\begin{cases} Z_{abcd} = 0 & \text{in } \mathbb{R}_+^n \\ \partial_n \mathcal{H}_{ij} = 0 & \text{on } \partial \mathbb{R}_+^n, \end{cases}$$

then $\mathcal{H} = 0$ in $\mathbb{R}_+^n$.

Proof. Observe that the hypothesis $\partial_n \mathcal{H}_{ij} = 0$ on $\partial \mathbb{R}_+^n$ implies that $h_{ij,\alpha} = 0$ for $\alpha = (0, \dots, 0, 1)$. In this case, the expression (3.5) can be written as

$$\mathcal{H}_{ab}(x) = \sum_{|\alpha|=2}^d h_{ab,\alpha} x^\alpha.$$

Now the result is just in [10, Proposition 2.3]. $\square$

Proposition 3.4. *Set $U_r = B_{r/4}(0, \dots, 0, \frac{3r}{2}) \subset \mathbb{R}_+^n$. Then there exists $C = C(n) > 0$ such that*

$$\sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 r^{2|\alpha|-4+n} \leq C \int_{U_r} Z_{abcd} Z_{abcd} + C r^{-1} \int_{D_{\frac{5r}{3}}(0) \setminus D_{\frac{4r}{3}}(0)} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij},$$

for all $r > 0$.

Proof. If $r = 1$, observe that the square roots of both sides of the inequality are norms in $\mathcal{H}$, due to Lemma 3.3. The general case follows by scaling. $\square$

Lemma 3.5. *There exists $C = C(n) > 0$ such that*

$$\begin{aligned} \epsilon^{n-2} r^{6-2n} \int_{U_r} Z_{abcd} Z_{abcd} &\leq \frac{C}{\theta} \int_{B_{2r}^+(0) \setminus B_r^+(0)} Q_{ab,c} Q_{ab,c} \\ &\quad + \theta \epsilon^{n-2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 r^{2|\alpha|+2-n} \end{aligned}$$

for all $0 < \theta < 1$ and all $r \geq \epsilon$.

Proof. This follows from the third formula in the proof of in [10, Proposition 2.5], by means of Young's inequality. Observe that, in our calculations, we are using the range $1 \leq |\alpha| \leq d$ in the summation formulas, instead of the range $2 \leq |\alpha| \leq d$ used in [10]. $\square$

Lemma 3.6. *There exists $C = C(n) > 0$ such that*

$$\begin{aligned} \epsilon^{n-2} r^{5-2n} \int_{D_{\frac{5r}{3}}(0) \setminus D_{\frac{4r}{3}}(0)} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} &\leq C \theta \epsilon^{n-2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 r^{2|\alpha|+2-n} \\ &\quad + \frac{C}{\theta} \int_{B_{2r}^+(0) \setminus B_r^+(0)} Q_{ij,n} Q_{ij,n} \end{aligned}$$

for all $0 < \theta < 1$ and all $r \geq \epsilon$.

Proof. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative smooth function such that $\chi(t) = 1$ for $t \in [4/3, 5/3]$ and $\chi(t) = 0$ for $t \notin [1, 2]$. For $r > 0$ and $x \in \mathbb{R}_+^n$ we define $\chi_r(x) = \chi(|x|/r)$. Observe that $\partial_n S_{ij} = -\frac{1}{n-1} \partial_n S_{nn} \delta_{ij}$ on $\partial \mathbb{R}_+^n$. On the other hand, (3.10) gives $\partial_n S_{nn} = -\partial_n T_{nn} = 0$. Thus, $\partial_n S_{ij} = 0$ and $\partial_n \mathcal{H}_{ij} = \partial_n T_{ij} = U_\epsilon^{-1} Q_{ij,n}$ on $\partial \mathbb{R}_+^n$. Integration by parts gives

$$\begin{aligned} \int_{\partial \mathbb{R}_+^n} U_\epsilon^{\frac{2(n-1)}{n-2}} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r &= \int_{\partial \mathbb{R}_+^n} U_\epsilon^{\frac{2}{n-2}} Q_{ij,n} Q_{ij,n} \chi_r \\ &= - \int_{\mathbb{R}_+^n} \partial_n (U_\epsilon^{\frac{2}{n-2}} Q_{ij,n} Q_{ij,n} \chi_r) \\ &= - \int_{\mathbb{R}_+^n} \partial_n (U_\epsilon^{\frac{2}{n-2}} Q_{ij,n} \chi_r) Q_{ij,n} \\ &\quad - \int_{\mathbb{R}_+^n} U_\epsilon^{\frac{2}{n-2}} \partial_n Q_{ij,n} Q_{ij,n} \chi_r. \end{aligned} \tag{3.17}$$

Using Young's inequality, the result now follows from the inequalities

$$U_\epsilon^{\frac{2(n-1)}{n-2}} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r \geq C^{-1} \epsilon^{n-1} r^{2-2n} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r$$

and

$$|\partial_n (U_\epsilon^{\frac{2}{n-2}} Q_{ij,n} \chi_r)| + |U_\epsilon^{\frac{2}{n-2}} \partial_n Q_{ij,n} \chi_r| \leq C \epsilon^{\frac{n}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| r^{|\alpha|-2-n}. \quad \square$$

Proposition 3.7. *There exists $\lambda = \lambda(n) > 0$ such that*

$$\lambda \epsilon^{n-2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 \int_{B_\rho^+(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \leq \frac{1}{4} \int_{B_\rho^+(0)} Q_{ab,c} Q_{ab,c} dx$$

for all $\rho \geq 2\epsilon$.

Proof. This follows from Proposition 3.4, Lemma 3.5, and Lemma 3.6. $\square$

3.2. Type A test functions $(\bar{u}_A; (x_0, \epsilon))$

In this subsection we use the same test functions as in [10] but we need to do some changes when estimating their energy by $Q(S_+^n)$ because the boundary does not need to be umbilical in our case.

For $\rho \in (0, P_0/2]$, the Fermi coordinates centered at $x_0 \in \partial M$ define a smooth map $\psi_{x_0} : B_\rho^+(0) \subset \mathbb{R}_+^n \rightarrow M$. We will sometimes omit the symbols ψ_{x_0} in order to simplify our notation, identifying $\psi_{x_0}(x) \in M$ with $x \in B_\rho^+(0)$. In those coordinates, we have the properties $g_{ab}(0) = \delta_{ab}$ and $g_{nb}(x) = \delta_{nb}$, for any $x \in B_\rho^+(0)$ and $a, b = 1, \dots, n$. If we write $g = \exp(h)$, where $\exp$ denotes the matrix exponential, then the symmetric 2-tensor h satisfies the following properties:

$$\begin{cases} h_{ab}(0) = 0 & \text{for } a, b = 1, \dots, n \\ h_{an}(x) = 0 & \text{for } x \in B_\rho^+(0) \text{ } a = 1, \dots, n \\ \partial_k h_{ij}(0) = 0 & \text{for } i, j, k = 1, \dots, n-1 \\ \sum_{j=1}^{n-1} x_j h_{ij}(x) = 0 & \text{for } x \in D_\rho(0), i = 1, \dots, n-1. \end{cases}$$

The last two properties follow from the fact that Fermi coordinates are normal on the boundary.

According to [22, Proposition 3.1], for each $x_0 \in \partial M$ we can find a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$, with $f_{x_0}(x_0) = 1$, and Fermi coordinates centered at x_0 such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$, where $d = \lfloor \frac{n-2}{2} \rfloor$. In particular, if we write $g_{x_0} = \exp(h_{x_0})$, we have $\text{tr}(h_{x_0})(x) = O(|x|^{2d+2})$. Moreover, $H_{g_{x_0}}$, the trace of the second fundamental form of ∂M , satisfies

$$H_{g_{x_0}}(x) = -\frac{1}{2} g^{ij} \partial_n g_{ij}(x) = -\frac{1}{2} \partial_n (\log \det(g_{x_0}))(x) = O(|x|^{2d+1}). \quad (3.18)$$

Since M is compact, we can assume that $1/2 \leq f_{x_0} \leq 3/2$ for any $x_0 \in \partial M$, choosing P_0 smaller if necessary.

Notation. In order to simplify our notation, in the coordinates above, we will write g_{ab} and g^{ab} instead of $(g_{x_0})_{ab}$ and $(g_{x_0})^{ab}$ respectively, and h_{ab} instead of $(h_{x_0})_{ab}$.

In this subsection we denote by

$$\mathcal{H}_{ab}(x) = \sum_{1 \leq |\alpha| \leq d} h_{ab, \alpha} x^\alpha$$

the Taylor expansion of order d associated with the function $h_{ab}(x)$. Thus, we have $h_{ab}(x) = \mathcal{H}_{ab}(x) + O(|x|^{d+1})$. Observe that $\mathcal{H}$ is a symmetric trace-free 2-tensor on $\mathbb{R}_+^n$, which satisfies the properties (3.4) and has the form (3.5). Then we can use the function $\phi = \phi_{\epsilon, \rho, \mathcal{H}}$ (see formula (3.12)) and the results obtained in Subsection 3.1.

Recall the definitions of U_ϵ in (3.1), χ_ρ in (3.6), and $\bar{R}_\infty$ in (2.9). Define

$$\begin{aligned} \bar{U}_{(x_0, \epsilon)}(x) = & \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x)) (U_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))) \\ & + \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}} (1 - \chi_\rho(\psi_{x_0}^{-1}(x))) G_{x_0}(x), \end{aligned} \quad (3.19)$$

for $x \in M$. Here, G_{x_0} is the Green's function of the conformal Laplacian $L_{g_{x_0}} = \Delta_{g_{x_0}} - \frac{n-2}{4(n-1)} R_{g_{x_0}}$, with pole at $x_0 \in \partial M$, satisfying the boundary condition

$$\frac{\partial}{\partial \eta_{g_{x_0}}} G_{x_0} - \frac{n-2}{2(n-1)} H_{g_{x_0}} G_{x_0} = 0, \quad (3.20)$$

on $\partial M \setminus \{x_0\}$, and the normalization $\lim_{|y| \rightarrow 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1$. This function, obtained in Proposition B.2, satisfies

$$\begin{aligned} |G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| & \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| |y|^{|\alpha|+2-n} \\ & + \begin{cases} C|y|^{d+3-n} & \text{if } n \geq 5 \\ C(1 + |\log |y||) & \text{if } n = 3, 4 \end{cases} \end{aligned} \quad (3.21)$$

$$\left| \frac{\partial}{\partial y_b} (G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}) \right| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| |y|^{|\alpha|+1-n} + C|y|^{d+2-n},$$

for all $b = 1, \dots, n$.

We define the test function

$$\bar{u}_{A; (x_0, \epsilon)} = f_{x_0} \bar{U}_{(x_0, \epsilon)}. \quad (3.22)$$

Observe that this function depends also on the radius ρ above, which will be fixed later in Section 4. Such constant will also be referred to as ρ_A in order to avoid confusion with test functions of the other subsections.

Our main result in this subsection is the following estimate for the energy of $\bar{u}_{A; (x_0, \epsilon)}$:

Proposition 3.8. *Under the hypotheses of Theorem 1.9, there exists $P_1 = P_1(M, g_0) > 0$ such that*

$$\begin{aligned} & \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{u}_{A;(x_0, \epsilon)}|_{g_0}^2 + R_{g_0} \bar{u}_{A;(x_0, \epsilon)}^2 \right\} dv_{g_0}}{\left(\int_M \bar{u}_{A;(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right)^{\frac{n-2}{n}}} \\ &\leq Q(S_+^n) \end{aligned}$$

for all $x_0 \in \partial M$ and $0 < 2\epsilon < \rho_A < P_1$.

Let λ be the constant obtained in Proposition 3.7.

Proposition 3.9. *There exist $C, P_1 > 0$, depending only on (M, g_0) , such that*

$$\begin{aligned} & \int_{B_\rho^+(0)} \left\{ \frac{4(n-1)}{n-2} |d(U_\epsilon + \phi)|_{g_{x_0}}^2 + R_{g_{x_0}} (U_\epsilon + \phi)^2 \right\} dx \\ &+ \int_{D_\rho(0)} 2H_{g_{x_0}} (U_\epsilon + \phi)^2 d\sigma \\ &\leq 4n(n-1) \int_{B_\rho^+(0)} U_\epsilon^{\frac{4}{n-2}} (U_\epsilon^2 + \frac{n+2}{n-2} \phi^2) dx \\ &+ \int_{\partial^+ B_\rho^+(0)} \left\{ \frac{4(n-1)}{n-2} U_\epsilon \partial_a U_\epsilon + U_\epsilon^2 \partial_b h_{ab} - \partial_b U_\epsilon^2 h_{ab} \right\} \frac{x_a}{|x|} d\sigma_\rho \\ &- \frac{\lambda}{2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\rho^+(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\ &+ C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n} \end{aligned}$$

for all $0 < 2\epsilon \leq \rho \leq P_1$.

Proof. Following the steps in [10, Proposition 3.6] we obtain

$$\begin{aligned}
 & \int_{B_\rho^+(0)} \left\{ \frac{4(n-1)}{n-2} |d(U_\epsilon + \phi)|_{g_{x_0}}^2 + R_{g_{x_0}}(U_\epsilon + \phi)^2 \right\} dx \\
 & + \int_{D_\rho(0)} 2H_{g_{x_0}}(U_\epsilon + \phi)^2 d\sigma \\
 & \leq \int_{B_\rho^+(0)} \frac{4(n-1)}{n-2} |dU_\epsilon|^2 dx + \int_{B_\rho^+(0)} \frac{4(n-1)}{n-2} n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\
 & + \int_{\partial^+ B_\rho^+(0)} \left(U_\epsilon^2 \partial_b h_{ab} - \partial_b U_\epsilon^2 h_{ab} \right) \frac{x_a}{|x|} d\sigma_\rho - \frac{1}{4} \int_{B_\rho^+(0)} Q_{ab,c} Q_{ab,c} dx \\
 & + \frac{\lambda}{2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\rho^+(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\
 & + C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n}.
 \end{aligned}$$

The result follows by making use of Proposition 3.7 and

$$|dU_\epsilon|^2 = \partial_a(U_\epsilon \partial_a U_\epsilon) - U_\epsilon \Delta U_\epsilon = \partial_a(U_\epsilon \partial_a U_\epsilon) + n(n-2) U_\epsilon^{\frac{2n}{n-2}}. \quad \square$$

As in [10, p. 1006], we define the flux integral

$$\begin{aligned}
 \mathcal{I}(x_0, \rho) &= \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+(0)} (|x|^{2-n} \partial_a G_{x_0} - \partial_a |x|^{2-n} G_{x_0}) \frac{x_a}{|x|} d\sigma_\rho \\
 &\quad - \int_{\partial^+ B_\rho^+(0)} |x|^{2-2n} (|x|^2 \partial_b h_{ab} - 2n x_b h_{ab}) \frac{x_a}{|x|} d\sigma_\rho,
 \end{aligned} \tag{3.23}$$

for $\rho > 0$ sufficiently small.

Proposition 3.10. *There exists $P_1 = P_1(M, g_0) > 0$ such that*

$$\begin{aligned}
 & \int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\
 & \leq Q(S_+^n) \left\{ \int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right\}^{\frac{n-2}{n}} - \epsilon^{n-2} \mathcal{I}(x_0, \rho) \\
 & \quad - \frac{\lambda}{4} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\rho^+(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\
 & \quad + C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n} + C \epsilon^n \rho^{-n}
 \end{aligned}$$

for all $0 < 2\epsilon \leq \rho \leq P_1$.

Proof. As in [9, Proposition 15], we get

$$\begin{aligned}
 & 4n(n-1) \int_{B_\rho^+(0)} U_\epsilon^{\frac{4}{n-2}} \left(U_\epsilon^2 + \frac{n+2}{n-2} \phi^2 \right) dx \\
 & \leq Q(S_+^n) \left(\int_{B_\rho^+(0)} (U_\epsilon + \phi)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} + \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| \rho^{|\alpha|-n} \epsilon^n \quad (3.24) \\
 & \quad + C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}|^2 \epsilon^{n-1} \int_{B_\rho^+(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx
 \end{aligned}$$

for all $0 < 2\epsilon \leq \rho \leq P_1$ and P_1 sufficiently small. Now, with Proposition 3.9 at hand, our proof is analogous to the one in [10, Proposition 4.1]. $\square$

Corollary 3.11. *There exist P_1 , θ , $C_0 > 0$, depending only on (M, g_0) , such that*

$$\begin{aligned}
 & \int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0,\epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 d\sigma_{g_{x_0}} \\
 & \leq Q(S_+^n) \left\{ \int_M \bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right\}^{\frac{n-2}{n}} - \epsilon^{n-2} \mathcal{I}(x_0, \rho) \\
 & \quad - \theta \epsilon^{n-2} \int_{B_\rho^+(0)} |W_{g_0}(x)|^2 (\epsilon + |x|)^{6-2n} dx \\
 & \quad - \theta \epsilon^{n-2} \int_{D_\rho(0)} |\pi_{g_0}(x)|^2 (\epsilon + |x|)^{5-2n} d\sigma + C_0 \epsilon^{n-2} \rho^{2d+4-n} \\
 & \quad + C_0 \left(\frac{\epsilon}{\rho} \right)^{n-2} \frac{1}{|\log(\rho/\epsilon)|}
 \end{aligned}$$

for all $0 < 2\epsilon \leq \rho \leq P_1$. Here, we denote by W_{g_0} the Weyl tensor of (M, g_0) and by π_{g_0} the trace-free 2nd fundamental form of ∂M .

Proof. Similar to [1, Corollary 3.10]. $\square$

Recall that we denote by $\mathcal{Z}_{\partial M}$ the set of all points $x_0 \in \partial M$ such that

$$\limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)| = \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{1-d} |\pi_{g_0}(x)| = 0.$$

Proposition 3.12. *The functions $\mathcal{I}(x_0, \rho)$ converge uniformly to a continuous function $I : \mathcal{Z}_{\partial M} \rightarrow \mathbb{R}$ as $\rho \rightarrow 0$.*

Proof. As in [1, Proposition 3.11] we can prove that

$$\sup_{x_0 \in \mathcal{Z}_{\partial M}} |\mathcal{I}(x_0, \rho) - \mathcal{I}(x_0, \tilde{\rho})| \leq \begin{cases} C\rho^{2d+4-n} & \text{if } n \geq 5 \\ C\rho^{2d+4-n} |\log \rho| & \text{if } n = 3, 4, \end{cases}$$

for all $0 < \tilde{\rho} < \rho$. The result follows. $\square$

The following proposition, which is [1, Proposition 3.12]¹, relates $\mathcal{I}(x_0)$ with the mass defined by (1.2):

Proposition 3.13. *Let $x_0 \in \mathcal{Z}_{\partial M}$ and consider inverted coordinates $y = x/|x|^2$, where $x = (x_1, \dots, x_n)$ are Fermi coordinates centered at x_0 . If we define the metric $\bar{g} = G_{x_0}^{\frac{n-2}{2}} g_{x_0}$ on $M \setminus \{x_0\}$, then the following statements hold:*

- (i) *$(M \setminus \{x_0\}, \bar{g})$ is an asymptotically flat manifold with order $p > \frac{n-2}{2}$ (in the sense of Definition 1.4), and satisfies $R_{\bar{g}} \equiv 0$ and $H_{\bar{g}} \equiv 0$;*
- (ii) *We have*

$$\mathcal{I}(x_0) = \lim_{R \rightarrow \infty} \left\{ \int_{\partial^+ B_R^+(0)} \frac{y_a}{|y|} \frac{\partial \bar{g}}{\partial y_b} \left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b} \right) d\sigma_R - \int_{\partial^+ B_R^+(0)} \frac{y_a}{|y|} \frac{\partial \bar{g}}{\partial y_a} \left(\frac{\partial}{\partial y_b}, \frac{\partial}{\partial y_b} \right) d\sigma_R \right\}.$$

In particular, $\mathcal{I}(x_0)$ is the mass $m(\bar{g})$ of $(M \setminus \{x_0\}, \bar{g})$.

Proof of Proposition 3.8. Once we have proved Corollary 3.11, and Propositions 3.12 and 3.13, this proof follows the same lines as [1, Proposition 3.7]. $\square$

We now prove some further results for later use.

Proposition 3.14.² *For $x \in M$ and $\epsilon < \rho$,*

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + \bar{R}_{\infty} \bar{U}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \right| (x) \\ & \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{1}{2}} 1_{B_{2\rho}^+(0)}(x) \\ & \quad + C \left(\frac{\epsilon}{\epsilon^2 + d_{g_{x_0}}(x, x_0)^2} \right)^{\frac{n+2}{2}} 1_{M \setminus B_{\rho}^+(0)}(x) \\ & \quad + C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{1-n} |\log \rho|) 1_{B_{2\rho}^+(0) \setminus B_{\rho}^+(0)}(x). \end{aligned}$$

¹ In [1, Propositions 3.11 and 3.12] a $\log \rho$ must be included in the arguments for dimensions 3 and 4, when the Green function has $\log$ in its expansion; see (3.21).

² The $(\epsilon^2 + |x|^2)^{-\frac{1}{2}}$ term in this proposition is necessary only in dimension 3, when $d = 0$ and so $\mathcal{H} = 0$. On the other hand, the $|\log \rho|$ term is necessary only in dimensions 3 and 4, because of (3.21). The same terms are also necessary in the first inequality of [1, Proposition 3.13], but this does not affect any other results in that paper because weaker estimates similar to the ones obtained in Subsection 3.5 are also enough to [1].

Proof. Note that after scaling, we are assuming $\bar{R}_\infty = 4n(n-1)$. Then

$$\begin{aligned}
& \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - \frac{n-2}{4(n-1)} R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + \frac{n-2}{4(n-1)} \bar{R}_\infty \bar{U}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \\
&= (\Delta_{g_{x_0}} \chi_\rho)(U_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}) + 2 \langle d\chi_\rho, d(U_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}) \rangle_{g_{x_0}} \\
&\quad - (\Delta_{g_{x_0}} \chi_\rho) \epsilon^{\frac{n-2}{2}} (G_{x_0} - |x|^{2-n}) - 2\epsilon^{\frac{n-2}{2}} \langle d\chi_\rho, d(G_{x_0} - |x|^{2-n}) \rangle_{g_{x_0}} \\
&\quad + \chi_\rho \left(\Delta_{g_{x_0}} (U_\epsilon + \phi) - \frac{n-2}{4(n-1)} R_{g_{x_0}} (U_\epsilon + \phi) + n(n-2)(U_\epsilon + \phi)^{\frac{n+2}{n-2}} \right) \\
&\quad + n(n-2) \left(\left(\chi_\rho (U_\epsilon + \phi) + (1 - \chi_\rho) \epsilon^{\frac{n-2}{2}} G_{x_0} \right)^{\frac{n+2}{n-2}} - \chi_\rho (U_\epsilon + \phi)^{\frac{n+2}{n-2}} \right) \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where $I_i, i = 1, 2, 3, 4$, denote the corresponding row.

To estimate I_1 , notice that for $|x| \geq \rho > \epsilon$ we have

$$|(\epsilon^2 + |x|^2)^{\frac{2-n}{2}} - |x|^{2-n}| \leq C\epsilon^2 |x|^{-n} \quad (3.25)$$

and, equivalently, $|U_\epsilon - \epsilon^{\frac{n-2}{2}} |x|^{2-n}| \leq C\epsilon^{\frac{n+2}{2}} |x|^{-n}$. Then I_1 can be estimated as

$$|I_1| \leq C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{1-n}) 1_{B_{2\rho}^+(0) \setminus B_\rho^+(0)}.$$

Recall the properties (3.21) of G_{x_0} . Then $|I_2| \leq C\epsilon^{\frac{n-2}{2}} \rho^{1-n} |\log \rho| 1_{B_{2\rho}^+(0) \setminus B_\rho^+(0)}$.

In order to estimate I_3 , first observe that

$$\begin{aligned}
I_3 &= \chi_\rho \left((\Delta_{g_{x_0}} - \Delta) U_\epsilon - \partial_i (H_{ij} \partial_j U_\epsilon) - \frac{n-2}{4(n-1)} R_{g_{x_0}} U_\epsilon + \frac{n-2}{4(n-1)} \partial_i \partial_j H_{ij} U_\epsilon \right) \\
&\quad + \chi_\rho \left((\Delta_{g_{x_0}} - \Delta) \phi - \frac{n-2}{4(n-1)} R_{g_{x_0}} \phi \right) \\
&\quad + \chi_\rho \left(n(n-2)(U_\epsilon + \phi)^{\frac{n+2}{n-2}} - n(n-2) U_\epsilon^{\frac{n+2}{n-2}} - n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi \right),
\end{aligned}$$

where we have used (3.2) and (3.13). Using [1, inequality (3.20)],

$$|(\Delta_{g_{x_0}} - \Delta) U_\epsilon + \partial_i (H_{ij} \partial_j U_\epsilon)| + |R_{g_{x_0}} U_\epsilon - \partial_i \partial_j H_{ij} U_\epsilon| \leq C\epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{1-n},$$

$$|(\Delta_{g_{x_0}} - \Delta) \phi + \partial_i (H_{ij} \partial_j \phi)| + |R_{g_{x_0}} \phi - \partial_i \partial_j H_{ij} \phi| \leq C\epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{2-n}$$

and

$$\left| (U_\epsilon + \phi)^{\frac{n+2}{n-2}} - U_\epsilon^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} U_\epsilon^{\frac{4}{n-2}} \phi \right| \leq C U_\epsilon^{\frac{n+2}{n-2}} (\phi U_\epsilon^{-1})^2 \leq C\epsilon^{\frac{n+2}{2}} (\epsilon + |x|)^{-n}.$$

This leads to

$$|I_3| \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{1}{2}} 1_{B_{2\rho}^+(0)}.$$

Finally we consider I_4 , using the elementary inequality

$$|a^{\frac{n+2}{n-2}} - b^{\frac{n+2}{n-2}}| \leq C b^{\frac{4}{n-2}} |a - b| + C |a - b|^{\frac{n+2}{n-2}},$$

which holds for any $a, b > 0$, and where $C = C(n)$. Letting $a = \chi_\rho(U_\epsilon + \phi) + (1 - \chi_\rho)\epsilon^{\frac{n-2}{2}} G_{x_0}$ and $b = \chi_\rho^{\frac{n+2}{n-2}}(U_\epsilon + \phi)$, and applying the bound (3.21) for G_{x_0} , one gets the estimate

$$|I_4| \leq C \left(\frac{\epsilon}{\epsilon^2 + d_{g_{x_0}}(x, x_0)^2} \right)^{\frac{n+2}{2}} 1_{M \setminus B_\rho^+(0)}.$$

Combining all the estimates above, we get the conclusion. $\square$

Proposition 3.15. *For $x \in \partial M$,*

$$\left| \frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} - H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right| (x) \leq C \rho \left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2} \right)^{\frac{n-2}{2}} 1_{D_{2\rho}(0)}(x).$$

Proof. Observe that

$$\begin{aligned} & \frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} - \frac{n-2}{2(n-1)} H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \\ &= \chi_\rho \frac{\partial}{\partial \eta_{g_{x_0}}} (U_\epsilon + \phi) + \frac{n-2}{2(n-1)} \chi_\rho H_{g_{x_0}} (U_\epsilon + \phi) \\ &+ (1 - \chi_\rho) \epsilon^{\frac{n-2}{2}} \left(\frac{\partial}{\partial \eta_{g_{x_0}}} G_{x_0} - \frac{n-2}{2(n-1)} H_{g_{x_0}} G_{x_0} \right). \end{aligned}$$

Recall that we were using Fermi coordinates, thus $\eta_{g_{x_0}} = \partial_n$. The first and third terms are zero by the equations (3.2) and (3.13) while the middle one can be bounded as

$$|\chi_\rho H_{g_{x_0}} (U_\epsilon + \phi)| \leq C \rho \left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2} \right)^{\frac{n-2}{2}} 1_{D_{2\rho}(0)}.$$

$\square$

3.3. Type B test functions $(\tilde{u}_B; (x_0, \epsilon))$

In this case the test functions we use are essentially the same as in [9]. However, when trying to control their energy by $Y(S^n)$, due to the proximity to the boundary, the argument in that paper cannot be directly applied. We are able to overcome this difficulty by exploiting the sign of $\partial_n U_\epsilon(0)$ (see the definition in (3.1)). Since all the argument is local, we do not make use of the positive mass theorem in this subsection.

Fix $x_0 \in M_{2\delta_0} \setminus \partial M$ and let $\psi_{x_0} : \tilde{B}_{x_0, 2\rho} \subset \mathbb{R}^n \rightarrow M$ be normal coordinates centered at x_0 (see Definition 3.2) where $0 < \rho \leq P_0$. We will sometimes omit the symbols ψ_{x_0} in order to simplify our notation, identifying $\psi_{x_0}(x) \in M$ with $x \in \tilde{B}_{x_0, 2\rho}$. In those coordinates, we have the properties $g_{ab}(0) = \delta_{ab}$ and $\partial_c g_{ab}(0) = 0$, for $a, b, c = 1, \dots, n$. If we write $g = \exp(h)$, where $\exp$ denotes the matrix exponential, then the symmetric 2-tensor h satisfies the following properties:

$$\begin{cases} h_{ab}(0) = 0 & \text{for } a, b = 1, \dots, n \\ \partial_c h_{ab}(0) = 0 & \text{for } a, b, c = 1, \dots, n \\ \sum_{b=1}^n x_b h_{ab}(x) = 0 & \text{for } x \in \tilde{B}_{x_0, \rho}, a = 1, \dots, n. \end{cases}$$

According to [20], we can find a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$, with $f_{x_0}(x_0) = 1$, such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$ in normal coordinates centered at x_0 , again written $\psi_{x_0} : \tilde{B}_{x_0, 2\rho} \rightarrow M$ for simplicity. We can suppose that $1/2 \leq f_{x_0} \leq 3/2$.

Notation. In order to simplify notation, in the coordinates above, we will write g_{ab} and g^{ab} instead of $(g_{x_0})_{ab}$ and $(g_{x_0})^{ab}$ respectively, h_{ab} instead of $(h_{x_0})_{ab}$, and η^a instead of $(\eta_{g_{x_0}})^a$. We denote by $\nu = \nu_{x_0}$ the unit normal vector to $\tilde{D}_{x_0, \rho}$ with respect to the Euclidean metric δ_{ab} , pointing the same way as η_{g_0} and $\eta_{g_{x_0}}$, and write $\nu = \nu^a \partial_a$ and $\eta = \eta^a \partial_a$.

Set $\delta = d_{g_{x_0}}(x_0, \partial M)$. If $\tilde{x}_0 \in \partial M$ is chosen such that $d_{g_{x_0}}(x_0, \tilde{x}_0) = \delta$ then we can assume that ψ_{x_0} takes $(-\delta, 0, \dots, 0) \in \mathbb{R}^n$ to $\tilde{x}_0$ and thus both $\eta_{g_{x_0}}$ and ν_{x_0} coincide at $\tilde{x}_0$ with the coordinate vector ∂_n . So, there exists $C_0 = C_0(M, g_0) > 2$ such that

$$|\eta^a(x) - \delta_{an}| \leq C_0 |\bar{x}|, \quad \text{and} \quad (3.26)$$

$$|\nu^a(x) - \delta_{an}| \leq C_0 |\bar{x}|, \quad \text{for all } x \in \tilde{D}_{x_0, 2\rho}, \quad (3.27)$$

where $x = (x_1, \dots, x_n) = (\bar{x}, x_n) \in \mathbb{R}^n$. We will also assume that $\tilde{D}_{x_0, 2\rho}$ is the graph of a smooth function $\gamma = \gamma_{x_0}$ so that

$$\tilde{D}_{x_0, 2\rho} = \{x = (\bar{x}, \gamma(\bar{x})) \mid |\bar{x}| < 2\rho\}.$$

We can write $\gamma(\bar{x}) = -\delta + O(|\bar{x}|^2)$ and choose C_0 larger if necessary such that

$$|\gamma(\bar{x}) + \delta| \leq C_0 |\bar{x}|^2, \quad \text{for all } x \in \tilde{D}_{x_0, 2\rho}. \quad (3.28)$$

See Figure 1.

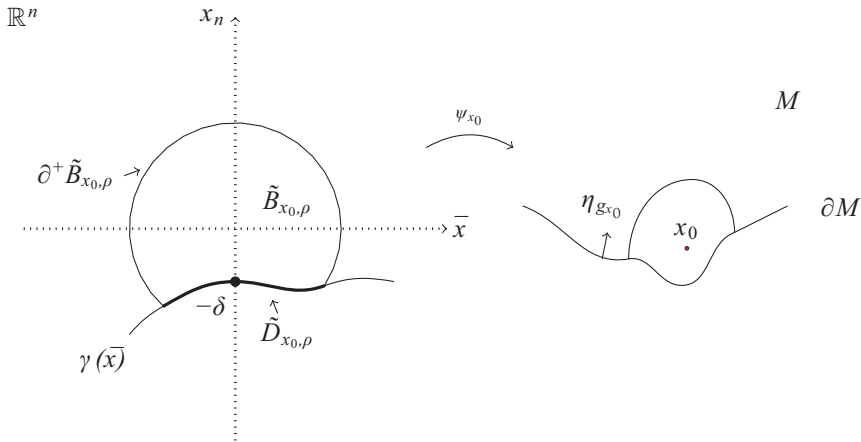


Figure 3.1. Some notation.

In this subsection, we denote by

$$\mathcal{H}_{ab}(x) = \sum_{2 \leq |\alpha| \leq d} h_{ab,\alpha} x^\alpha$$

the Taylor expansion of order $d = \lfloor \frac{n-2}{2} \rfloor$ associated with the function $h_{ab}(x)$. Thus, $h_{ab}(x) = \mathcal{H}_{ab}(x) + O(|x|^{d+1})$. We define ϕ , S , T and $Q_{ab,c}$ as in Subsection 3.1 (see (3.12), (3.9) and (3.16)), except for the fact that, as in [9], the whole construction is done in $\mathbb{R}^n$ instead of $\mathbb{R}^n_+$. Then the first equation of (3.13) and the estimates (3.14) and (3.15) also hold, with $2 \leq |\alpha| \leq d$ replacing $1 \leq |\alpha| \leq d$.

Lemma 3.16. *There exists $\lambda = \lambda(n) > 0$ such that*

$$\lambda \epsilon^{n-2} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \int_{B_\rho(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \leq \frac{1}{4} \int_{B_\rho(0)} Q_{ab,c} Q_{ab,c}$$

for all $\rho \geq 2\epsilon$.

Proof. See [9, Corollary 10]. □

Recall the definitions of U_ϵ in (3.1), χ_ρ in (3.6), and $\bar{R}_\infty$ in (2.9). Set

$$\begin{aligned} \bar{U}_{(x_0,\epsilon)}(x) &= \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x)) (U_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))) \\ &\quad + \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}} (1 - \chi_\rho(\psi_{x_0}^{-1}(x))) G_{x_0}(x), \end{aligned}$$

for $x \in M$. Here, G_{x_0} is the Green function of the conformal Laplacian $L_{g_{x_0}}$ with pole at $x_0 \in M \setminus \partial M$, satisfying the boundary condition (3.20) and the normalization $\lim_{|y| \rightarrow 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1/2$. This function is obtained in Proposition B.4 and satisfies, for some $C = C(M, g_0)$,

$$|G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| \leq \begin{cases} C|y|^{3-n} + C\delta|y|^{1-n} & \text{if } n \geq 4 \\ C(1 + |\log |y||) + C\delta|y|^{1-n} & \text{if } n = 3, \end{cases} \quad (3.29)$$

$$\left| \frac{\partial}{\partial y_b} (G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}) \right| \leq C|y|^{2-n} + C\delta|y|^{-n},$$

for all $b = 1, \dots, n$ and $\psi_{x_0}(y) \in M_{\tilde{\delta}}$ for some small $\tilde{\delta} = \tilde{\delta}(M, g_0)$.

Define the test function

$$\bar{u}_{B;(x_0, \epsilon)} = f_{x_0} \bar{U}_{(x_0, \epsilon)}. \quad (3.30)$$

Observe that this function also depends on the radius ρ above, which will be fixed later in Section 4. Such a constant will also be referred to as ρ_B in order to avoid confusion with test functions of the other subsections.

The main result of this subsection is the following:

Proposition 3.17. *Under the hypothesis of Theorem 1.9, there exist positive P_2 and C_B , depending only on (M, g_0) , such that for any $\rho_B \leq P_2$ one can choose $\delta_0 < C_B \rho_B^2$ satisfying*

$$\begin{aligned} & \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{u}_{B;(x_0, \epsilon)}|_{g_0}^2 + R_{g_0} \bar{u}_{B;(x_0, \epsilon)}^2 \right\} dv_{g_0}}{\left(\int_M \bar{u}_{B;(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right)^{\frac{n-2}{n}}} \\ &\leq Y(S^n) \end{aligned}$$

for all $x_0 \in M_{2\delta_0} \setminus \partial M$ and $0 < \epsilon < C_B^{-1} d_{g_0}(x_0, \partial M)$.

We will prove several lemmas before proceeding to the proof of Proposition 3.17.

Lemma 3.18. *If $|\bar{x}| \leq 1/(2C_0)$, then for $\epsilon > 0$ and $0 < \delta < 1$ we have*

$$\frac{1}{2C_0} (\epsilon^2 + |\bar{x}|^2 + \delta^2) < \epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 < 2(\epsilon^2 + |\bar{x}|^2 + \delta^2). \quad (3.31)$$

Proof. First assume $\delta \geq C_0|\bar{x}|^2$. Since $|\gamma(\bar{x})| \geq \delta - C_0|\bar{x}|^2 \geq 0$ by (3.28), Cauchy's inequality implies

$$\gamma(\bar{x})^2 \geq \left(\delta - C_0|\bar{x}|^2\right)^2 \geq \delta^2 - 2C_0\delta|\bar{x}|^2 \geq \frac{1}{2}\delta^2 - 2C_0^2|\bar{x}|^4.$$

So,

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \geq \epsilon^2 + (1 - 2C_0^2|\bar{x}|^2)|\bar{x}|^2 + \frac{1}{2}\delta^2,$$

and our assumption $|\bar{x}|^2 \leq 1/(4C_0^2)$ gives

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \geq \epsilon^2 + \frac{1}{2}|\bar{x}|^2 + \frac{1}{2}\delta^2 > \frac{1}{2}(\epsilon^2 + |\bar{x}|^2 + \delta^2).$$

If $\delta < C_0|\bar{x}|^2$ we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 + \epsilon^2 > \frac{\delta^2}{2C_0} + \frac{|\bar{x}|^2}{2} + \epsilon^2 > \frac{1}{2C_0}(\delta^2 + |\bar{x}|^2 + \epsilon^2).$$

so the left-hand side of (3.31) is proved.

As for the right-hand side, notice that

$$\gamma(\bar{x})^2 \leq (\delta + C_0|\bar{x}|^2)^2 \leq 2\delta^2 + 2C_0^2|\bar{x}|^4.$$

Consequently,

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \leq \epsilon^2 + (1 + 2C_0^2|\bar{x}|^2)|\bar{x}|^2 + 2\delta^2 < 2(\epsilon^2 + |\bar{x}|^2 + \delta^2),$$

because our assumption on $|\bar{x}|$ implies $1 + 2C_0^2|\bar{x}|^2 \leq 2$. □

Lemma 3.19. *If $0 < \rho < 1/C_0$ and $0 < \delta \leq \rho/4$ then*

$$\sqrt{|\bar{x}|^2 + \gamma(\bar{x})^2} < \rho, \quad \text{for all } |\bar{x}| \leq \rho/2.$$

Proof. From our assumption it is easy to get $\delta/\rho + C_0\rho/4 < 1/2$. Since

$$|\gamma(\bar{x})| \leq \delta + C_0|\bar{x}|^2 \leq \delta + C_0\rho^2/4,$$

we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 \leq \frac{\rho^2}{4} + \left(\delta + \frac{C_0\rho^2}{4}\right)^2 < \frac{\rho^2}{4} + \left(\frac{\rho}{2}\right)^2 = \frac{\rho^2}{2}. \quad \square$$

Lemma 3.20. *If $0 < \rho \leq 1/C_0$ and $0 < \delta < 1$ then*

$$\sqrt{|\bar{x}|^2 + \gamma(\bar{x})^2} > \delta/\sqrt{C_0}, \quad \text{for all } |\bar{x}| < \rho.$$

Proof. First assume $\delta \geq C_0|\bar{x}|^2$. Then $|\gamma(\bar{x})| \geq \delta - C_0|\bar{x}|^2 \geq 0$, which yields

$$\begin{aligned}\gamma(\bar{x})^2 &\geq (\delta - C_0|\bar{x}|^2)^2 \\ &\geq \delta^2 - 2\delta C_0|\bar{x}|^2 + C_0^2|\bar{x}|^4 = \frac{\delta^2}{2} - C_0^2|\bar{x}|^4.\end{aligned}$$

Therefore, by the assumption $|\bar{x}| < \rho \leq 1/C_0$, we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 \geq (1 - C_0^2|\bar{x}|^2)|\bar{x}|^2 + \delta^2/2 \geq \delta^2/2 > \delta^2/C_0,$$

because $C_0 > 2$.

If $\delta < C_0|\bar{x}|^2$, since $0 < \delta < 1$, we have $\delta^2 < \delta < C_0|\bar{x}|^2$. Obviously

$$|\bar{x}|^2 + \gamma(\bar{x})^2 > \delta^2/C_0,$$

proving the result. □

Lemma 3.21. *There exists $C = C(n)$ such that*

$$\int_{\{\bar{x} \in \mathbb{R}^{n-1} \mid |\bar{x}| \leq \rho\}} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n} d\bar{x} \leq C\rho\delta^{2-n}, \quad \text{for } 0 < \delta \leq \rho.$$

Proof. Just observe that

$$\begin{aligned}\int_{|\bar{x}| \leq \rho} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n} d\bar{x} &\leq \int_{|\bar{x}| \leq \rho} (|\bar{x}|^2 + \delta^2)^{2-n} d\bar{x} \\ &\leq \sqrt{2}\rho \int_{\mathbb{R}^{n-1}} (|\bar{x}|^2 + \delta^2)^{\frac{3-2n}{2}} d\bar{x} \\ &= \sqrt{2}\rho\delta^{2-n} \int_{\mathbb{R}^{n-1}} (|\bar{y}|^2 + 1)^{\frac{3-2n}{2}} d\bar{y}. \quad \square\end{aligned}$$

Lemma 3.22. *There exist $\tilde{c}$, K , $P_2 > 0$, depending only on (M, g_0) , such that*

$$\frac{4(n-1)}{n-2} \int_{\tilde{D}_{x_0, \rho}} U_\epsilon \partial_v U_\epsilon d\sigma \geq \tilde{c}\epsilon^{n-2}\delta^{2-n}$$

when $0 < \epsilon < \delta < K\rho$ and $\rho < P_2$.

Proof. Observe that $U_\epsilon \partial_a U_\epsilon = -(n-2)\epsilon^{n-2}(\epsilon^2 + |x|^2)^{1-n}x_a$ and, on $\tilde{D}_{x_0, \rho}$,

$$U_\epsilon \partial_v U_\epsilon = U_\epsilon v^a \partial_a U_\epsilon = U_\epsilon \partial_n U_\epsilon + U_\epsilon (v^a - \delta_{an}) \partial_a U_\epsilon.$$

Using (3.27) and Lemma 3.18, we have

$$\begin{aligned}|U_\epsilon (v^a - \delta_{an}) \partial_a U_\epsilon|(x) &\leq (n-2)C\epsilon^{n-2}(\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{2-n} \\ &\leq (2C_0)^{n-2}(n-2)C\epsilon^{n-2}(\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n}\end{aligned}$$

when $x = (\bar{x}, \gamma(\bar{x})) \in \tilde{D}_{x_0, \rho}$ with $|\bar{x}| \leq (2C_0)^{-1}$. Hence if $\rho \leq (2C_0)^{-1}$ and $0 < \delta \leq \rho$, then

$$\int_{\tilde{D}_{x_0, \rho}} U_\epsilon \partial_v U_\epsilon d\sigma \geq \int_{\tilde{D}_{x_0, \rho}} U_\epsilon \partial_n U_\epsilon d\sigma - C\rho \left(\frac{\epsilon}{\delta}\right)^{n-2},$$

where we used Lemma 3.21.

In order to estimate from below the right-hand side of this last inequality, we see that

$$\begin{aligned} U_\epsilon \partial_n U_\epsilon(x) &= -(n-2)\epsilon^{n-2}(\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{1-n} \gamma(\bar{x}) \\ &\geq (n-2)\epsilon^{n-2}(\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{1-n} (\delta - C_0 |\bar{x}|^2) \\ &\geq (n-2)\epsilon^{n-2} \delta (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{1-n} \\ &\quad - (n-2)C_0 \epsilon^{n-2} (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{2-n} \\ &\geq (n-2)2^{1-n} \epsilon^{n-2} \delta (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{1-n} - C \epsilon^{n-2} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n} \end{aligned}$$

for $x = (\bar{x}, \gamma(\bar{x})) \in \tilde{D}_{x_0, \rho}$ with $|\bar{x}| \leq (2C_0)^{-1}$, where we used Lemma 3.18 in the last step.

Assume $0 < \rho < (2C_0)^{-1}$ and $0 < \delta \leq \rho/4$. According to Lemma 3.19,

$$\{(\bar{x}, \gamma(\bar{x})) \mid |\bar{x}| \leq \rho/2\} \subset \tilde{D}_{x_0, \rho}.$$

Then

$$\begin{aligned} \int_{\tilde{D}_{x_0, \rho}} U_\epsilon \partial_n U_\epsilon d\sigma &\geq (n-2)2^{1-n} \epsilon^{n-2} \delta \int_{|\bar{x}| \leq \rho/2} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{1-n} d\bar{x} \\ &\quad - C \epsilon^{n-2} \int_{|\bar{x}| < \rho} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n} d\bar{x} \\ &= I - II. \end{aligned}$$

Notice that

$$\begin{aligned} \delta \int_{|\bar{x}| \leq \rho/2} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{1-n} d\bar{x} &= \delta^{2-n} \int_{|\bar{y}| \leq \rho/2\delta} \left(\left(\frac{\epsilon}{\delta}\right)^2 + |\bar{y}|^2 + 1 \right)^{1-n} d\bar{y} \\ &\geq 2^{1-n} \delta^{2-n} \int_{|\bar{y}| \leq \rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y} \end{aligned}$$

for $0 < \epsilon < \delta$, because $(\epsilon/\delta)^2 + |\bar{y}|^2 + 1 < 2(|\bar{y}|^2 + 1)$.

Set $\alpha(n) = \int_{\mathbb{R}^{n-1}} (|\bar{y}|^2 + 1)^{1-n} d\bar{y}$ and observe that

$$\int_{|\bar{y}| \leq \rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y} = \alpha(n) - \int_{|\bar{y}| > \rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y} \geq \alpha(n) - C \left(\frac{\delta}{\rho}\right)^{n-1}.$$

Hence,

$$I \geq (n-2)2^{2-2n}\alpha(n)\left(\frac{\epsilon}{\delta}\right)^{n-2} - C\left(\frac{\delta}{\rho}\right)^{n-1}\left(\frac{\epsilon}{\delta}\right)^{n-2}.$$

On the other hand, $II \leq C\rho(\epsilon/\delta)^{n-2}$, by Lemma 3.21.

Putting things together, we obtain

$$\int_{\tilde{D}_{x_0,\rho}} U_\epsilon \partial_\nu U_\epsilon d\sigma \geq (n-2)2^{2-2n}(\alpha(n) - C(\delta/\rho)^{n-1} - C\rho)(\epsilon/\delta)^{n-2},$$

from which the result follows. $\square$

Proposition 3.23. *There exists $P_2 = P_2(M, g_0) > 0$ such that if $0 < \delta \leq \rho \leq P_2$*

$$\begin{aligned} & \int_{\tilde{B}_{x_0,\rho}} \left\{ \frac{4(n-1)}{n-2} |d(U_\epsilon + \phi)|^2 + R_{g_{x_0}}(U_\epsilon + \phi)^2 \right\} dx \\ & \leq \frac{4(n-1)}{n-2} \int_{\tilde{B}_{x_0,\rho}} |dU_\epsilon|^2 dx + \int_{\tilde{B}_{x_0,\rho}} \frac{4(n-1)}{n-2} n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\ & \quad + \frac{\lambda}{2} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\ & \quad - \frac{1}{4} \int_{\tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx + C\rho \left(\frac{\epsilon}{\delta}\right)^{n-2} + C\rho \left(\frac{\epsilon}{\rho}\right)^{n-2} \end{aligned}$$

for all $\epsilon \in (0, \rho/2]$. Here, λ is the constant obtained in Lemma 3.16.

Proof. As in [10, Proposition 3.6], we can choose $0 < P_2 < 1$ such that

$$\begin{aligned} & \int_{\tilde{B}_{x_0,\rho}} \left\{ \frac{4(n-1)}{n-2} |d(U_\epsilon + \phi)|^2 + R_{g_{x_0}}(U_\epsilon + \phi)^2 \right\} dx \\ & \leq \frac{4(n-1)}{n-2} \int_{\tilde{B}_{x_0,\rho}} |dU_\epsilon|^2 dx + \int_{\tilde{B}_{x_0,\rho}} \frac{4(n-1)}{n-2} n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\ & \quad + \int_{\partial^+ \tilde{B}_{x_0,\rho}} \left(U_\epsilon^2 \partial_b h_{ab} - \partial_b U_\epsilon^2 h_{ab} \right) \frac{x_a}{|x|} d\sigma_\rho - \frac{1}{4} \int_{\tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx \\ & \quad + \frac{\lambda}{2} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\ & \quad + C \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n} + \int_{\tilde{D}_{x_0,\rho}} \Psi d\sigma \end{aligned}$$

holds for all $0 < 2\epsilon \leq \rho \leq P_2$, where

$$\begin{aligned} \Psi = & -\frac{8(n-1)}{n-2} \left(\partial_a U_\epsilon \phi + \frac{(n-2)^2}{2} U_\epsilon^{\frac{2n}{n-2}} V_a \right) v^a \\ & - U_\epsilon^2 \partial_b h_{ab} v^a + 2U_\epsilon (\partial_b U_\epsilon) h_{ab} v^a + U_\epsilon^2 \mathcal{H}_{ab} \partial_c \mathcal{H}_{ab} v^b - v^a \xi_a \end{aligned}$$

comes from integration by parts. Here, ξ_a is a 1-tensor controlled by

$$|\xi_a(x)| \leq C \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^{n-2} (\epsilon + |x|)^{3+2|\alpha|-2n}.$$

It is easy to estimate the following term on $\tilde{D}_{x_0,\rho}$

$$\begin{aligned} |U_\epsilon^{\frac{2n}{n-2}} V_a|(x) & \leq C \epsilon^n (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{1-n} \\ & \leq C \epsilon^{n-2} (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{2-n}, \end{aligned} \quad (3.32)$$

and all the other terms in Ψ can also be estimated by the right-hand side of (3.32).

Choosing P_2 possibly smaller, from Lemmas 3.18 and 3.21 we get

$$\int_{\tilde{D}_{x_0,\rho}} \Psi d\sigma \leq C \left(\frac{\epsilon}{\delta} \right)^{n-2} \rho, \quad (3.33)$$

for $0 < \delta \leq \rho$, from which the result follows. $\square$

Proposition 3.24. *There exist $P_2, C > 0$, depending only on (M, g_0) , such that*

$$\begin{aligned} & \int_{\tilde{B}_{x_0,\rho}} \left\{ \frac{4(n-1)}{n-2} |d(U_\epsilon + \phi)|^2 + R_{g_{x_0}}(U_\epsilon + \phi)^2 \right\} dx \\ & \leq Y(S^n) \left(\int_{\tilde{B}_{x_0,\rho}} (U_\epsilon + \phi)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} - (\tilde{c} - C\rho - C(\delta/\rho)^{n-2}) \left(\frac{\epsilon}{\delta} \right)^{n-2} \\ & \quad - \frac{\lambda}{4} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \end{aligned}$$

for all $0 < \rho \leq P_2$ and $0 < \epsilon < \delta < K\rho$, where K and $\tilde{c}$ are the constants obtained in Lemma 3.22.

Proof. This result is a consequence of Proposition 3.23 and Lemma 3.16. Observe that

$$\begin{aligned}
 & \frac{4(n-1)}{n-2} \int_{\tilde{B}_{x_0, \rho}} |dU_\epsilon|^2 dx + \int_{\tilde{B}_{x_0, \rho}} \frac{4(n-1)}{n-2} n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\
 &= \int_{\tilde{B}_{x_0, \rho}} \frac{4(n-1)}{n-2} \left(n(n-2) U_\epsilon^{\frac{2n}{n-2}} + n(n+2) U_\epsilon^{\frac{4}{n-2}} \phi^2 \right) dx \\
 & \quad - \int_{\tilde{D}_{x_0, \rho}} \frac{4(n-1)}{n-2} U_\epsilon \partial_v U_\epsilon d\sigma + \int_{\partial^+ \tilde{B}_{x_0, \rho}} \frac{4(n-1)}{n-2} U_\epsilon \partial_a U_\epsilon \frac{x_a}{|x|} d\sigma \quad (3.34) \\
 &\leq \int_{\tilde{B}_{x_0, \rho}} 4n(n-1) U_\epsilon^{\frac{4}{n-2}} (U_\epsilon^2 + \frac{n+2}{n-2} \phi^2) dx \\
 & \quad - \int_{\tilde{D}_{x_0, \rho}} \frac{4(n-1)}{n-2} U_\epsilon \partial_v U_\epsilon d\sigma + C \left(\frac{\epsilon}{\rho} \right)^{n-2}.
 \end{aligned}$$

We shall handle the first two terms of the right-hand side of (3.34) separately. As in [9, Proposition 14], we have

$$\begin{aligned}
 & \left(U_\epsilon^2 + \frac{n+2}{n-2} \phi^2 \right)^{\frac{n}{n-2}} - (U_\epsilon + \phi)^{\frac{2n}{n-2}} + \frac{2n}{n-2} U_\epsilon^{\frac{n+2}{n-2}} \phi \\
 &\leq C \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^n (\epsilon + |x|)^{2|\alpha|+2-2n}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\tilde{B}_{x_0, \rho}} \frac{2n}{n-2} U_\epsilon^{\frac{n+2}{n-2}} \phi dx &\geq \int_{\tilde{B}_{x_0, \rho}} \partial_a (U_\epsilon^{\frac{2n}{n-2}} V_a) dx \\
 &= \int_{\partial^+ \tilde{B}_{x_0, \rho}} U_\epsilon^{\frac{2n}{n-2}} V_a \frac{x_a}{|x|} d\sigma - \int_{\tilde{D}_{x_0, \rho}} U_\epsilon^{\frac{2n}{n-2}} V_a v^a d\sigma \\
 &\geq -C\rho^{1-n} \epsilon^n - C\rho \left(\frac{\epsilon}{\delta} \right)^{n-2}.
 \end{aligned}$$

Here, in the last step we estimated the integral on $\tilde{D}_{x_0, \rho}$ by (3.32) and Lemmas 3.18 and 3.21. So,

$$\begin{aligned}
 & \int_{\tilde{B}_{x_0, \rho}} 4n(n-1) U_\epsilon^{\frac{4}{n-2}} (U_\epsilon^2 + \frac{n+2}{n-2} \phi^2) dx \\
 &\leq Y(S^n) \left(\int_{\tilde{B}_{x_0, \rho}} (U_\epsilon^2 + \frac{n+2}{n-2} \phi^2)^{\frac{n}{n-2}} dx \right)^{\frac{n-2}{n}} \\
 &\leq Y(S^n) \left(\int_{\tilde{B}_{x_0, \rho}} (U_\epsilon + \phi)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} + C\rho \left(\frac{\epsilon}{\rho} \right)^n + C\rho \left(\frac{\epsilon}{\delta} \right)^{n-2} \quad (3.35) \\
 &+ C \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^n \int_{\tilde{B}_{x_0, \rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx.
 \end{aligned}$$

Recall that Lemma 3.22 says

$$-\int_{\tilde{D}_{x_0,\rho}} \frac{4(n-1)}{n-2} U_\epsilon \partial_\nu U_\epsilon d\sigma \leq -\tilde{c} \left(\frac{\epsilon}{\delta}\right)^{n-2} \quad (3.36)$$

if $0 < \epsilon < \delta < K\rho$ and $0 < \rho < P_2$, for P_2 small enough.

Now it follows from Lemma 3.16 that

$$\lambda \epsilon^{n-2} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \int_{\tilde{B}_{x_0,\rho}(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \leq \frac{1}{4} \int_{B_\rho(0)} Q_{ab,c} Q_{ab,c} dx.$$

We claim that we can choose $P_2 > 0$ possibly smaller such that

$$\int_{B_\rho(0) \setminus \tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx \leq C\rho^2 \left(\frac{\epsilon}{\delta}\right)^{n-2}$$

for all $\rho < P_2$. In fact, from Lemma 3.20 we can choose P_2 small such that

$$B_\rho(0) \setminus \tilde{B}_{x_0,\rho} \subset B_\rho(0) \setminus B_{\delta/\sqrt{C_0}}(0)$$

for any $\rho < P_2$. Then using $Q_{ab,c} Q_{ab,c} \leq C\epsilon^{n-2}(\epsilon + |x|)^{4-2n}$ we get

$$\begin{aligned} \int_{B_\rho(0) \setminus \tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx &\leq C\epsilon^{n-2} \int_{B_\rho(0) \setminus \tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{4-2n} dx \\ &\leq C\epsilon^{n-2} \rho^2 \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{C_0}}} (\epsilon + |x|)^{2-2n} dx \leq C\epsilon^{n-2} \rho^2 \delta^{2-n}. \end{aligned}$$

In particular,

$$\begin{aligned} &\lambda \epsilon^{n-2} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\ &\leq \frac{1}{4} \int_{\tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx + C\rho^2 \left(\frac{\epsilon}{\delta}\right)^{n-2}. \end{aligned} \quad (3.37)$$

Now the result follows from Proposition 3.23 and estimates (3.34), (3.35), (3.36) and (3.37). $\square$

Proposition 3.25. *There exist P_2 and K such that*

$$\begin{aligned} &\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0,\epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 d\sigma_{g_{x_0}} \\ &\leq Y(S^n) \left(\int_M \bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right)^{\frac{n-2}{n}} - \frac{\lambda}{4} \sum_{a,b=1}^n \sum_{|\alpha|=2}^d |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\ &\quad - \frac{\tilde{c}}{2} \left(\frac{\epsilon}{\delta}\right)^{n-2}. \end{aligned}$$

for all $0 < \epsilon < \delta < K\rho$ and $0 < \rho < P_2$.

Proof. We have

$$\begin{aligned} & \int_{M \setminus \tilde{B}_{x_0, \rho}} \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right\} dv_{g_{x_0}} \\ & + \int_{\partial M \setminus \tilde{D}_{x_0, \rho}} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\ & \leq C \left(\frac{\epsilon}{\rho} \right)^{n-2}. \end{aligned}$$

As in the proof of Proposition 3.23,

$$\int_{\tilde{D}_{x_0, \rho}} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \leq C\rho \left(\frac{\epsilon}{\delta} \right)^{n-2}.$$

The result now follows from Proposition 3.24 and the fact that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$. $\square$

Proof of Proposition 3.17. Let P_2 and K be as in Proposition 3.25. Choose P_2 maybe smaller such that $P_2 < K$. Given $\rho_B \leq P_2$ choose $K' \leq \rho_B$ and $\delta'_0 \in (0, K'\rho_B)$. Observe that, in particular, one has $\delta'_0 < \rho_B^2$ and $\delta'_0 < K\rho_B$. By Proposition 3.25, the inequality we want to prove holds for all $0 < \epsilon < \delta < \delta'_0$ and $0 < \rho = \rho_B \leq P_2$, where $\delta = d_{g_0}(x_0, \partial M)$.

Now choose $C_B = C_B(M, g_0)$ such that $C_B^{-1}\delta \leq d_{g_0}(x_0, \partial M) \leq C_B\delta$, and take any $\delta_0 < C_B\delta'_0$. Then, because $\delta'_0 < \rho_B^2$, we have

$$\delta_0 < C_B\rho_B^2.$$

For any $\epsilon < C_B^{-1}d_{g_0}(x_0, \partial M)$ we have $\epsilon < C_B^{-1}d_{g_0}(x_0, \partial M) < \delta < \delta'_0$ and the inequality in Proposition 3.17 holds. $\square$

We finally prove some results for later use.

Proposition 3.26. *For $x \in M$, $\epsilon < \rho$ and $\delta \leq C\rho^2$,*

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + \bar{R}_\infty \bar{U}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \right| (x) \\ & \leq C\rho^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{B}_{x_0, \rho}}(x) + C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n+2}{2}} 1_{M \setminus \tilde{B}_{x_0, \rho}}(x) \\ & + C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{1-n} |\log \rho|) 1_{\tilde{B}_{x_0, 2\rho} \setminus \tilde{B}_{x_0, \rho}}(x). \end{aligned}$$

Proof. The proof goes like that of Proposition 3.14 with I_1, I_2, I_3, I_4 being the same. Observing that we are using normal coordinates, we have

$$|I_3| \leq C\rho^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{B}_{x_0, 2\rho}}.$$

Using (3.29) we obtain

$$|I_2| \leq C\epsilon^{\frac{n-2}{2}} \rho^{1-n} |\log \rho| 1_{\tilde{B}_{x_0, 2\rho} \setminus \tilde{B}_{x_0, \rho}} + C\epsilon^{\frac{n-2}{2}} \delta \rho^{-1-n} 1_{\tilde{B}_{x_0, 2\rho} \setminus \tilde{B}_{x_0, \rho}},$$

the $|\log \rho|$ being necessary only in dimension $n = 3$.

With the same estimate for I_1 and I_4 as in Proposition 3.14, we get the result. $\square$

Proposition 3.27. *For $x \in \partial M$, $\epsilon < \rho$ and $\delta \leq C\rho^2$,*

$$\begin{aligned} & \left| \frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} - H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right| (x) \\ & \leq C \frac{\delta}{\epsilon} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n}{2}} 1_{\tilde{D}_{x_0, 2\rho}}(x) + C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{D}_{x_0, 2\rho}}(x) \\ & \quad + C(\epsilon^{\frac{n+2}{2}} \rho^{-1-n} + \epsilon^{\frac{n-2}{2}} \rho^{2-n} |\log \rho|) 1_{\tilde{D}_{x_0, 2\rho} \setminus \tilde{D}_{x_0, \rho}}(x). \end{aligned}$$

Proof. Observe that, on ∂M ,

$$\begin{aligned} & \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \eta_{g_{x_0}}} - \frac{n-2}{2(n-1)} H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \\ & = \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} (U_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}) + \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} \epsilon^{\frac{n-2}{2}} (|x|^{2-n} - G_{x_0}) \\ & \quad + \chi_\rho \frac{\partial}{\partial \eta_{g_{x_0}}} (U_\epsilon + \phi) - \frac{n-2}{2(n-1)} \chi_\rho H_{g_{x_0}} (U_\epsilon + \phi) \\ & \quad + (1 - \chi_\rho) \epsilon^{\frac{n-2}{2}} \left(\frac{\partial G_{x_0}}{\partial \eta_{g_{x_0}}} - \frac{n-2}{2(n-1)} H_{g_{x_0}} G_{x_0} \right), \end{aligned}$$

where the last term is zero by the definition of G_{x_0} . Set

$$\begin{aligned} J_1 &= \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} (U_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}), & J_2 &= \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} \epsilon^{\frac{n-2}{2}} (|x|^{2-n} - G_{x_0}), \\ J_3 &= \chi_\rho \frac{\partial U_\epsilon}{\partial \eta_{g_{x_0}}}, & J_4 &= \chi_\rho \left(\frac{\partial \phi}{\partial \eta_{g_{x_0}}} - \frac{n-2}{2(n-1)} H_{g_{x_0}} (U_\epsilon + \phi) \right). \end{aligned}$$

Recall (3.25) to bound

$$|J_1| \leq \left| \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} \right| (|U_\epsilon - \epsilon^{\frac{n-2}{2}} |x|^{2-n}| + |\phi|) \leq C(\epsilon^{\frac{n+2}{2}} \rho^{-1-n} + \epsilon^{\frac{n-2}{2}} \rho^{3-n}) 1_{\tilde{D}_{x_0, 2\rho} \setminus \tilde{D}_{x_0, \rho}}.$$

For J_2 , we use the properties (3.29) of the Green function and the hypothesis $\delta \leq C\rho^2$ to obtain

$$|J_2| \leq \epsilon^{\frac{n-2}{2}} \left| \frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}} \right| (|x|^{2-n} - G_{x_0}) \leq C\epsilon^{\frac{n-2}{2}} \rho^{2-n} |\log \rho| 1_{\tilde{D}_{x_0, 2\rho} \setminus \tilde{D}_{x_0, \rho}}.$$

In order to estimate J_3 , let us calculate $\partial U_\epsilon / \partial \eta_{g_{x_0}}$. Suppose $x = (\bar{x}, \gamma(\bar{x})) \in \tilde{D}_{x_0, \rho}$, then

$$\begin{aligned} \partial U_\epsilon / \partial \eta_{g_{x_0}}(x) &= -(n-2)\epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{n}{2}} x_a \eta^a(x) \\ &= -(n-2)\epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{n}{2}} (\gamma(\bar{x}) + (\eta^a(x) - \delta_{an})x_a). \end{aligned} \quad (3.38)$$

Recall the properties (3.28) and (3.26) of γ and $\eta_{g_{x_0}}$. So,

$$\begin{aligned} |\partial U_\epsilon / \partial \eta_{g_{x_0}}|(x) &\leq C\epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{n}{2}} (\delta + C|\bar{x}|^2) \\ &\leq C\frac{\delta}{\epsilon} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n}{2}} + C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} \end{aligned}$$

for $x \in \tilde{D}_{x_0, \rho}$. Consequently,

$$|J_3| \leq C\frac{\delta}{\epsilon} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n}{2}} 1_{\tilde{D}_{x_0, 2\rho}} + C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{D}_{x_0, 2\rho}}.$$

Easily we can get

$$|J_4| \leq C\chi_\rho \left(\left| \frac{\partial \phi}{\partial \eta_{g_{x_0}}} \right| + U_\epsilon + |\phi| \right) \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{D}_{x_0, 2\rho}}.$$

Combining all the results, we get the conclusion. $\square$

Proposition 3.28. *For $x \in \partial M$, $\epsilon < \rho$ and $\delta \leq C\rho^2$,*

$$\begin{aligned} &\left(\frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} - H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right)(x) \\ &\geq -C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{\tilde{D}_{x_0, 2\rho}}(x) - C(\epsilon^{\frac{n+2}{2}} \rho^{-1-n} \\ &\quad + \epsilon^{\frac{n-2}{2}} \rho^{2-n} |\log \rho|) 1_{\tilde{D}_{x_0, 2\rho} \setminus \tilde{D}_{x_0, \rho}}(x). \end{aligned}$$

Proof. By (3.38) we have

$$\begin{aligned}\chi_\rho \partial U_\epsilon / \partial \eta_{g_{x_0}} &\geq \chi_\rho (n-2)(\epsilon^2 + |x|^2)^{-\frac{n}{2}} (\delta - C|\bar{x}|^2) \\ &\geq -C\epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{\frac{2-n}{2}} 1_{\tilde{D}_{x_0, 2\rho}}.\end{aligned}$$

Now the result follows as in Proposition 3.27. $\square$

3.4. Type C test functions ($\bar{u}_{C; (x_0, \epsilon)}$)

Our test functions in this case are the ones in [9], which are controlled by $Y(S^n)$ the same way as in that paper.

Recall that we assume that the background metric g_0 on M satisfies $H_{g_0} \equiv 0$ on ∂M . Fix $x_0 \in M \setminus M_{\delta_0}$ and let $\psi_{x_0} : B_{2\rho}(0) \subset \mathbb{R}^n \rightarrow B_{2\rho}(x_0) \subset M$ be normal coordinates centered at x_0 , where ρ is small such that $0 < \rho \leq \delta_0/4$. As in Subsection 3.3, we choose a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$ such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$ in normal coordinates centered at x_0 , still denoted by ψ_{x_0} . We assume $f_{x_0} \equiv 1$ in $M \setminus B_{2\rho}(x_0)$, which implies $H_{g_{x_0}} \equiv 0$ on ∂M .

Define ϕ as in Subsection 3.3 and set

$$\begin{aligned}\bar{U}_{(x_0, \epsilon)}(x) &= \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x)) (U_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))) \\ &\quad + \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}} (1 - \chi_\rho(\psi_{x_0}^{-1}(x))) G_{x_0}(x)\end{aligned}\tag{3.39}$$

for $x \in M$. Here, G_{x_0} is the Green's function of the conformal Laplacian $L_{g_{x_0}} = \Delta_{g_{x_0}} - \frac{n-2}{4(n-1)} R_{g_{x_0}}$, with pole at $x_0 \in M \setminus M_{\delta_0}$, boundary condition (3.20) and the normalization $\lim_{|y| \rightarrow 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1$. This function, obtained in Proposition B.2, satisfies

$$\begin{aligned}|G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| &\leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| |y|^{|\alpha|+2-n} \\ &\quad + \begin{cases} C|y|^{d+3-n} & \text{if } n \geq 5 \\ C(1 + |\log |y||) & \text{if } n = 3, 4, \end{cases}\end{aligned}\tag{3.40}$$

$$\left| \frac{\partial}{\partial y_b} (G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}) \right| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ij,\alpha}| |y|^{|\alpha|+1-n} + C|y|^{d+2-n},$$

for some $C = C(M, g_0, \delta_0)$ for all $b = 1, \dots, n$ and $x_0 \in M \setminus M_{\delta_0}$.

We define the test function

$$\bar{u}_{C; (x_0, \epsilon)} = f_{x_0} \bar{U}_{(x_0, \epsilon)}.\tag{3.41}$$

Observe that this function also depends on the radius ρ above, which will be fixed later in Section 4. Such constant will also be referred to as ρ_C in order to avoid confusion with test functions of the other subsections.

For later use we observe that $\frac{\partial}{\partial \eta_{g_0}} \bar{U}_{(x_0, \epsilon)} = B_{g_0} \bar{u}_{C; (x_0, \epsilon)} = B_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} = 0$ on ∂M .

Our main result in this subsection is the following:

Proposition 3.29. *Under the hypothesis of Theorem 1.9, there exists $P_3 = P_3(M, g_0, \delta_0)$ such that*

$$\begin{aligned} & \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{u}_{C; (x_0, \epsilon)}|_{g_0}^2 + R_{g_0} \bar{u}_{C; (x_0, \epsilon)}^2 \right\} dv_{g_0}}{\left(\int_M \bar{u}_{C; (x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right)^{\frac{n-2}{n}}} \\ &\leq Y(S^n) \end{aligned}$$

for all $x_0 \in M \setminus M_{\delta_0}$ and $0 < 2\epsilon < \rho_C < P_3$.

Proof. Choose P_3 small such that for any $x_0 \in M \setminus M_{\delta_0}$ we have $d_{g_{x_0}}(x_0, \partial M) > 2P_3$. Choosing P_3 smaller if necessary (also depending on δ_0 because of the above estimates for G_{x_0}) the result is Corollary 3 and Proposition 19 in [9] with some obvious modifications, by making use of Theorem 1.6. $\square$

For later use we state the following result, which is proved as Proposition 3.26:

Proposition 3.30. *We can choose $P_3 = P_3(M, g_0, \delta_0)$ maybe smaller such that there is $C = C(M, g_0)$ satisfying*

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + \bar{R}_\infty \bar{U}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \right| \\ &\leq C\rho^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} 1_{B_{2\rho}(0)} + C \left(\frac{\epsilon}{\epsilon^2 + d_{g_{x_0}}(x, x_0)^2} \right)^{\frac{n+2}{2}} 1_{M \setminus B_\rho(0)} \\ &\quad + C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{3/4-n} |\log \rho|) 1_{B_{2\rho}(0) \setminus B_\rho(0)} \end{aligned}$$

for all $x_0 \in M \setminus M_{\delta_0}$ and $\epsilon < \rho \leq P_3$.

Proof. As in Proposition 3.26, the proof follows the lines of Proposition 3.14, but the term I_2 is estimated by $|I_2| \leq C\epsilon^{\frac{n-2}{2}} \rho^{1-n} |\log \rho|$, where C depends on δ_0 . Choose $P_3 < C^{-4}$. $\square$

3.5. Further estimates

The results of this subsection are consequences of what was proved in Subsections 3.2, 3.3 and 3.4.

In this subsection, unless otherwise stated, if $x_0 \in \partial M$, $x_0 \in M_{\delta_0} \setminus \partial M$ or $x_0 \in M \setminus M_{2\delta_0}$, $\bar{u}_{(x_0, \epsilon)}$ will stand for $\bar{u}_{A; (x_0, \epsilon)}$, $\bar{u}_{B; (x_0, \epsilon)}$ or $\bar{u}_{C; (x_0, \epsilon)}$, respectively. If $x_0 \in M_{2\delta_0} \setminus M_{\delta_0}$, $\bar{u}_{(x_0, \epsilon)}$ will stand for $\bar{u}_{B; (x_0, \epsilon)}$ and $\bar{u}_{C; (x_0, \epsilon)}$, the results below holding for either. By the "radius" ρ of $\bar{u}_{(x_0, \epsilon)}$, we mean ρ_A , ρ_B or ρ_C , if $\bar{u}_{(x_0, \epsilon)} = \bar{u}_{A; (x_0, \epsilon)}$, $\bar{u}_{(x_0, \epsilon)} = \bar{u}_{B; (x_0, \epsilon)}$ or $\bar{u}_{(x_0, \epsilon)} = \bar{u}_{C; (x_0, \epsilon)}$, respectively.

We observe that whenever $\bar{u}_{(x_0, \epsilon)} = \bar{u}_{B; (x_0, \epsilon)}$ we have $d_{g_0}(x_0, \partial M) \leq \delta_0 \leq C\rho^2$, according to Proposition 3.17, because $x_0 \in M_{\delta_0} \setminus \partial M$ in this case. Hence, we can make use of Propositions 3.26, 3.27 and 3.28.

Corollary 3.31. *There exists $C = C(M, g_0)$ such that, for $\epsilon < \rho$,*

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_0, \epsilon)} - R_{g_0} \bar{u}_{(x_0, \epsilon)} + \bar{R}_\infty \bar{u}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \right| \\ & \leq C\rho^{-1/2} \left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x, x_0)^2} \right)^{\frac{n-2}{2}} (\epsilon^2 + d_{g_0}(x, x_0)^2)^{-\frac{1}{2}} 1_{B_{4\rho}(x_0)} \\ & \quad + C \left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x, x_0)^2} \right)^{\frac{n+2}{2}} 1_{M \setminus B_{\rho/2}(x_0)}. \end{aligned}$$

Proof. It is a consequence of Propositions 3.14, 3.26 and 3.30. $\square$

Corollary 3.32. ³ *There exists $C = C(M, g_0)$ such that, if ρ is the radius of $\bar{u}_{(x_2, \epsilon_2)}$ and $\epsilon_1 \leq \epsilon_2 < \rho$, we have*

$$\begin{aligned} & \int_M \bar{u}_{(x_1, \epsilon_1)} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_2, \epsilon_2)} - R_{g_0} \bar{u}_{(x_2, \epsilon_2)} + \bar{R}_\infty \bar{u}_{(x_2, \epsilon_2)}^{\frac{n+2}{n-2}} \right| dv_{g_0} \\ & \leq C \left(\rho^{1/2} + \frac{\epsilon_2^2}{\rho^2} \right) \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}}. \end{aligned}$$

Proof. As in [8, Lemma B.5] we get

$$\begin{aligned} & \int_{\{d_{g_0}(y, x_2) \geq \rho/2\}} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n+2}{2}} dv_{g_0} \\ & \leq C \frac{\epsilon_2^2}{\rho^2} \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}}. \end{aligned} \tag{3.42}$$

³ For types A and B test functions in dimensions $n \geq 5$, the coefficient $\rho^{1/2}$ in this inequality can be improved to ρ . Indeed, ρ was worsened to $\rho^{1/2}$ due to the $\log \rho$ terms in Propositions 3.14 and 3.26, which are necessary only for $n = 3$ or 4, as observed in the footnote in Proposition 3.14.

We claim that

$$\begin{aligned} & \int_{\{d_{g_0}(y, x_2) \leq 4\rho\}} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2, y)^2)^{-\frac{1}{2}} dv_{g_0} \\ & \leq C\rho \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}}. \end{aligned} \quad (3.43)$$

Set

$$A = \{2d_{g_0}(x_1, y) \leq \epsilon_2 + d_{12}\} \cap \{d_{g_0}(y, x_2) \leq 4\rho\}$$

and

$$B = \{2d_{g_0}(x_1, y) \geq \epsilon_2 + d_{12}\} \cap \{d_{g_0}(y, x_2) \leq 4\rho\}$$

where $d_{12} = d_{g_0}(x_1, x_2)$. Observe that on A we have

$$\epsilon_2 + d_{g_0}(y, x_2) \geq \epsilon_2 + d_{12} - d_{g_0}(y, x_1) \geq \frac{1}{2}(\epsilon_2 + d_{12}) \geq d_{g_0}(y, x_1)$$

$$\text{and} \quad d_{g_0}(y, x_1) \leq \frac{1}{2}(\epsilon_2 + d_{12}) \leq \epsilon_2 + d_{g_0}(y, x_2) \leq 5\rho.$$

Then

$$\begin{aligned} & \int_A \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2, y)^2)^{-\frac{1}{2}} dv_{g_0} \\ & \leq C \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{12}^2} \right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y, x_1) \leq 5\rho\}} (\epsilon_1^2 + d_{g_0}(x_1, y)^2)^{\frac{2-n}{2}} d_{g_0}(x_1, y)^{-1} dv_{g_0} \quad (3.44) \\ & \leq C \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{12}^2} \right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y, x_1) \leq 5\rho\}} d_{g_0}(x_1, y)^{1-n} dv_{g_0} \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_B \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2, y)^2)^{-\frac{1}{2}} dv_{g_0} \\ & \leq C \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{12}^2} \right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y, x_2) \leq 4\rho\}} d_{g_0}(x_2, y)^{1-n} dv_{g_0}. \end{aligned} \quad (3.45)$$

The estimate (3.43) follows from (3.44) and (3.45) observing that the integrals on the right-hand sides of those inequalities are bounded by $C\rho$.

The result now follows from (3.42), (3.43) and Corollary 3.31. $\square$

Corollary 3.33. ⁴ *There exists $C = C(M, g_0)$ such that, if ρ is the radius of $\bar{u}_{(x_2, \epsilon_2)}$ and $\epsilon_1 \leq \epsilon_2 < \rho$,*

$$\int_{\partial M} \bar{u}_{(x_1, \epsilon_1)} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_2, \epsilon_2)} d\sigma_{g_0} \geq -C \left(\rho^{1/2} + \frac{\epsilon_2}{\rho} \right) \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}}.$$

Proof. Observe that the above integral vanishes when $\bar{u}_{(x_2, \epsilon_2)}$ is a type C test function. For type B test functions we obtain

$$\begin{aligned} & \frac{\partial}{\partial \eta_{g_{x_2}}} \bar{U}_{(x_2, \epsilon_2)} - \frac{n-2}{2(n-1)} H_{g_{x_2}} \bar{U}_{(x_2, \epsilon_2)} \\ & \geq -C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} \rho^{-1/2} 1_{\tilde{D}_{x_2, 2\rho}} - C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n}{2}} 1_{\tilde{D}_{x_2, 2\rho} \setminus \tilde{D}_{x_2, \rho}} \end{aligned}$$

from Proposition 3.28. Then, using (2.3) and (3.30), we estimate

$$\begin{aligned} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_2, \epsilon_2)} & \geq -C \rho^{-1/2} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n-2}{2}} 1_{\{d_{g_0}(y, x_2) \leq 4\rho\} \cap \partial M} \\ & \quad - C \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n}{2}} 1_{\{d_{g_0}(y, x_2) \geq \rho/2\} \cap \partial M}. \end{aligned}$$

The same (actually a better) estimate as above can be obtained for type A test functions by means of Proposition 3.15.

As in [8, p.274-275] we can prove

$$\begin{aligned} & \int_{\{d_{g_0}(y, x_2) \leq 4\rho\} \cap \partial M} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n-2}{2}} d\sigma_{g_0} \\ & \leq C \rho \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}} \end{aligned}$$

and

$$\begin{aligned} & \int_{\{d_{g_0}(y, x_2) \geq \rho/2\} \cap \partial M} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1, y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2, y)^2} \right)^{\frac{n}{2}} d\sigma_{g_0} \\ & \leq C \frac{\epsilon_2}{\rho} \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1, x_2)^2} \right)^{\frac{n-2}{2}}. \end{aligned}$$

The result now follows. $\square$

⁴ Similarly to the footnote in Corollary 3.32, for types A and B test functions the coefficient $\rho^{1/2}$ can be improved to ρ if $n \geq 5$.

Corollary 3.34. *For $\epsilon < \rho$ we have*

$$\begin{aligned} & \left(\int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_0, \epsilon)} - R_{g_0} \bar{u}_{(x_0, \epsilon)} + \bar{R}_\infty \bar{u}_{(x_0, \epsilon)}^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dv_{g_0} \right)^{\frac{n+2}{2n}} \\ & \leq C \left(\frac{\epsilon}{\rho} \right)^{\frac{n+2}{2}} + C \begin{cases} \epsilon \rho^{-1/2} & n \geq 5 \\ \epsilon \rho^{-1/2} |\log(\rho/\epsilon)| & n = 4 \\ \epsilon^{1/2} & n = 3. \end{cases} \end{aligned}$$

Proof. The result follows easily from Corollary 3.31. $\square$

Corollary 3.35. *If $\bar{u}_{(x_0, \epsilon)} = \bar{u}_{B; (x_0, \epsilon)}$ we have*

$$\begin{aligned} & \left(\int_{\partial M} \left| \frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_0, \epsilon)} - H_{g_0} \bar{u}_{(x_0, \epsilon)} \right|^{\frac{2(n-1)}{n}} d\sigma_{g_0} \right)^{\frac{n}{2(n-1)}} \\ & \leq \begin{cases} C \left(\frac{\epsilon}{\delta} \right)^{\frac{n-2}{2}} |\log \rho| + \frac{\epsilon}{\rho} & n \geq 5 \\ C \left(\frac{\epsilon}{\delta} \right) |\log \rho| + \frac{\epsilon}{\rho} |\log(\rho/\epsilon)| & n = 4 \\ C \left(\frac{\epsilon}{\delta} \right)^{1/2} |\log \rho| + C \left(\frac{\epsilon}{\rho} \right)^{1/2} & n = 3, \end{cases} \end{aligned}$$

for $\epsilon < \rho$, where $\delta = d_{g_0}(x_0, \partial M)$.

Proof. From Proposition 3.27, on ∂M we have

$$\begin{aligned} & \left| \frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_0, \epsilon)} - H_{g_0} \bar{u}_{(x_0, \epsilon)} \right| \\ & \leq C \frac{\delta}{\epsilon} \left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x, x_0)^2} \right)^{\frac{n}{2}} 1_{\{d_{g_0}(x, x_0) \leq 4\rho\}} \\ & \quad + C \rho^{-1} \left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x, x_0)^2} \right)^{\frac{n-2}{2}} 1_{\{d_{g_0}(x, x_0) \leq 4\rho\}}. \end{aligned}$$

Using $\delta \leq C\rho^2$, which in particular implies $\delta \leq C\rho$, the first term on the right-hand side above is estimated by $C(\delta/\epsilon)^{(n-2)/2}(\epsilon + d_{g_0}(x, x_0))^{-n/2} 1_{\{d_{g_0}(x, x_0) \leq 4\rho\}}$, and the result follows easily. $\square$

4. Blow-up analysis

In this section, we carry out the blow-up analysis for sequences of solutions to the equations (2.4) that will be necessary for the proof of Theorem 1.9. Although the

analysis goes along the lines of [8, Sections 4, 5 and 6], here we have to consider the possibility of both interior and boundary blow-up points, thus differing from the situation in [1, Section 4]. As we will see in Proposition 4.2 below, type A test functions are used to approximate solutions near boundary blow-up points. As for interior blow-up points, we make use of type B test functions if those points accumulate on the boundary, and type C ones otherwise.

Remark 4.1. Before proceeding to the blow-up analysis, we observe that one can choose ρ_A , ρ_B and ρ_C in Propositions 3.8, 3.17 and 3.29 in such a way that the inequalities of those propositions hold the three at the same time. To that end, choose δ_0 according to a small ρ_B in Proposition 3.17 and then ρ_C according to δ_0 in Proposition 3.29. Moreover, observe that given $C = C(M, g_0)$ one can always assume $\rho_A, \rho_B, \rho_C \leq C$. This last remark will be used in the proofs of Propositions 4.10 and 4.22 below.

Let $u(t)$, $t \geq 0$, be the solution of (2.4) obtained in Section 2, and let $\{t_v\}_{v=1}^\infty$ be any sequence satisfying $\lim_{v \rightarrow \infty} t_v = \infty$. We set $u_v = u(t_v)$ and $g_v = g(t_v) = u_v^{\frac{4}{n-2}} g_0$. Then

$$\int_M u_v^{\frac{2n}{n-2}} dv_{g_0} = \int_M dv_{g_v} = 1, \quad \text{for all } v.$$

It follows from Corollary 2.3 that

$$\int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u_v - R_{g_0} u_v + \bar{R}_\infty u_v^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dv_{g_0} = \int_M |R_{g_v} - \bar{R}_\infty|^{\frac{2n}{n+2}} dv_{g_v} \rightarrow 0$$

as $v \rightarrow \infty$.

The next proposition is an application of the decomposition result in [24], which plays the same role here as [30] did in [8, Proposition 4.1].

Proposition 4.2. *After passing to a subsequence, there exist an integer $m \geq 0$, a smooth function $u_\infty \geq 0$, and a sequence of m -tuplets $\{(x_{k,v}^*, \epsilon_{k,v}^*)_{1 \leq k \leq m}\}_{v=1}^\infty$, such that:*

(i) *The function u_∞ satisfies*

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta_{g_0} u_\infty - R_{g_0} u_\infty + \bar{R}_\infty u_\infty^{\frac{n+2}{n-2}} = 0, & \text{in } M \\ \partial u_\infty / \partial \eta_{g_0} = 0, & \text{on } \partial M; \end{cases}$$

(ii) *For all $i \neq j$,*

$$\lim_{v \rightarrow \infty} \left\{ \frac{\epsilon_{i,v}^*}{\epsilon_{j,v}^*} + \frac{\epsilon_{j,v}^*}{\epsilon_{i,v}^*} + \frac{d_{g_0}(x_{i,v}^*, x_{j,v}^*)^2}{\epsilon_{i,v}^* \epsilon_{j,v}^*} \right\} = \infty;$$

- (iii) *There are integers m_1, m_2 , with $0 \leq m_1 \leq m_2 \leq m$, such that $x_{k,v}^* \in \partial M$ for $1 \leq k \leq m_1$, $x_{k,v}^* \in M_{3\delta_0/2} \setminus \partial M$ for $m_1 + 1 \leq k \leq m_2$, $x_{k,v}^* \in M \setminus M_{3\delta_0/2}$ for $m_2 + 1 \leq k \leq m$, and*

$$\lim_{v \rightarrow \infty} d_{g_0}(x_{k,v}^*, \partial M) / \epsilon_{k,v}^* = \infty \quad \text{if } k \geq m_1 + 1;$$

- (iv) *If*

$$\bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)} = \begin{cases} \bar{u}_A; (x_{k,v}^*, \epsilon_{k,v}^*) & \text{if } k \leq m_1 \\ \bar{u}_B; (x_{k,v}^*, \epsilon_{k,v}^*) & \text{if } m_1 + 1 \leq k \leq m_2 \\ \bar{u}_C; (x_{k,v}^*, \epsilon_{k,v}^*) & \text{if } k \geq m_2 + 1, \end{cases} \quad (4.1)$$

(see equations (3.22), (3.30) and (3.41)) then

$$\lim_{v \rightarrow \infty} \|u_v - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)}\|_{H^1(M)} = 0.$$

Proof. By modifying the arguments in [24, Section 3] to the case of Riemannian manifolds, we can prove the existence of u_∞ and $\bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)}$ satisfying (i) and (iv) except for, instead of using equations (4.1), the $\bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)}$ are defined by

$$\bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)}(x) = \left(\frac{4n(n-1)}{\bar{R}_\infty} \right)^{\frac{n-2}{4}} (\epsilon_{k,v}^*)^{-\frac{n-2}{2}} \chi_\rho(\psi_{x_{k,v}^*}^{-1}(x)) u((\epsilon_{k,v}^*)^{-1} \psi_{x_{k,v}^*}^{-1}(x)).$$

Here, $\psi_{x_{k,v}^*}$ are coordinates centered at $x_{k,v}^*$ and u satisfies

$$\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n \quad (4.2)$$

if $\lim_{v \rightarrow \infty} d_{g_0}(x_{k,v}^*, \partial M) / \epsilon_{k,v}^* = \infty$, and

$$\begin{cases} \Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0 & \text{in } \{y = (y_1, \dots, y_n) \mid y_n \geq t\} \\ \frac{\partial}{\partial y_n} u = 0 & \text{on } \{y = (y_1, \dots, y_{n-1}, t)\}, \end{cases} \quad (4.3)$$

for some $t \in \mathbb{R}$ if $d_{g_0}(x_{k,v}^*, \partial M) / \epsilon_{k,v}^*$ is bounded.

Rearrange the indices and choose m_1 such that $k \geq m_1 + 1$ should (4.2) holds and $k \leq m_1$ should (4.3) holds.

As in [14, Lemma 3.3], we can prove that $u \geq 0$ and also that (ii) holds. The classification results in [11, 21] (regularity was established in [12]) imply that $u(y) = U_\epsilon(y - z)$ (see (3.1)), for some $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ (with $z_n = t$ if $k \leq m_1$).

The points $x_{k,v}^*$ are now redefined as $\psi_{x_{k,v}^*}(z)$.⁵ This establishes (iii).

For each pair $(x_{k,v}^*, \epsilon_{k,v}^*)$, one can check that the difference between each function obtained above and the corresponding one defined by (4.1) converges to zero in $H^1(M)$. This proves (iv). $\square$

Proposition 4.3. *If $u_\infty(x) = 0$ for some $x \in M$, then $u_\infty \equiv 0$.*

Proof. This is just a consequence of the maximum principle. $\square$

Define the functionals

$$E(u) = \frac{\frac{4(n-1)}{n-2} \int_M |du|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} u^2 dv_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}}$$

and

$$F(u) = \frac{\frac{4(n-1)}{n-2} \int_M |du|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} u^2 dv_{g_0}}{\int_M u^{\frac{2n}{n-2}} dv_{g_0}}.$$

Observe that $\bar{R}_\infty = F(u_\infty)$. Hence,

$$1 = \lim_{v \rightarrow \infty} \int_M u_v^{\frac{2n}{n-2}} dv_{g_0} = \lim_{v \rightarrow \infty} \left\{ \int_M u_\infty^{\frac{2n}{n-2}} dv_{g_0} + \sum_{k=1}^m \int_M \bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)}^{\frac{2n}{n-2}} dv_{g_0} \right\}.$$

The right-hand side of this equation is $(E(u_\infty)/\bar{R}_\infty)^{\frac{n}{2}} + m_1(Q(S_+^n)/\bar{R}_\infty)^{\frac{n}{2}} + (m-m_1)(Y(S^n)/\bar{R}_\infty)^{\frac{n}{2}}$ if $u_\infty > 0$ and $m_1(Q(S_+^n)/\bar{R}_\infty)^{\frac{n}{2}} + (m-m_1)(Y(S^n)/\bar{R}_\infty)^{\frac{n}{2}}$ if $u_\infty \equiv 0$. Thus,

$$\bar{R}_\infty = \left(E(u_\infty)^{n/2} + m_1 Q(S_+^n)^{n/2} + (m-m_1) Y(S^n)^{n/2} \right)^{2/n} \quad \text{if } u_\infty > 0, \quad (4.4)$$

$$\text{and } \bar{R}_\infty = \left(m_1 Q(S_+^n)^{n/2} + (m-m_1) Y(S^n)^{n/2} \right)^{2/n} \quad \text{if } u_\infty \equiv 0.$$

⁵ To see that changing the centers $x_{j,v}^*$ as above does not change the limit in (ii), we consider, for fixed j , new centers $\bar{x}_{j,v}^*$ satisfying $d_{g_0}(x_{j,v}^*, \bar{x}_{j,v}^*)/\epsilon_{j,v}^* \leq C$ (the term $\epsilon_{j,v}^*$ in the quotient comes from the rescaling). If the limit in (ii) holds with $\epsilon_{j,v}^*/\epsilon_{i,v}^* \rightarrow \infty$, that relation does not change after replacing the centers. So, let us assume $\epsilon_{j,v}^*/\epsilon_{i,v}^* \leq C$ without loss of generality. The triangle inequality gives

$$d_{g_0}(x_{i,v}^*, \bar{x}_{j,v}^*)^2 \geq \left(d_{g_0}(x_{i,v}^*, x_{j,v}^*) - d_{g_0}(x_{j,v}^*, \bar{x}_{j,v}^*) \right)^2 \geq \frac{1}{2} d_{g_0}(x_{i,v}^*, x_{j,v}^*)^2 - C d_{g_0}(x_{j,v}^*, \bar{x}_{j,v}^*)^2.$$

Hence,

$$\frac{d_{g_0}(x_{i,v}^*, \bar{x}_{j,v}^*)^2}{\epsilon_{i,v}^* \epsilon_{j,v}^*} \geq \frac{1}{2} \frac{d_{g_0}(x_{i,v}^*, x_{j,v}^*)^2}{\epsilon_{i,v}^* \epsilon_{j,v}^*} - C \frac{\epsilon_{j,v}^*}{\epsilon_{i,v}^*} \left(\frac{d_{g_0}(x_{j,v}^*, \bar{x}_{j,v}^*)}{\epsilon_{j,v}^*} \right)^2 \geq \frac{1}{2} \frac{d_{g_0}(x_{i,v}^*, x_{j,v}^*)^2}{\epsilon_{i,v}^* \epsilon_{j,v}^*} - C,$$

so that (ii) still holds with $\bar{x}_{j,v}^*$ replacing $x_{j,v}^*$.

4.1. The case $u_\infty \equiv 0$

We set

$$\mathcal{A}_v = \left\{ (x_k, \epsilon_k, \alpha_k)_{k=1, \dots, m} \in (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m, \text{ such that} \right. \\ \left. \begin{aligned} & x_k \in \partial M \text{ if } k \leq m_1, \quad x_k \in M \setminus \partial M \text{ if } k \geq m_1 + 1, \\ & d_{g_0}(x_k, x_{k,v}^*) \leq \epsilon_{k,v}^*, \quad \frac{1}{2} \leq \frac{\epsilon_k}{\epsilon_{k,v}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2 \end{aligned} \right\}. \quad (4.5)$$

For each v , we can choose a triplet $(x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})_{k=1, \dots, m} \in \mathcal{A}_v$ such that

$$\begin{aligned} & \int_M \frac{4(n-1)}{n-2} \left| d(u_v - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \right|_{g_0}^2 dv_{g_0} \\ & + \int_M R_{g_0} (u_v - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})})^2 dv_{g_0} \\ & \leq \int_M \frac{4(n-1)}{n-2} \left| d(u_v - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)}) \right|_{g_0}^2 dv_{g_0} \\ & + \int_M R_{g_0} (u_v - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)})^2 dv_{g_0} \end{aligned}$$

for all $(x_k, \epsilon_k, \alpha_k)_{k=1, \dots, m} \in \mathcal{A}_v$. Here, $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{A; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{A; (x_k, \epsilon_k)}$ if $k \leq m_1$, $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{B; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{B; (x_k, \epsilon_k)}$ if $m_1 + 1 \leq k \leq m_2$, and $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{C; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{C; (x_k, \epsilon_k)}$ if $k \geq m_2 + 1$; see (3.22), (3.30) and (3.41).

Proposition 4.4. *If $k \geq m_1 + 1$, then $\lim_{v \rightarrow \infty} d_{g_0}(x_{k,v}, \partial M) / \epsilon_{k,v} = \infty$.*

Proof. It follows from the triangle inequality and (4.5) that

$$\frac{d_{g_0}(x_{k,v}, \partial M)}{\epsilon_{k,v}} \geq \frac{d_{g_0}(x_{k,v}, \partial M)}{2\epsilon_{k,v}^*} \geq \frac{d_{g_0}(x_{k,v}^*, \partial M)}{2\epsilon_{k,v}^*} - \frac{1}{2}.$$

Now the right-hand side goes to infinity as $v \rightarrow \infty$ by (iii) of Proposition 4.2. $\square$

Proposition 4.5. *We have:*

(i) *For all $i \neq j$,*

$$\lim_{v \rightarrow \infty} \left\{ \frac{\epsilon_{i,v}}{\epsilon_{j,v}} + \frac{\epsilon_{j,v}}{\epsilon_{i,v}} + \frac{d_{g_0}(x_{i,v}, x_{j,v})^2}{\epsilon_{i,v} \epsilon_{j,v}} \right\} = \infty.$$

(ii) We have

$$\lim_{v \rightarrow \infty} \left\| u_v - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \right\|_{H^1(M)} = 0.$$

Proof. This is a simple consequence of Proposition 4.2 and the definition of $(x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})$; see [8, Proposition 5.1] for details. $\square$

Proposition 4.6. *We have*

$$d_{g_0}(x_{k,v}, x_{k,v}^*) \leq o(1) \epsilon_{k,v}^*, \quad \frac{\epsilon_{k,v}}{\epsilon_{k,v}^*} = 1 + o(1), \quad \text{and} \quad \alpha_{k,v} = 1 + o(1),$$

for all $k = 1, \dots, m$. In particular, $(x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})_{k=1, \dots, m}$ is an interior point of A_v for v sufficiently large.

Proof. It follows from Propositions 4.2 and 4.5 that

$$\begin{aligned} & \left\| \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} - \sum_{k=1}^m \bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)} \right\|_{H^1(M)} \\ & \leq \left\| u_v - \sum_{k=1}^m \bar{u}_{(x_{k,v}^*, \epsilon_{k,v}^*)} \right\|_{H^1(M)} + \left\| u_v - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \right\|_{H^1(M)} = o(1). \end{aligned}$$

Now the result follows. $\square$

Notation. We write $u_v = v_v + w_v$, where

$$v_v = \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \quad \text{and} \quad w_v = u_v - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}. \quad (4.6)$$

Observe that by Proposition 4.5 we have

$$\int_M \frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} w_v^2 dv_{g_0} = o(1). \quad (4.7)$$

Set

$$C_v = \left(\int_{\partial M} |w_v|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{2(n-1)}} + \left(\int_M |w_v|^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{2n}}.$$

Proposition 4.7. *Fix $\rho \leq P_0$. Let $\psi_{k,v} : \Omega_{k,v} = B_\rho^+(0) \subset \mathbb{R}_+^n \rightarrow M$ be Fermi coordinates centered at $x_{k,v}$ if $1 \leq k \leq m_1$, and let $\psi_{k,v} : \Omega_{k,v} = \tilde{B}_{x_{k,v}, \rho} \subset \mathbb{R}^n \rightarrow M$ be normal coordinates centered at $x_{k,v}$ if $m_1 + 1 \leq k \leq m$ (see Definitions 3.1 and 3.2). We have:*

$$(i) \quad \left| \int_M \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} w_v dv_{g_0} \right| \leq o(1) C_v;$$

$$\begin{aligned}
\text{(ii)} \quad & \left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v}^2 - |\psi_{k,v}^{-1}(x)|^2}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v; \\
\text{(iii)} \quad & \left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v} \psi_{k,v}^{-1}(x)}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v, \quad \text{if } m_1 + 1 \leq k \leq m, \\
& \text{and } \left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v} \psi_{k,v}^{-1}(x)}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v, \quad \text{if } k \leq m_1,
\end{aligned}$$

where we are denoting $\bar{y} = (y_1, \dots, y_{n-1})$ for any $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Proof. It follows from the definition of $(x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})$ that

$$\int_M \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{k,v}, \epsilon_{k,v})}, dw_v \rangle_{g_0} + R_{g_0} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} w_v \right) dv_{g_0} = 0.$$

Integrating by parts, we obtain

$$\begin{aligned}
& \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} - R_{g_0} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \right) w_v dv_{g_0} \\
& + \int_{\partial M} \frac{4(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} w_v d\sigma_{g_0} = 0.
\end{aligned}$$

We claim that

$$\left\| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} - R_{g_0} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} + \bar{R}_\infty \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

and

$$\left\| \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \right\|_{L^{\frac{2(n-1)}{n}}(\partial M)} = o(1).$$

The first statement follows from Corollary 3.34. As for the second one, observe first that

$$\partial \bar{u}_{(x_{k,v}, \epsilon_{k,v})} / \partial \eta_{g_0} = 0$$

on ∂M if $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{C; (x_{k,v}, \epsilon_{k,v})}$. If $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{A; (x_{k,v}, \epsilon_{k,v})}$ this statement follows easily from Proposition 3.15 and (2.1), and if $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{B; (x_{k,v}, \epsilon_{k,v})}$ this is Corollary 3.35, also making use of Proposition 4.4.

This proves (i). The remaining statements follow similarly. $\square$

Proposition 4.8. *There exists $c > 0$ such that*

$$\begin{aligned}
& \frac{n+2}{n-2} \bar{R}_\infty \int_M \sum_{k=1}^m \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{4}{n-2}} w_v^2 dv_{g_0} \\
& \leq (1-c) \left\{ \int_M \frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} w_v^2 dv_{g_0} \right\}
\end{aligned}$$

for all v sufficiently large.

Proof. Once we have proved Proposition 4.7, this proof is a contradiction argument similar to [8, Proposition 5.4] and [1, Proposition 4.6] and we will omit the details. Assume by contradiction that there is a sequence $\{\tilde{w}_v\}$ satisfying

$$\int_M \frac{4(n-1)}{n-2} |d\tilde{w}_v|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} \tilde{w}_v^2 dv_{g_0} = 1$$

and

$$\lim_{v \rightarrow \infty} \frac{n+2}{n-2} \bar{R}_\infty \int_M \sum_{k=1}^m \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{4}{n-2}} \tilde{w}_v^2 dv_{g_0} \geq 1.$$

After rescaling around $x_{k,v}$, the new sequence obtained converges (weakly in $H_{\text{loc}}^1(\mathbb{R}_+^n)$ if $k \leq m_1$ and in $H_{\text{loc}}^1(\mathbb{R}^n)$ if $k \geq m_1 + 1$) to a certain $\hat{w}$. It turns out that one can choose $k \in \{1, \dots, m\}$ in such way that $\hat{w}$ satisfies

$$\int_{\mathbb{R}_+^n} \left(\frac{1}{1+|y|^2} \right)^2 \hat{w}^2(y) dy > 0$$

and

$$\int_{\mathbb{R}_+^n} |d\hat{w}(y)|^2 dy \leq n(n+2) \int_{\mathbb{R}_+^n} \left(\frac{1}{1+|y|^2} \right)^2 \hat{w}^2(y) dy$$

if $k \leq m_1$, or the same two inequalities with $\mathbb{R}_+^n$ replaced by $\mathbb{R}^n$ if $k \geq m_1 + 1$.

On the other hand, if $k \leq m_1$, due to Proposition 4.7, $\hat{w}$ satisfies

$$\begin{aligned} \int_{\mathbb{R}_+^n} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \hat{w}(y) dy &= 0, \\ \int_{\mathbb{R}_+^n} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \frac{1-|y|^2}{1+|y|^2} \hat{w}(y) dy &= 0, \\ \int_{\mathbb{R}_+^n} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \frac{y_j}{1+|y|^2} \hat{w}(y) dy &= 0, \end{aligned}$$

where $y = (y_1, \dots, y_n)$, and $j = 1, \dots, n-1$. By considering the corresponding equations on the round hemisphere we obtain a contradiction as in [1, Proposition 4.6]. If $k \geq m_1 + 1$, $\hat{w}$ satisfies the same last three equations (with $j = 1, \dots, n$ for the last), but with $\mathbb{R}_+^n$ replaced by $\mathbb{R}^n$, and the same contradiction is reached by considering corresponding equations on the round sphere instead of the hemisphere. $\square$

Corollary 4.9. *There exists $c > 0$ such that*

$$\frac{n+2}{n-2} \bar{R}_\infty \int_M v_v^{\frac{4}{n-2}} w_v^2 dv_{g_0} \leq (1-c) \left\{ \int_M \frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} w_v^2 dv_{g_0} \right\}$$

for all v sufficiently large.

Proof. By the definition of v_ν (equation (4.6)), we have

$$\lim_{\nu \rightarrow \infty} \int_M \left| v_\nu^{\frac{4}{n-2}} - \sum_{k=1}^m \bar{u}_{(x_k, \nu, \epsilon_k, \nu)}^{\frac{4}{n-2}} \right|^{n/2} dv_{g_0} = 0.$$

Hence, the assertion follows from Proposition 4.8. $\square$

Proposition 4.10. *For all ν sufficiently large, we have*

$$E(v_\nu) \leq \left(\sum_{k=1}^m E(\bar{u}_{(x_k, \epsilon_k)})^{n/2} \right)^{2/n}.$$

Proof. Choose a permutation $\sigma : \{1, \dots, m\}$ such that $\epsilon_{\sigma(i), \nu} \leq \epsilon_{\sigma(j), \nu}$ for all $i < j$. During this proof we will omit the symbol σ , writing $\epsilon_{i, \nu}$ instead of $\epsilon_{\sigma(i), \nu}$, so that $\epsilon_{i, \nu} \leq \epsilon_{j, \nu}$ for all $i < j$. After calculations similar to the ones in [8, Proposition 5.6] we obtain

$$\begin{aligned} & E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}} \\ & \leq \left(\sum_{k=1}^m E(\bar{u}_{(x_k, \nu, \epsilon_k, \nu)})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}} - c \sum_{i < j} \left(\frac{\epsilon_{i, \nu} \epsilon_{j, \nu}}{\epsilon_{j, \nu}^2 + d_{g_0}(x_{i, \nu}, x_{j, \nu})^2} \right)^{\frac{n-2}{2}} \\ & \quad - 2 \int_M \sum_{i < j} \alpha_{i, \nu} \alpha_{j, \nu} \bar{u}_{(x_{i, \nu}, \epsilon_{i, \nu})} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})} - R_{g_0} \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})} \right. \\ & \quad \left. + \bar{R}_\infty \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}^{\frac{n+2}{n-2}} \right) dv_{g_0} \\ & \quad - \frac{8(n-1)}{n-2} \int_{\partial M} \sum_{i < j} \alpha_{i, \nu} \alpha_{j, \nu} \bar{u}_{(x_{i, \nu}, \epsilon_{i, \nu})} \frac{\partial \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}}{\partial \eta_{g_0}} d\sigma_{g_0} \\ & \quad - 2 \sum_{i < j} \alpha_{i, \nu} \alpha_{j, \nu} (F(\bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}) - \bar{R}_\infty) \int_M \bar{u}_{(x_{i, \nu}, \epsilon_{i, \nu})} \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}^{\frac{n+2}{n-2}} dv_{g_0}. \end{aligned}$$

It is not difficult to see that $F(\bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}) = \bar{R}_\infty + o(1)$. This is more subtle in the case $\bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})} = \bar{u}_{B; (x_{j, \nu}, \epsilon_{j, \nu})}$, when we make use of Proposition 4.4 and Lemma 3.20. Then, because of [8, Lemma B.4], we have

$$|F(\bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}) - \bar{R}_\infty| \int_M \bar{u}_{(x_{i, \nu}, \epsilon_{i, \nu})} \bar{u}_{(x_{j, \nu}, \epsilon_{j, \nu})}^{\frac{n+2}{n-2}} dv_{g_0} \leq o(1) \left(\frac{\epsilon_{i, \nu} \epsilon_{j, \nu}}{\epsilon_{j, \nu}^2 + d_{g_0}(x_{i, \nu}, x_{j, \nu})^2} \right)^{\frac{n-2}{2}}.$$

Then, using Corollaries 3.32 and 3.33,

$$\begin{aligned} & E(v_v) \left(\int_M v_v^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}} \\ & \leq \left(\sum_{k=1}^m E(\bar{u}_{(x_{k,v}, \epsilon_{k,v})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_v^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ & \quad - \sum_{i < j} (c - C \max\{\rho_A, \rho_B, \rho_C\}^{1/2} - o(1)) \left(\frac{\epsilon_{i,v} \epsilon_{j,v}}{\epsilon_{j,v}^2 + d_{g_0}(x_{i,v}, x_{j,v})^2} \right)^{\frac{n-2}{2}}. \end{aligned}$$

Hence, the assertion follows by choosing ρ_A, ρ_B and ρ_C smaller if necessary (see Remark 4.1). $\square$

Corollary 4.11. *Under the hypothesis of Theorem 1.9, we have*

$$E(v_v) \leq \bar{R}_\infty, \quad \text{for all } v \text{ sufficiently large.}$$

Proof. Using Propositions 3.8, 3.17 and 3.29, we obtain $E(\bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \leq Q(S_+^n)$ for $k \leq m_1$, and $E(\bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \leq Y(S^n)$ for $k \geq m_1 + 1$. Then the result follows from Proposition 4.10 and (4.4). $\square$

4.2. The case $u_\infty > 0$

Proposition 4.12. *There exist sequences $\{\psi_a\}_{a \in \mathbb{N}} \subset C^\infty(M)$ and $\{\lambda_a\}_{a \in \mathbb{N}} \subset \mathbb{R}$, with $\lambda_a > 0$, satisfying:*

(i) *For all $a \in \mathbb{N}$,*

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta_{g_0} \psi_a - R_{g_0} \psi_a + \lambda_a u_\infty^{\frac{4}{n-2}} \psi_a = 0 & \text{in } M \\ \frac{\partial}{\partial \eta_{g_0}} \psi_a = 0 & \text{on } \partial M; \end{cases}$$

(ii) *For all $a, b \in \mathbb{N}$,*

$$\int_M \psi_a \psi_b u_\infty^{\frac{4}{n-2}} dv_{g_0} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b; \end{cases}$$

(iii) *The span of $\{\psi_a\}_{a \in \mathbb{N}}$ is dense in $L^2(M)$;*

(iv) *We have $\lim_{a \rightarrow \infty} \lambda_a = \infty$.*

Proof. Since we are assuming $R_{g_0} > 0$, for each $f \in L^2(M)$ we can define $T(f) = u$, where $u \in H^1(M)$ is the unique solution of

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u = f u_\infty^{\frac{4}{n-2}} & \text{in } M \\ \frac{\partial}{\partial \eta_{g_0}} u = 0 & \text{on } \partial M. \end{cases}$$

Since $H^1(M)$ is compactly embedded in $L^2(M)$, the operator $T : L^2(M) \rightarrow L^2(M)$ is compact. Integrating by parts, we see that T is symmetric with respect to the inner product $(\psi_1, \psi_2) \mapsto \int_M \psi_1 \psi_2 u_\infty^{\frac{4}{n-2}} dv_{g_0}$. Then the result follows from the spectral theorem for compact operators. $\square$

Let $A \subset \mathbb{N}$ be a finite set such that $\lambda_a > \frac{n+2}{n-2} \bar{R}_\infty$ for all $a \notin A$, and define the projection

$$\Gamma(f) = \sum_{a \notin A} \left(\int_M \psi_a f dv_{g_0} \right) \psi_a u_\infty^{\frac{4}{n-2}} = f - \sum_{a \in A} \left(\int_M \psi_a f dv_{g_0} \right) \psi_a u_\infty^{\frac{4}{n-2}}.$$

Lemma 4.13. *There exists $\zeta > 0$ with the following significance: for all $z \in \mathbb{R}^A$ with $|z| \leq \zeta$, there exists a smooth function $\bar{u}_z$ satisfying $\partial \bar{u}_z / \partial \eta_{g_0} = 0$ on ∂M ,*

$$\int_M u_\infty^{\frac{4}{n-2}} (\bar{u}_z - u_\infty) \psi_a dv_{g_0} = z_a \quad \text{for all } a \in A, \quad (4.8)$$

and

$$\Gamma \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + \bar{R}_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) = 0. \quad (4.9)$$

Moreover, the mapping $z \mapsto \bar{u}_z$ is real analytic.

Proof. This is just an application of the implicit function theorem. $\square$

Lemma 4.14. *There exists $0 < \gamma < 1$ such that*

$$E(\bar{u}_z) - E(u_\infty) \leq C \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + \bar{R}_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) dv_{g_0} \right|^{1+\gamma},$$

if $|z|$ is sufficiently small.

Proof. Observe that the function $z \mapsto E(\bar{u}_z)$ is real analytic. According to results of Łojasiewicz (see equation (2.4) in [29, page 538]), there exists $0 < \gamma < 1$ such that

$$|E(\bar{u}_z) - E(u_\infty)| \leq \sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{u}_z) \right|^{1+\gamma},$$

if $|z|$ is sufficiently small. Now we can follow the lines in [8, Lemma 6.5] to obtain the result. $\square$

We set

$$\begin{aligned} \mathcal{A}_v = \Big\{ (z, (x_k, \epsilon_k, \alpha_k)_{k=1, \dots, m}) \in \mathbb{R}^A \times (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m, \text{ such that} \\ x_k \in \partial M \text{ if } k \leq m_1, \ x_k \in M \setminus \partial M \text{ if } k \geq m_1 + 1, \\ |z| \leq \zeta, \ d_{g_0}(x_k, x_{k,v}^*) \leq \epsilon_{k,v}^*, \ \frac{1}{2} \leq \frac{\epsilon_k}{\epsilon_{k,v}^*} \leq 2, \ \frac{1}{2} \leq \alpha_k \leq 2 \Big\}. \end{aligned}$$

For each v , we can choose a pair $(z_v, (x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})_{k=1,\dots,m}) \in \mathcal{A}_v$ such that

$$\begin{aligned} & \int_M \frac{4(n-1)}{n-2} \left| d(u_v - \bar{u}_{z_v} - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \right|_{g_0}^2 dv_{g_0} \\ & + \int_M R_{g_0} (u_v - \bar{u}_{z_v} - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})})^2 dv_{g_0} \\ & \leq \int_M \frac{4(n-1)}{n-2} \left| d(u_v - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)}) \right|_{g_0}^2 dv_{g_0} \\ & + \int_M R_{g_0} (u_v - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)})^2 dv_{g_0} \end{aligned}$$

for all $(z, (x_k, \epsilon_k, \alpha_k)_{k=1,\dots,m}) \in \mathcal{A}_v$. Here, $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{A; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{A; (x_k, \epsilon_k)}$ if $k \leq m_1$, $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{B; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{B; (x_k, \epsilon_k)}$ if $m_1 + 1 \leq k \leq m_2$, and $\bar{u}_{(x_{k,v}, \epsilon_{k,v})} = \bar{u}_{C; (x_{k,v}, \epsilon_{k,v})}$ and $\bar{u}_{(x_k, \epsilon_k)} = \bar{u}_{C; (x_k, \epsilon_k)}$ if $k \geq m_2 + 1$; see (3.22), (3.30) and (3.41).

The proofs of the next three propositions are similar to Propositions 4.4, 4.5 and 4.6.

Proposition 4.15. *If $k \geq m_1 + 1$, then $\lim_{v \rightarrow \infty} d_{g_0}(x_{k,v}, \partial M) / \epsilon_{k,v} = \infty$.*

Proposition 4.16. *We have:*

(i) *For all $i \neq j$,*

$$\lim_{v \rightarrow \infty} \left\{ \frac{\epsilon_{i,v}}{\epsilon_{j,v}} + \frac{\epsilon_{j,v}}{\epsilon_{i,v}} + \frac{d_{g_0}(x_{i,v}, x_{j,v})^2}{\epsilon_{i,v} \epsilon_{j,v}} \right\} = \infty;$$

(ii) *We have*

$$\lim_{v \rightarrow \infty} \left\| u_v - \bar{u}_{z_v} - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \right\|_{H^1(M)} = 0.$$

Proposition 4.17. *We have $|z_v| = o(1)$, and*

$$d_{g_0}(x_{k,v}, x_{k,v}^*) \leq o(1) \epsilon_{k,v}^*, \quad \frac{\epsilon_{k,v}}{\epsilon_{k,v}^*} = 1 + o(1), \quad \text{and} \quad \alpha_{k,v} = 1 + o(1),$$

for all $k = 1, \dots, m$. In particular, $(z_v, (x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})_{k=1,\dots,m})$ is an interior point of $\mathcal{A}_v$ for v sufficiently large.

Notation. We write $u_v = v_v + w_v$, where

$$v_v = \bar{u}_{z_v} + \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})} \quad \text{and} \quad w_v = u_v - \bar{u}_{z_v} - \sum_{k=1}^m \alpha_{k,v} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}. \quad (4.10)$$

Observe that by Proposition 4.16 we have

$$\int_M \frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} w_v^2 dv_{g_0} = o(1). \quad (4.11)$$

Set

$$C_v = \left(\int_{\partial M} |w_v|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{2(n-1)}} + \left(\int_M |w_v|^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{2n}}.$$

Proposition 4.18. *Fix $\rho \leq P_0$. Let $\psi_{k,v} : \Omega_{k,v} = B_\rho^+(0) \subset \mathbb{R}_+^n \rightarrow M$ be Fermi coordinates centered at $x_{k,v}$ if $1 \leq k \leq m_1$, and let $\psi_{k,v} : \Omega_{k,v} = \tilde{B}_{x_{k,v}\rho} \subset \mathbb{R}^n \rightarrow M$ be normal coordinates centered at $x_{k,v}$ if $m_1 + 1 \leq k \leq m$ (see Definitions 3.1 and 3.2). We have:*

- (i) $\left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a w_v dv_{g_0} \right| \leq o(1) \int_M |w_v| dv_{g_0}, \quad \text{for } a \in A;$
 - (ii) $\left| \int_M \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} w_v dv_{g_0} \right| \leq o(1) C_v;$
 - (iii) $\left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v}^2 - |\psi_{k,v}^{-1}(x)|^2}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v;$
 - (iv) $\left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v} \psi_{k,v}^{-1}(x)}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v, \quad \text{if } m_1 + 1 \leq k \leq m,$
- and $\left| \int_{\Omega_{k,v}} \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{n+2}{n-2}} \frac{\epsilon_{k,v} \psi_{k,v}^{-1}(x)}{\epsilon_{k,v}^2 + |\psi_{k,v}^{-1}(x)|^2} w_v dv_{g_0} \right| \leq o(1) C_v, \quad \text{if } k \leq m_1,$

where we are denoting $\bar{y} = (y_1, \dots, y_{n-1})$ for any $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Proof. (i) Set $\tilde{\psi}_{a,z} = \partial \tilde{u}_z / \partial z_a$. It follows from the identities (4.8) and (4.9) that $\tilde{\psi}_{a,0} = \psi_a$ for all $a \in A$. By the definition of $(z_v, (x_{k,v}, \epsilon_{k,v}, \alpha_{k,v})_{1 \leq k \leq m})$, we have

$$\int_M \frac{4(n-1)}{n-2} \langle d\tilde{\psi}_{a,z_v}, w_v \rangle_{g_0} dv_{g_0} + \int_M R_{g_0} \tilde{\psi}_{a,z_v} w_v dv_{g_0} = 0.$$

Hence,

$$\begin{aligned} \lambda_a \int_M u_\infty^{\frac{4}{n-2}} \psi_a w_v dv_{g_0} &= - \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \psi_a - R_{g_0} \psi_a \right) w_v dv_{g_0} \\ &= \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} (\tilde{\psi}_{a,z_v} - \psi_a) - R_{g_0} (\tilde{\psi}_{a,z_v} - \psi_a) \right) w_v dv_{g_0} \\ &\quad + \int_{\partial M} \frac{\partial \tilde{\psi}_{a,z_v}}{\partial \eta_{g_0}} w_v d\sigma_{g_0}. \end{aligned}$$

However, we know that $\partial \tilde{\psi}_{a,z_v} / \partial \eta_{g_0} = 0$ on ∂M . Then, since $\lambda_a > 0$ and $|z_v| \rightarrow 0$ as $v \rightarrow \infty$, we conclude that the assertion (i) follows.

The proofs of (ii), (iii), and (iv) are similar to Proposition 4.7. $\square$

Proposition 4.19. *There exists $c > 0$ such that*

$$\begin{aligned} & \frac{n+2}{n-2} \bar{R}_\infty \int_M \left(u_\infty^{\frac{4}{n-2}} + \sum_{k=1}^m \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{4}{n-2}} \right) w_v^2 dv_{g_0} \\ & \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 + R_{g_0} w_v^2 \right) dv_{g_0} \end{aligned}$$

for all v sufficiently large.

Proof. As in Proposition 4.8, once Proposition 4.18 is established, this proof is a contradiction argument similar to [8, Proposition 6.8] and [1, Proposition 4.18]. $\square$

Corollary 4.20. *There exists $c > 0$ such that*

$$\frac{n+2}{n-2} \bar{R}_\infty \int_M v_v^{\frac{4}{n-2}} w_v^2 dv_{g_0} \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |dw_v|_{g_0}^2 + R_{g_0} w_v^2 \right) dv_{g_0}$$

for all v sufficiently large.

Proof. By the definition of v_v (see (4.10)), we have

$$\lim_{v \rightarrow \infty} \int_M \left| v_v^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}} - \sum_{k=1}^m \bar{u}_{(x_{k,v}, \epsilon_{k,v})}^{\frac{4}{n-2}} \right|^{\frac{n}{2}} dv_{g_0} = 0.$$

Hence, the assertion follows from Proposition 4.19. $\square$

The next two propositions are similar to Propositions 6.14 and 6.15 of [8] and we will just outline their proofs.

Proposition 4.21. *There exist $C > 0$ and $0 < \gamma < 1$ such that*

$$E(\bar{u}_{z_v}) - E(u_\infty) \leq C \left\{ \int_M u_v^{\frac{2n}{n-2}} |R_{g_v} - \bar{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} + C \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}(1+\gamma)}$$

if v is sufficiently large.

Proof. As in [8, Lemmas 6.11 and 6.12], because $\partial u_v / \partial \eta_{g_0} = \partial \bar{u}_{z_v} / \partial \eta_{g_0} = 0$ on ∂M , we can show that there exists $C > 0$ such that

$$\|u_v - \bar{u}_{z_v}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} \leq C \|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}} \quad (4.12)$$

and

$$\|u_v - \bar{u}_{z_v}\|_{L^1(M)} \leq C \|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}}, \quad (4.13)$$

for v sufficiently large.

We will prove the estimate

$$\begin{aligned} & \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + \bar{R}_\infty \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right) dv_{g_0} \right| \\ & \leq C \left\{ \int_M u_v^{\frac{2n}{n-2}} |R_{g_v} - \bar{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}} + C \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}} \end{aligned} \quad (4.14)$$

for v is sufficiently large.

Integrating by parts, we obtain

$$\begin{aligned} & \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + \bar{R}_\infty \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right) dv_{g_0} \\ & = \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u_v - R_{g_0} u_v + \bar{R}_\infty u_v^{\frac{n+2}{n-2}} \right) dv_{g_0} \\ & \quad + \lambda_a \int_M u_v^{\frac{4}{n-2}} \psi_a (u_v - \bar{u}_{z_v}) dv_{g_0} - \bar{R}_\infty \int_M \psi_a (u_v^{\frac{n+2}{n-2}} - \bar{u}_{z_v}^{\frac{n+2}{n-2}}) dv_{g_0}. \end{aligned}$$

Using the fact that $\frac{4(n-1)}{n-2} \Delta_{g_0} u_v - R_{g_0} u_v + \bar{R}_\infty u_v^{\frac{n+2}{n-2}} = -(R_{g_v} - \bar{R}_\infty) u_v^{\frac{n+2}{n-2}}$ and the pointwise estimate

$$|u_v^{\frac{n+2}{n-2}} - \bar{u}_{z_v}^{\frac{n+2}{n-2}}| \leq C \bar{u}_{z_v}^{\frac{4}{n-2}} |u_v - \bar{u}_{z_v}| + C |u_v - \bar{u}_{z_v}|^{\frac{n+2}{n-2}},$$

we obtain

$$\begin{aligned} & \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + \bar{R}_\infty \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right) dv_{g_0} \right| \\ & \leq C \|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C \|u_v - \bar{u}_{z_v}\|_{L^1(M)} + C \|u_v - \bar{u}_{z_v}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}}. \end{aligned}$$

Then it follows from (4.12) and (4.13) that

$$\begin{aligned} & \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + \bar{R}_\infty \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right) dv_{g_0} \right| \\ & \leq C \|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ & \quad + C \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}}. \end{aligned} \quad (4.15)$$

On the other hand, by Corollary 2.3 we can assume

$$\|u_v^{\frac{n+2}{n-2}} (R_{g_v} - \bar{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{2n}} = \left(\int_M |R_{g_v} - \bar{R}_\infty|^{\frac{2n}{n+2}} dv_{g_v} \right)^{\frac{n+2}{2n}} < 1. \quad (4.16)$$

The estimate (4.14) now follows using the inequality (4.16) in (4.15). Proposition 4.21 is a consequence of Lemma 4.14 and the estimate (4.14). $\square$

Proposition 4.22. *There exists $c > 0$ such that*

$$E(v_v) \leq \left(E(\bar{u}_{z_v})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{x_k, \epsilon_{k,v}})^{\frac{n}{2}} \right)^{\frac{2}{n}} - c \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}}$$

if v is sufficiently large.

Proof. Choose a permutation $\sigma : \{1, \dots, m\}$ such that $\epsilon_{\sigma(i),v} \leq \epsilon_{\sigma(j),v}$ for all $i < j$. During this proof we will omit the symbol σ , writing $\epsilon_{i,v}$ instead of $\epsilon_{\sigma(i),v}$, so that $\epsilon_{i,v} \leq \epsilon_{j,v}$ for all $i < j$. After calculations similar to the ones in [8, Proposition 6.15], we obtain

$$\begin{aligned} E(v_v) & \left(\int_M v_v^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}} \\ & \leq \left(E(\bar{u}_{z_v})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{(x_k, v, \epsilon_{k,v})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_v^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}} \\ & \quad - \sum_{k=1}^m 2\alpha_{k,v} \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + F(\bar{u}_{z_v}) \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right) \bar{u}_{(x_k, v, \epsilon_{k,v})} dv_{g_0} \\ & \quad - \sum_{i < j} 2\alpha_{i,v} \alpha_{j,v} \int_M \frac{4(n-1)}{n-2} \frac{\partial \bar{u}_{(x_j, v, \epsilon_{j,v})}}{\partial \eta_{g_0}} \bar{u}_{(x_i, v, \epsilon_{i,v})} dv_{g_0} \\ & \quad - \sum_{i < j} 2\alpha_{i,v} \alpha_{j,v} \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_j, v, \epsilon_{j,v})} - R_{g_0} \bar{u}_{(x_j, v, \epsilon_{j,v})} \right. \\ & \quad \quad \left. + F(\bar{u}_{(x_j, v, \epsilon_{j,v})}) \bar{u}_{(x_j, v, \epsilon_{j,v})}^{\frac{n+2}{n-2}} \right) \bar{u}_{(x_i, v, \epsilon_{i,v})} dv_{g_0} \\ & \quad - c \sum_{k=1}^m \epsilon_{k,v}^{\frac{n-2}{2}} - c \sum_{i < j} \left(\frac{\epsilon_{i,v} \epsilon_{j,v}}{\epsilon_{j,v}^2 + d_{g_0}(x_{i,v}, x_{j,v})^2} \right)^{\frac{n-2}{2}}. \end{aligned}$$

Since $F(\bar{u}_{z_v}) \rightarrow F(u_\infty) = \bar{R}_\infty$ as $v \rightarrow \infty$, we have the estimate

$$\int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_v} - R_{g_0} \bar{u}_{z_v} + F(\bar{u}_{z_v}) \bar{u}_{z_v}^{\frac{n+2}{n-2}} \right| \bar{u}_{(x_k, v, \epsilon_{k,v})} dv_{g_0} \leq o(1) \epsilon_{k,v}^{\frac{n-2}{2}}.$$

Now the assertion follows as in the proof of Proposition 4.10. $\square$

Corollary 4.23. *Under the hypothesis of Theorem 1.9, there exist $C > 0$ and $0 < \gamma < 1$ such that*

$$E(v_v) \leq \overline{R}_\infty + C \left(\int_M u_v^{\frac{2n}{n-2}} |R_{g_v} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)},$$

if v is sufficiently large.

Proof. Using Propositions 3.8, 3.17 and 3.29, we obtain $E(\bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \leq Q(S_+^n)$ for all $k = 1, \dots, m_1$ and $E(\bar{u}_{(x_{k,v}, \epsilon_{k,v})}) \leq Y(S^n)$ for all $k = m_1 + 1, \dots, m$. Then the result follows from Propositions 4.21 and 4.22 and (4.4). $\square$

5. Proof of the main theorem

As in [8, Sections 3 and 7], the proof of Theorem 1.9 is carried out in several propositions, whose proofs will be only sketched in what follows.

Let $u(t)$, $t \geq 0$, be the solution of (2.4) obtained in Section 2. The next proposition, which is analogous to [8, Proposition 3.3], is a crucial step in the argument.

Proposition 5.1. *Let $\{t_v\}_{v=1}^\infty$ be a sequence such that $\lim_{v \rightarrow \infty} t_v = \infty$. Then we can choose $0 < \gamma < 1$ and $C > 0$ such that, after passing to a subsequence, we have*

$$\overline{R}_{g(t_v)} - \overline{R}_\infty \leq C \left\{ \int_M u(t_v)^{\frac{2n}{n-2}} |R_{g(t_v)} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)}$$

for all v .

Proof. It is a long computation using Corollaries 4.9, 4.11, 4.20 and 4.23; see [8, Section 7]. $\square$

Proposition 5.2. *There exists $C > 0$ such that*

$$\int_0^\infty \left\{ \int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - \overline{R}_{g(t)})^2 dv_{g_0} \right\}^{\frac{1}{2}} dt \leq C$$

for all $t \geq 0$.

Proof. A simple contradiction argument using Corollary 2.3 and Proposition 5.1 (see [8, Proposition 3.4]) shows that there exist $0 < \gamma < 1$ and $t_0 > 0$ such that

$$\overline{R}_{g(t)} - \overline{R}_\infty \leq C \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)}$$

for all $t \geq t_0$. Then it follows that

$$\begin{aligned} \bar{R}_{g(t)} - \bar{R}_\infty &\leq C \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \bar{R}_{g(t)}|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} \\ &\quad + C(\bar{R}_{g(t)} - \bar{R}_\infty)^{1+\gamma}, \end{aligned}$$

hence

$$\bar{R}_{g(t)} - \bar{R}_\infty \leq C \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \bar{R}_{g(t)}|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} \quad (5.1)$$

for $t > 0$ sufficiently large. By (2.7) and (5.1), there exists $c > 0$ such that

$$\begin{aligned} \frac{d}{dt}(\bar{R}_{g(t)} - \bar{R}_\infty) &= -\frac{n-2}{2} \int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} \\ &\leq -\frac{n-2}{2} \left\{ \int_M |R_{g(t)} - \bar{R}_{g(t)}|^{\frac{2n}{n+2}} u(t)^{\frac{2n}{n-2}} dv_{g_0} \right\}^{\frac{n+2}{n}} \\ &\leq -c(\bar{R}_{g(t)} - \bar{R}_\infty)^{\frac{2}{1+\gamma}} \end{aligned}$$

for $t > 0$ sufficiently large. Hence, $\frac{d}{dt}(\bar{R}_{g(t)} - \bar{R}_\infty)^{-\frac{1+\gamma}{1-\gamma}} \geq c$, which implies

$$\bar{R}_{g(t)} - \bar{R}_\infty \leq Ct^{-\frac{1+\gamma}{1-\gamma}}, \quad \text{for } t > 0 \text{ sufficiently large.}$$

Then using Hölder's inequality and the equation (2.7) we obtain

$$\begin{aligned} &\int_T^{2T} \left(\int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{1}{2}} dt \\ &\leq \left(\int_T^{2T} dt \right)^{\frac{1}{2}} \left(\int_T^{2T} \int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} dt \right)^{\frac{1}{2}} \\ &= \left\{ \frac{2}{n-2} T(\bar{R}_{g(T)} - \bar{R}_{g(2T)}) \right\}^{\frac{1}{2}} \\ &\leq CT^{-\frac{\gamma}{1-\gamma}} \end{aligned}$$

for T sufficiently large. This implies

$$\begin{aligned} & \int_0^\infty \left(\int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{1}{2}} dt \\ &= \int_0^1 \left(\int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{1}{2}} dt \\ &\quad + \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \left(\int_M (R_{g(t)} - \bar{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{1}{2}} dt \\ &\leq C \sum_{k=0}^\infty 2^{-\frac{\gamma}{1-\gamma}k} \leq C, \end{aligned}$$

which concludes the proof. $\square$

Proposition 5.3. *There exist $C, c > 0$ such that*

$$\sup_M u(t) \leq C \quad \text{and} \quad \inf_M u(t) \geq c, \quad \text{for all } t \geq 0. \quad (5.2)$$

Proof. We first claim that, given $\gamma_0 > 0$, there exists $r > 0$ such that

$$\int_{B_r(x)} u(t)^{\frac{2n}{n-2}} dv_{g_0} \leq \gamma_0, \quad \text{for all } t \geq 0, x \in M. \quad (5.3)$$

Indeed, we can make use of Proposition 5.2 as in [8, Proposition 3.6] to obtain the above inequality.

Fix $n/2 < q < p < (n+2)/2$. According to Corollary 2.3 there is $C_2 > 0$ such that

$$\int_M |R_{g(t)}|^p dv_{g(t)} \leq C_2, \quad \text{for all } t \geq 0.$$

Set $\gamma_0 = \gamma_1^{\frac{p}{p-q}} C_2^{-\frac{q}{p-q}}$, where γ_1 is the constant obtained in Proposition A.3. By (5.3), there is $r > 0$ such that

$$\int_{B_r(x)} dv_{g(t)} \leq \gamma_0, \quad \text{for all } t \geq 0, x \in M.$$

Then

$$\int_{B_r(x)} |R_{g(t)}|^q dv_{g(t)} \leq \left\{ \int_{B_r(x)} dv_{g(t)} \right\}^{\frac{p-q}{p}} \left\{ \int_{B_r(x)} |R_{g(t)}|^p dv_{g(t)} \right\}^{\frac{q}{p}} \leq \gamma_1.$$

Hence, the first assertion of (5.2) follows from Proposition A.3. The second one follows exactly as in the proof of the second estimate of (2.8). $\square$

Proof of Theorem 1.9. Once we have proved Proposition 5.3, it follows as in [8, p.229] that all higher order derivatives of u are uniformly bounded. The uniqueness of the asymptotic limit of $R_{g(t)}$ follows from Proposition 5.2. $\square$

Appendix

A. Some elliptic estimates

Let (M^n, g) be a complete Riemannian manifold with boundary ∂M and dimension $n \geq 3$, and let η_g be its unit normal vector pointing inwards.

Definition A.1. We say that $u \in H^1(M)$ is a *subsolution* (resp. *supersolution*) of

$$\begin{cases} \Delta_g u + Pu = f & \text{in } M \\ \partial u / \partial \eta_g + \bar{P}u = \bar{f} & \text{on } \partial M \end{cases} \quad (\text{A.1})$$

if, for all $0 \leq v \in C_c^1(M)$, the following quantity is nonpositive (resp. nonnegative)

$$\int_M (\langle du, dv \rangle_g - Puv + fv) dv_g + \int_{\partial M} (-\bar{P}uv + \bar{f}v) d\sigma_g.$$

The next proposition is similar to [17, Theorems 8.17 and 8.18]; see also [19, Lemma A.1].

Proposition A.2. Let $q > n$, $s > n - 1$ and $P \in L^{q/2}(M)$, $\bar{P} \in L^s(\partial M)$ with $\|P\|_{L^{q/2}(M)} + \|\bar{P}\|_{L^s(\partial M)} \leq \Lambda$.

(a) For any $p > 1$, there exists $C = C(n, p, q, s, g, \Lambda)$ and $r_0 = r_0(M, g)$ such that

$$\begin{aligned} \sup_{B_r^+(x)} u &\leq Cr^{-\frac{n}{p}} \|u\|_{L^p(B_{2r}^+(x))} + Cr^{2-\frac{2n}{q}} \|f\|_{L^{q/2}(B_{4r}^+(x))} \\ &\quad + Cr^{1-\frac{n-1}{s}} \|\bar{f}\|_{L^s(D_{4r}(x))} \end{aligned}$$

for any $x \in \partial M$, $r < r_0$ and $0 \leq u \in H^1(M)$ subsolution of (A.1).

(b) If $1 \leq p < \frac{n}{n-2}$, there exists $C = C(n, p, q, s, g, \Lambda)$ and $r_0 = r_0(M, g)$ such that

$$\begin{aligned} r^{-\frac{n}{p}} \|u\|_{L^p(B_{2r}^+(x))} &\leq C \inf_{B_r^+(x)} u + Cr^{2-\frac{2n}{q}} \|f\|_{L^{q/2}(B_{4r}^+(x))} \\ &\quad + Cr^{1-\frac{n-1}{s}} \|\bar{f}\|_{L^s(D_{4r}(x))} \end{aligned}$$

for any $x \in \partial M$, $r < r_0$ and $0 \leq u \in H^1(M)$ supersolution of (A.1).

Proof. After rescaling we can assume $r = 1$. Let $\beta \neq 0$, $k = \|f\|_{L^{q/2}(B_4^+)} + \|\bar{f}\|_{L^s(D_4)}$ and $0 \leq \chi \in C_c^1(B_4^+)$. We will assume that $k > 0$. The general case will follow by tending k to zero. Set $\bar{u} = u + k$.

If u is a subsolution, by definition we have

$$\int_M \langle du, d(\chi^2 \bar{u}^\beta) \rangle_g dv_g \leq \int_M (Pu - f) \chi^2 \bar{u}^\beta dv_g + \int_{\partial M} (\bar{P}u - \bar{f}) \chi^2 \bar{u}^\beta d\sigma_g,$$

and we have the opposite inequality in case u is a supersolution. Choosing $\beta > 0$ should u be a subsolution and $\beta < 0$ should u be a supersolution, in both cases we obtain

$$\begin{aligned} \int_M \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2 dv_g &\leq |\beta|^{-1} \int_M 2\chi \bar{u}^\beta |d\chi|_g |d\bar{u}|_g dv_g \\ &\quad + |\beta|^{-1} \int_M \chi^2 (|P| + k^{-1}|f|) \bar{u}^{\beta+1} dv_g \\ &\quad + |\beta|^{-1} \int_{\partial M} \chi^2 (|\bar{P}| + k^{-1}|\bar{f}|) \bar{u}^{\beta+1} d\sigma_g \end{aligned} \quad (\text{A.2})$$

by means of $\langle du, d(\chi^2 \bar{u}^\beta) \rangle_g = 2\chi \bar{u}^\beta \langle d\chi, d\bar{u} \rangle_g + \beta \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2$. Applying Young's inequality to the last term of (A.2) we arrive at

$$\begin{aligned} \int_M \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2 dv_g &\leq C|\beta|^{-2} \int_M |d\chi|_g^2 \bar{u}^{\beta+1} dv_g \\ &\quad + C|\beta|^{-1} \int_M \chi^2 (|P| + k^{-1}|f|) \bar{u}^{\beta+1} dv_g \\ &\quad + C|\beta|^{-1} \int_{\partial M} \chi^2 (|\bar{P}| + k^{-1}|\bar{f}|) \bar{u}^{\beta+1} d\sigma_g. \end{aligned} \quad (\text{A.3})$$

Set $h = |P| + k^{-1}|f|$, $\bar{h} = |\bar{P}| + k^{-1}|\bar{f}|$ and

$$w = \begin{cases} \bar{u}^{\frac{\beta+1}{2}} & \text{if } \beta \neq -1 \\ \log \bar{u} & \text{if } \beta = -1. \end{cases}$$

Then (A.3) can be rewritten as

$$\begin{aligned} \int_M \chi^2 |dw|_g^2 dv_g &\leq C \frac{(\beta+1)^2}{|\beta|^2} \int_M |d\chi|_g^2 w^2 dv_g \\ &\quad + C \frac{(\beta+1)^2}{|\beta|} \int_M \chi^2 h w^2 dv_g \\ &\quad + C \frac{(\beta+1)^2}{|\beta|} \int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g \end{aligned} \quad (\text{A.4})$$

if $\beta \neq -1$ and

$$\int_M \chi^2 |dw|_g^2 dv_g \leq C \int_M |d\chi|_g^2 dv_g + C \int_M \chi^2 h dv_g + C \int_{\partial M} \chi^2 \bar{h} d\sigma_g \quad (\text{A.5})$$

if $\beta = -1$. It follows from $\chi^2 |dw|_g^2 \geq \frac{1}{2} |d(\chi w)|_g^2 - w^2 |d\chi|_g^2$ and Sobolev inequalities that

$$\left(\int_M (\chi w)^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} - C \int_M |d\chi|_g^2 w^2 dv_g \leq C \int_M \chi^2 |dw|_g^2 dv_g \quad (\text{A.6})$$

In order to handle the right-hand side of (A.4) we use Hölder's and interpolation inequalities to get

$$\begin{aligned} \int_M \chi^2 h w^2 dv_g &\leq \|h\|_{L^{q/2}(B_4^+)} \|\chi w\|_{L^{2q/(q-2)}(B_4^+)}^2 \\ &\leq \|h\|_{L^{q/2}(B_4^+)} (\epsilon^{1/2} \|\chi w\|_{L^{2n/(n-2)}(B_4^+)} + \epsilon^{-\mu_1/2} \|\chi w\|_{L^2(B_4^+)})^2 \quad (\text{A.7}) \\ &\leq 2\|h\|_{L^{q/2}(B_4^+)} (\epsilon \|\chi w\|_{L^{2n/(n-2)}(B_4^+)}^2 + \epsilon^{-\mu_1} \|\chi w\|_{L^2(B_4^+)}^2) \end{aligned}$$

where $\mu_1 = n/(q - n)$, and

$$\begin{aligned} \int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g &\leq \|\bar{h}\|_{L^s(D_4)} \|\chi w\|_{L^{2s/(s-1)}(D_4)}^2 \\ &\leq \|\bar{h}\|_{L^s(D_4)} (\epsilon^{1/2} \|\chi w\|_{L^{2(n-1)/(n-2)}(D_4)} + \epsilon^{-\mu_2/2} \|\chi w\|_{L^2(D_4)})^2 \quad (\text{A.8}) \\ &\leq 2\|\bar{h}\|_{L^s(D_4)} (\epsilon \|\chi w\|_{L^{2(n-1)/(n-2)}(D_4)}^2 + \epsilon^{-\mu_2} \|\chi w\|_{L^2(D_4)}^2) \end{aligned}$$

where $\mu_2 = (n - 1)/(s + 1 - n)$. It follows from the Sobolev embedding theorems that

$$\epsilon^{-\mu_2} \int_{D_4} (\chi w)^2 d\sigma_g \leq \epsilon \int_{B_4^+} |d(\chi w)|_g^2 dv_g + \epsilon^{-2\mu_2-1} \int_{B_4^+} (\chi w)^2 dv_g$$

and

$$\left(\int_{D_4} (\chi w)^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}} \leq C \int_{B_4^+} |d(\chi w)|_g^2 dv_g.$$

Then the inequality (A.8) becomes

$$\begin{aligned} \int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g &\leq C \epsilon \|\bar{h}\|_{L^s(D_4)} \int_{B_4^+} |d(\chi w)|_g^2 dv_g \\ &\quad + C \epsilon^{-2\mu_2-1} \|\bar{h}\|_{L^s(D_4)} \int_{B_4^+} (\chi w)^2 dv_g. \end{aligned} \quad (\text{A.9})$$

Choosing $\epsilon = c|\beta|(\beta + 1)^{-2} \Lambda^{-1}$ with $c > 0$ small, we can make use of the inequalities (A.6), (A.7), (A.8) and (A.9) in (A.4) to obtain

$$\left(\int_{B_4^+} (\chi w)^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq C(1 + |\gamma|)^{2\mu} \int_{B_4^+} (|d\chi|_g^2 + \chi^2) w^2 dv_g. \quad (\text{A.10})$$

Here, $\gamma = \beta + 1$, $\mu = \max\{\mu_1 + 1, 2\mu_2 + 2\}$, and C depends on Λ and is bounded when $|\beta|$ is bounded away from zero.

For any $1 \leq r_a \leq r_b \leq 3$ we choose χ as a cut-off function satisfying $0 \leq \chi \leq 1$, $|d\chi| \leq 2/(r_b - r_a)$ and

$$\begin{cases} \chi \equiv 1 & \text{in } B_{r_a}^+ \\ \chi \equiv 0 & \text{in } B_4^+ \setminus B_{r_b}^+. \end{cases}$$

Using this in (A.10) we obtain

$$\left(\int_{B_{r_a}^+} \bar{u}^{\frac{\gamma n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq \frac{C(1+|\gamma|)^{2\mu}}{r_b - r_a} \int_{B_{r_b}^+} \bar{u}^\gamma dv_g. \quad (\text{A.11})$$

If we set $\Phi(e, r) = \left(\int_{B_r^+} \bar{u}^e dv_g \right)^{1/e}$ and $\delta = n/(n-2)$, the estimate (A.11) becomes

$$\begin{cases} \Phi(\delta\gamma, r_a) \leq \left(\frac{C(1+|\gamma|)^\mu}{r_b - r_a} \right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_b) & \text{if } \gamma > 0 \\ \Phi(\gamma, r_b) \leq \left(\frac{C(1+|\gamma|)^\mu}{r_b - r_a} \right)^{\frac{2}{|\gamma|}} \Phi(\delta\gamma, r_a) & \text{if } \gamma < 0. \end{cases} \quad (\text{A.12})$$

It is well known that $\lim_{e \rightarrow \infty} \Phi(e, r) = \sup_{B_r^+} \bar{u}$ and $\lim_{e \rightarrow -\infty} \Phi(e, r) = \inf_{B_r^+} \bar{u}$. The rest of the proof follows as in [17, p.197-198] by iterating the first inequality in (A.12) to prove (a), and by using (A.5) and iterating the second inequality in (A.12) to prove (b). $\square$

Once we have established Proposition A.2(a), the proof of the next proposition is similar to [1, Proposition A.3].

Proposition A.3. *Let (M^n, g_0) be a compact Riemannian manifold with boundary ∂M and with dimension $n \geq 3$. For each $q > n/2$ we can find positive constants $\gamma_1 = \gamma_1(M, g_0, q)$ and $C = C(M, g_0, q)$ with the following significance: if $g = u^{\frac{4}{n-2}} g_0$ is a conformal metric satisfying*

$$\int_M dv_g \leq 1 \quad \text{and} \quad \int_{B_r(x)} |R_g|^q dv_g \leq \gamma_1$$

for $x \in M$, then we have

$$u(x) \leq Cr^{-\frac{n-2}{2}} \left(\int_{B_r(x)} dv_g \right)^{\frac{n-2}{2n}}.$$

Using Proposition A.2(b) and interior Harnack estimates for elliptic linear equations (see [17, Theorem 8.18]), one can prove the next proposition by adapting the arguments in [8, Proposition A.2].

Proposition A.4. *Let (M, g_0) be a Riemannian manifold with boundary ∂M , P a smooth function on M , and suppose u that satisfies*

$$\begin{cases} -\Delta_{g_0} u(t) + Pu \geq 0 & \text{in } M \\ \frac{\partial}{\partial \eta_{g_0}} u = 0 & \text{on } \partial M. \end{cases}$$

Then there exists $C = C(P, g_0)$ such that

$$C \inf_M u \geq \int_M u dv_{g_0}.$$

In particular,

$$\int_M u^{\frac{2n}{n-2}} dv_{g_0} \leq C \inf_M u \left(\sup_M u \right)^{\frac{n+2}{n-2}}.$$

B. Construction of the Green function on manifolds with boundary

In this section we prove the existence of the Green function used in this paper and some of its properties. The construction performed here extends the one in [1, Proposition B-2]; see also [14, page 201] and [5, page 106].

Lemma B.1. *Let (M, g) be a connected Riemannian manifold of dimension $n \geq 2$ and fix $x \in M$ and $\alpha \in \mathbb{R}$. Let $u : M \setminus \{x\} \rightarrow \mathbb{R}$ be a function satisfying*

$$|u(y)| \leq C_0 d_g(x, y)^\alpha \quad \text{and} \quad |\nabla_g u(y)|_g \leq C_0 d_g(x, y)^{\alpha-1},$$

for any $y \in M$, with $x \neq y$. Then, for any $0 < \theta \leq 1$, there exists $C_1 = C_1(M, g, C_0, \alpha)$ such that

$$|u(y) - u(z)| \leq C_1 d_g(y, z)^\theta (d_g(x, y)^{\alpha-\theta} + d_g(x, z)^{\alpha-\theta})$$

for any $y, z \in M$, with $y \neq x \neq z$.

This is [1, Lemma B.1]. For the reader's convenience, we provide the proof here.

Proof. Let $y \neq x$ and $z \neq x$.

1st case: $d_g(y, z) \leq \frac{1}{2} d_g(x, y)$. Let $\gamma : [0, 1] \rightarrow M$ be a smooth curve such that $\gamma(0) = y, \gamma(1) = z$, and $\int_0^1 |\gamma'(t)|_g dt \leq \frac{3}{2} d_g(y, z)$.

Claim. We have $\frac{1}{4} d_g(x, y) \leq d_g(\gamma(t), x) \leq \frac{7}{4} d_g(x, y)$.

Indeed, since $d_g(y, \gamma(t)) \leq \frac{3}{2} d_g(y, z) \leq \frac{3}{4} d_g(x, y)$, we have

$$d_g(x, \gamma(t)) \geq d_g(x, y) - d_g(\gamma(t), y) \geq d_g(x, y) - \frac{3}{4} d_g(x, y) = \frac{1}{4} d_g(x, y).$$

Moreover,

$$d_g(\gamma(t), x) \leq d_g(\gamma(t), y) + d_g(y, x) \leq \frac{3}{4} d_g(x, y) + d_g(x, y) = \frac{7}{4} d_g(x, y).$$

This proves the claim.

Observe that $u(z) - u(y) = \int_0^1 g(\nabla_g u(\gamma(t)), \gamma'(t)) dt$. Thus,

$$\begin{aligned} |u(y) - u(z)| &\leq \sup_{t \in [0,1]} |\nabla_g u(\gamma(t))|_g \int_0^1 |\gamma'(t)|_g dt \\ &\leq C \sup_{t \in [0,1]} d_g(\gamma(t), x)^{\alpha-1} \frac{3}{2} d_g(y, z) \\ &\leq C(\alpha) d_g(x, y)^{\alpha-1} d_g(y, z) \leq C(\alpha) d_g(x, y)^{\alpha-\theta} d_g(y, z)^\theta. \end{aligned}$$

2nd case: $d_g(y, z) > \frac{1}{2} d_g(x, y)$. In this case, we have

$$\begin{aligned} |u(y) - u(z)| &\leq |u(y)| + |u(z)| \leq C d_g(y, x)^\alpha + C d_g(z, x)^\alpha \\ &\leq C d_g(y, x)^{\alpha-\theta} d_g(z, y)^\theta + C d_g(z, x)^{\alpha-\theta} (d_g(x, y) + d_g(y, z))^\theta \\ &\leq C d_g(y, z)^\theta (d_g(x, y)^{\alpha-\theta} + d_g(x, z)^{\alpha-\theta}). \quad \square \end{aligned}$$

Let (M, g) be a compact Riemannian manifold with boundary ∂M , dimension $n \geq 3$, and positive Sobolev quotient $Q(M)$.

Notation. We denote by L_g the conformal Laplacian $\Delta_g - \frac{n-2}{4(n-1)} R_g$, and by B_g the boundary conformal operator $\frac{\partial}{\partial \eta_g} - \frac{n-2}{2(n-1)} H_g$, where η_g is the inward unit normal vector to ∂M .

Set $d(x) = d_g(x, \partial M)$ for $x \in M$, and $M_\rho = \{x \in M; d(x) < \rho\}$ for $\rho > 0$. Choose $\tilde{\rho}_0 = \tilde{\rho}_0(M, g) > 0$ small such that the function

$$\begin{aligned} M_{2\tilde{\rho}_0} &\rightarrow \partial M \\ x &\mapsto \bar{x} \end{aligned}$$

is well defined and smooth, where $\bar{x}$ is defined by $d_g(x, \bar{x}) = d_g(x, \partial M)$, and $\tilde{\rho}_0/4$ is smaller than the injectivity radius of M . Then, for any $0 < t < 2\tilde{\rho}_0$, the set $\partial_t M = \{x \in M; d(x) = t\}$ is a smooth embedded $(n-1)$ -submanifold of M . For each $x \in M_{\tilde{\rho}_0}$, define the function

$$\begin{aligned} M_{2\tilde{\rho}_0} &\rightarrow \partial_{d(x)} M \\ y &\mapsto y_x, \end{aligned}$$

where y_x is defined by $d_g(y, y_x) = d_g(y, \partial_{d(x)} M)$.

For any $x \in M_{\rho_0}$ and $\rho_0 \in (0, \tilde{\rho}_0)$, we define the local coordinates $\psi_x(y) = (y_1, \dots, y_n)$ on $M_{2\rho_0}$, where $y_n = d(y)$, and $(y_1, \dots, y_{n-1})$ are normal coordinates of y_x , centered at x , with respect to the submanifold $\partial_{d(x)} M$. Then $(x, y) \mapsto \psi_x(y)$ is locally defined and smooth. Observe that $\psi_x(x) = (0, \dots, 0, d(x))$ for any $x \in M_{\rho_0}$, and that ψ_x are Fermi coordinates if $x \in \partial M$. Moreover, in those coordinates we have $g_{an} \equiv \delta_{an}$ and $g_{ab}(x) = \delta_{ab}$, for $a, b = 1, \dots, n$, and the inward normal unit vector to ∂M is $d\psi_x^{-1}(\partial/\partial y_n)$, see Figure B.1. Choosing $\tilde{\rho}_0$ possibly smaller, we can assume that, for any $x \in M_{\tilde{\rho}_0}$, $\psi_x(y) = (y_1, \dots, y_n)$ is defined for $0 \leq y_n < 2\tilde{\rho}_0$ and $|(y_1, \dots, y_{n-1})| < \tilde{\rho}_0$.

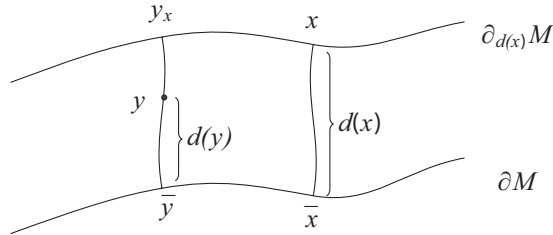


Figure B.1. Illustration of the notation.

Proposition B.2. Let $\rho_0 \in (0, \tilde{\rho}_0)$, $x_0 \in M$ and $d = \left\lfloor \frac{n-2}{2} \right\rfloor$. Suppose that one of the following conditions holds:

(a) $x_0 \in \partial M$ and there exist $C = C(M, g)$ and N sufficiently large such that

$$H_g(y) \leq C d_g(x_0, y)^N, \quad \text{for all } y \in \partial M; \quad (\text{B.1})$$

(b) $x_0 \in M_{\rho_0/2}$ and $H_g \equiv 0$ on ∂M ;

(c) $x_0 \in M \setminus M_{2\rho_0}$.

Then there exists a positive $G_{x_0} \in C^\infty(M \setminus \{x_0\})$ satisfying

$$\begin{cases} L_g G_{x_0} = 0 & \text{in } M \setminus \{x_0\} \\ B_g G_{x_0} = 0 & \text{on } \partial M \setminus \{x_0\}, \end{cases} \quad (\text{B.2})$$

$$\phi(x_0) = - \int_M G_{x_0}(y) L_g \phi(y) dv_g(y) - \int_{\partial M} G_{x_0}(y) B_g \phi(y) d\sigma_g(y) \quad (\text{B.3})$$

for any $\phi \in C^2(M)$. Moreover, the following properties hold:

(P1) There exists $C = C(M, g)$ such that, for any $y \in M$ with $y \neq x_0$,

$$|G_{x_0}(y)| \leq C d_g(x_0, y)^{2-n} \quad \text{and} \quad |\nabla_g G_{x_0}(y)| \leq C d_g(x_0, y)^{1-n};$$

(P2) If $x_0 \in \partial M$ consider Fermi coordinates $y = (y_1, \dots, y_n)$ centered at that point. In those coordinates, write $g_{ab} = \exp(h_{ab})$, $a, b = 1, \dots, n$, where

$$\left| h_{ab}(y) - \sum_{|\alpha|=1}^d h_{ab,\alpha} y^\alpha \right| \leq C(M, g) |y|^{d+1}, \quad (\text{B.4})$$

where $h_{ab,\alpha} \in \mathbb{R}$ and each α stands for a multi-index. Then there exists $C = C(M, g, \rho_0)$ such that⁶

$$\begin{aligned} \left| G_{x_0}(y) - \frac{2|y|^{2-n}}{(n-2)\sigma_{n-1}} \right| &\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ab,\alpha}| d_g(x_0, y)^{|\alpha|+2-n} \\ &\quad + \begin{cases} C d_g(x_0, y)^{d+3-n} & \text{if } n \geq 5 \\ C(1 + |\log d_g(x_0, y)|) & \text{if } n = 3, 4 \end{cases} \quad (\text{B.5}) \\ \left| \nabla_g \left(G_{x_0}(y) - \frac{2|y|^{2-n}}{(n-2)\sigma_{n-1}} \right) \right| &\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |h_{ab,\alpha}| d_g(x_0, y)^{|\alpha|+1-n} \\ &\quad + C d_g(x_0, y)^{d+2-n}; \end{aligned}$$

(P3) If $x_0 \in M_{\rho_0/2}$ consider the coordinate system ψ_{x_0} defined above. Then there exists $C = C(M, g, \rho_0)$ such that

$$\begin{aligned} \left| G_{x_0}(y) - \frac{1}{(n-2)\sigma_{n-1}} [|(y_1, \dots, y_{n-1}, y_n - d(x_0))|^{2-n} \right. \\ \left. + |(y_1, \dots, y_{n-1}, y_n + d(x_0))|^{2-n}] \right| &\leq C d_g(x_0, y)^{3-n}, \\ \left| \nabla_g \left(G_{x_0}(y) - \frac{1}{(n-2)\sigma_{n-1}} [|(y_1, \dots, y_{n-1}, y_n - d(x_0))|^{2-n} \right. \right. \\ \left. \left. + |(y_1, \dots, y_{n-1}, y_n + d(x_0))|^{2-n}] \right) \right| &\leq C d_g(x_0, y)^{2-n}, \end{aligned}$$

if $n \geq 4$ and

$$\begin{aligned} \left| G_{x_0}(y) - \frac{1}{(n-2)\sigma_{n-1}} [|(y_1, \dots, y_{n-1}, y_n - d(x_0))|^{2-n} \right. \\ \left. + |(y_1, \dots, y_{n-1}, y_n + d(x_0))|^{2-n}] \right| &\leq C(1 + |\log d_g(x_0, y)|), \\ \left| \nabla_g \left(G_{x_0}(y) - \frac{1}{(n-2)\sigma_{n-1}} [|(y_1, \dots, y_{n-1}, y_n - d(x_0))|^{2-n} \right. \right. \\ \left. \left. + |(y_1, \dots, y_{n-1}, y_n + d(x_0))|^{2-n}] \right) \right| &\leq C d_g(x_0, y)^{-1}, \end{aligned}$$

if $n = 3$;

⁶ The log term in dimensions 3 and 4 should also be included in [1, Proposition B-1]. However, that term does not affect the results in [1] as observed in the footnote in Proposition 3.14 above.

(P4) If $x_0 \in M \setminus M_{2\rho_0}$ consider normal coordinates $y = (y_1, \dots, y_n)$ centered at that point. As in (P2), write $g_{ab} = \exp(h_{ab})$ where h_{ab} satisfies (B.4). Then there exists $C = C(M, g, \rho_0)$ such that the estimates (B.5) hold. (Observe that in this case the sums range from $|\alpha| = 2$ to d instead of from $|\alpha| = 1$ to d .)

Remark B.3. The identity (B.3) and the estimates in (P2) and (P3) may change according to the normalization chosen for G_{x_0} . Notice that different ones have been used in the rest of the paper.

Proof. Let $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth cutoff function satisfying $\chi(t) = 1$ for $t < \rho_0/2$, and $\chi(t) = 0$ for $t \geq \rho_0$. For each $x \in M_{\rho_0}$, set

$$K_1(x, y) = \frac{\chi(y_n/2)\chi(|(y_1, \dots, y_{n-1})|)}{(n-2)\sigma_{n-1}} \cdot \left\{ |(y_1, \dots, y_{n-1}, y_n - d(x))|^{2-n} + |(y_1, \dots, y_{n-1}, y_n + d(x))|^{2-n} \right\},$$

where we are using the coordinates $\psi_x(y) = (y_1, \dots, y_n)$. Observe that

$$\sum_{a=1}^n \frac{\partial^2}{\partial y_a^2} K_1(x, y) = 0, \quad \text{for } |(y_1, \dots, y_{n-1})| < \rho_0/2, \quad 0 \leq y_n < \rho_0, \quad \text{and } x \neq y.$$

Moreover, $\partial K_1 / \partial y_n(x, y) = 0$ if $y \in \partial M$ with $x \neq y$.

For each $x \in M \setminus M_{\rho_0/2}$, set

$$K_2(x, y) = \frac{\chi(4d_g(y, x))}{(n-2)\sigma_{n-1}} d_g(y, x)^{2-n}, \quad \text{if } 0 < d_g(y, x) < \rho_0/4.$$

If we express $y \mapsto K_2(x, y)$ in normal coordinates $(y_1, \dots, y_n)$ centered at x , we have $K_2(x, y) = \chi(4|(y_1, \dots, y_n)|)|y_1, \dots, y_n|^{2-n}$, and thus

$$\sum_{a=1}^n \frac{\partial^2}{\partial y_a^2} K_2(x, y) = 0, \quad \text{for } 0 < d_g(y, x) < \rho_0/8.$$

Define $K : M \times M \setminus D_M \rightarrow \mathbb{R}$ by the expression

$$K(x, y) = \chi(d(x))K_1(x, y) + (1 - \chi(d(x)))K_2(x, y),$$

where $D_M = \{(x, x) \in M \times M; x \in M\}$. Thus, $K(x, y) = K_1(x, y)$ if $x \in M_{\rho_0/2}$, and $K(x, y) = K_2(x, y)$ if $x \in M \setminus M_{\rho_0}$. Observe that $\partial K / \partial \eta_{g,y}(x, y) = 0$ if $y \in \partial M$ with $y \neq x$.

Expressing $y \mapsto K_1(x, y)$ and $y \mapsto K_2(x, y)$ in their respective coordinate systems (as described above) one can check that there exists $C = C(M, g, \rho_0)$ such that

$$|L_{g,y}K(x, y)| \leq C d_g(x, y)^{1-n}.$$

For any $\phi \in C^2(M)$ and $x \in M$, we have

$$\begin{aligned} \phi(x) &= \int_M (\Delta_{g,y} K(x, y) \phi(y) - K(x, y) \Delta_g \phi(y)) dv_g(y) \\ &\quad - \int_{\partial M} K(x, y) \frac{\partial}{\partial \eta_g} \phi(y) d\sigma_g(y). \end{aligned} \quad (\text{B.6})$$

Indeed, this expression holds with $K_1(x, y)$ replacing $K(x, y)$ when $x \in M_{\rho_0/2}$, and with $K_2(x, y)$ replacing $K(x, y)$ when $x \in M \setminus M_{\rho_0}$.

We define $\Gamma_k : M \times M \setminus D_M \rightarrow \mathbb{R}$ inductively by setting

$$\Gamma_1(x, y) = L_{g,y} K(x, y)$$

and

$$\Gamma_{k+1}(x, y) = \int_M \Gamma_k(x, z) \Gamma_1(z, y) dv_g(z).$$

According to [5, Proposition 4.12], which is a result due to Giraud ([18, p.50]), we have

$$|\Gamma_k(x, y)| \leq \begin{cases} C d_g(x, y)^{k-n} & \text{if } k < n \\ C(1 + |\log d_g(x, y)|) & \text{if } k = n \\ C & \text{if } k > n, \end{cases} \quad (\text{B.7})$$

for some $C = C(M, g, \rho_0)$. Moreover, Γ_k is continuous on $M \times M$ for $k > n$, and on $M \times M \setminus D_M$ for $k \leq n$.

If (a) or (b) holds we can refine the estimate (B.7) around the point x_0 , using the expansion $g_{ab} = \exp(h_{ab})$. Since $K(x, y) = K_1(x, y)$ for $x \in M_{\rho_0/2}$ and $K(x, y) = K_2(x, y)$ for $x \in M \setminus M_{\rho_0}$, one can see that

$$|L_{g,y} K(x_0, y)| \leq C \sum_{a,b=1}^n \sum_{|\alpha|=1}^d |h_{ab,\alpha}| d_g(x_0, y)^{|\alpha|-n} + C d_g(x_0, y)^{d+1-n},$$

for some $C = C(M, g, \rho_0)$, if (a) or (b) holds. Then Giraud's result implies

$$\begin{aligned} |\Gamma_k(x_0, y)| &\leq C \sum_{a,b=1}^n \sum_{|\alpha|=1}^d |h_{ab,\alpha}| d_g(x_0, y)^{k-1+|\alpha|-n} \\ &\quad + d_g(x_0, y)^{k+d-n}, \quad \text{if } k < n - d. \end{aligned} \quad (\text{B.8})$$

Claim 1. Given $0 < \theta < 1$, there exists $C = C(M, g, \rho_0, \theta)$ such that

$$|\Gamma_{n+1}(x, y) - \Gamma_{n+1}(x, y')| \leq C d_g(y, y')^\theta, \quad \text{for any } y \neq x \neq y'. \quad (\text{B.9})$$

In particular, $\Gamma_{n+1}(x_0, \cdot) \in C^{0,\theta}(M)$.

Indeed, observe that $|\Gamma_1(x, y) - \Gamma_1(x, y')| \leq C d_g(y, y')^\theta (d_g(x, y)^{1-\theta-n} + d_g(x, y')^{1-\theta-n})$, according to Lemma B.1. So, Claim 1 follows from the estimates (B.7) and Giraud's result.

Set

$$F_k(x, y) = K(x, y) + \sum_{j=1}^k \int_M \Gamma_j(x, z) K(z, y) dv_g(z).$$

Claim 2. For any $\phi \in C^2(M)$ and $x \in M$, and for all $k = 1, 2, \dots$, we have

$$\begin{aligned} \phi(x) &= - \int_M F_k(x, y) L_g \phi(y) dv_g(y) - \int_{\partial M} F_k(x, y) B_g \phi(y) d\sigma_g(y) \\ &\quad + \int_M \Gamma_{k+1}(x, y) \phi(y) dv_g(y) \\ &\quad - \int_{\partial M} \frac{n-2}{2(n-1)} H_g(y) F_k(x, y) \phi(y) d\sigma_g(y). \end{aligned} \quad (\text{B.10})$$

Claim 2 can be proved by induction on k .

Claim 3. For any $x \in M$ and $0 < \theta < 1$, the function $y \mapsto F_n(x, y)$ is in $C^{1,\theta}(M \setminus \{x\})$ and satisfies

$$|F_n(x, y)| \leq C d_g(x, y)^{2-n}, \quad |\nabla_{g,y} F_n(x, y)|_g \leq C d_g(x, y)^{1-n}, \quad (\text{B.11})$$

and

$$\frac{|\nabla_{g,y} F_n(x, y) - \nabla_{g,y'} F_n(x, y')|_g}{d_g(y, y')^\theta} \leq C d_g(x, y)^{1-\theta-n} + C d_g(x, y')^{1-\theta-n}, \quad (\text{B.12})$$

for some $C = C(M, g, \rho_0)$. In particular, for any $x \in \partial M$, $y \mapsto \partial F_n / \partial \eta_{g,y}(x, y)$ defines a continuous function on $\partial M \setminus \{x\}$.

As a consequence of Claim 3, if $x_0 \in \partial M$ we can choose N large enough in the hypothesis (a) such that $y \mapsto H_g(y) F_n(x_0, y)$ is in $C^{1,\theta}(\partial M)$ for $0 < \theta < 1$ and satisfies

$$\|H_g(\cdot) F_n(x_0, \cdot)\|_{C^{1,\theta}(\partial M)} \leq C(M, g, \rho_0, \theta). \quad (\text{B.13})$$

It is clear that (B.13) also holds if $x_0 \in M \setminus M_{\rho_0}$ with no assumptions on H_g , and that its left-hand side vanishes under the hypothesis (b). In particular (B.13) holds should (a), (b) or (c) holds.

Let us prove Claim 3. Choose $y \neq x$ and a smooth curve y_t such that $y_0 = y$. Then, for any $r > 0$,

$$\frac{d}{dt} \int_{M \setminus B_r(y)} \Gamma_j(x, z) K(z, y_t) dv_g(z) = \int_{M \setminus B_r(y)} \Gamma_j(x, z) \frac{d}{dt} K(z, y_t) dv_g(z)$$

For any $r > 0$ such that $2r < d_g(x, y)$ and t small, we have

$$\begin{aligned} & \int_{B_r(y)} \Gamma_j(x, z) \left| \frac{K(z, y_t) - K(z, y)}{t} \right| dv_g(z) \\ & \leq C \int_{B_r(y)} d_g(x, z)^{1-n} (d_g(z, y_t)^{1-n} + d_g(z, y)^{1-n}) dv_g(z) \\ & \leq C 2^{n-1} d_g(x, y)^{1-n} \int_{B_r(y)} (d_g(z, y_t)^{1-n} + d_g(z, y)^{1-n}) dv_g(z) \end{aligned}$$

and the right-hand side goes to 0 as $r \rightarrow 0$. Here, $B_r(y)$ stands for the geodesic ball centered at y . Hence,

$$\frac{d}{dt} \int_M \Gamma_j(x, z) K(z, y_t) dv_g(z) = \int_M \Gamma_j(x, z) \frac{d}{dt} K(z, y_t) dv_g(z) \quad (\text{B.14})$$

and the estimates in (B.11) follow from Giraud's result.

Now,

$$\begin{aligned} & \frac{1}{d_g(y, y')^\theta} \left| \int_M \Gamma_j(x, z) \frac{\partial}{\partial y_i} K(z, y) dv_g(z) - \int_M \Gamma_j(x, z) \frac{\partial}{\partial y_i} K(z, y') dv_g(z) \right| \\ & \leq \int_M \Gamma_j(x, z) \left| \frac{\frac{\partial}{\partial y_i} K(z, y) - \frac{\partial}{\partial y_i} K(z, y')}{d_g(y, y')^\theta} \right| dv_g(z) \\ & \leq C \int_M d_g(x, z)^{1-n} (d_g(z, y)^{1-\theta-n} + d_g(z, y')^{1-\theta-n}) dv_g(z) \\ & \leq C (d_g(x, y)^{2-\theta-n} + d_g(x, y')^{2-\theta-n}), \end{aligned}$$

where we used Lemma B.1 in the second inequality, and Giraud's result in the last one.

This proves Claim 3.

Using the hypothesis $Q(M) > 0$, we define $u_{x_0} \in C^{2,\theta}(M)$ as the unique solution of

$$\begin{cases} L_g u_{x_0}(y) = -\Gamma_{n+1}(x_0, y) & \text{in } M \\ B_g u_{x_0}(y) = \frac{n-2}{2(n-1)} H_g(y) F_n(x_0, y) & \text{on } \partial M. \end{cases} \quad (\text{B.15})$$

It satisfies

$$\begin{aligned} \|u_{x_0}\|_{C^{2,\theta}(M)} & \leq C \|u_{x_0}\|_{C^0(M)} + C \|\Gamma_{n+1}(x_0, \cdot)\|_{C^{0,\theta}(M)} \\ & \quad + C \|H_g(\cdot) F_n(x_0, \cdot)\|_{C^{1,\theta}(\partial M)} \end{aligned} \quad (\text{B.16})$$

where $C = C(M, g, \rho_0, \theta)$ (see [17, Theorems 6.30 and 6.31]).

Claim 4. There exists $C = C(M, g, \rho_0, \theta)$ such that $\|u_{x_0}\|_{C^{2,\theta}(M)} \leq C$.

Indeed, using (B.10) with $k = n$ and any $\phi \in C^2(M)$, one can see that

$$\sup_M |\phi| \leq C \sup_M |L_g \phi| + C \sup_{\partial M} |B_g \phi| + C \|\phi\|_{L^2(M)} + C \|\phi\|_{L^2(\partial M)}.$$

Since $Q(M) > 0$, there exists $C = C(M, g)$ such that

$$\int_M \phi^2 dv_g + \int_{\partial M} \phi^2 d\sigma_g \leq C \int_M |L_g(\phi)\phi| dv_g + C \int_{\partial M} |B_g(\phi)\phi| d\sigma_g.$$

Thus, Young's inequality implies

$$\int_M \phi^2 dv_g + \int_{\partial M} \phi^2 d\sigma_g \leq C \int_M L_g(\phi)^2 dv_g + C \int_{\partial M} B_g(\phi)^2 d\sigma_g.$$

Hence, $\|\phi\|_{C^0(M)} \leq C \|L_g \phi\|_{C^0(M)} + C \|B_g \phi\|_{C^0(\partial M)}$. Setting $\phi = u_{x_0}$ and using the equations (B.15), we see that

$$\|u_{x_0}\|_{C^0(M)} \leq C \|\Gamma_{n+1}(x_0, \cdot)\|_{C^0(M)} + C \|H_g(\cdot)F_n(x_0, \cdot)\|_{C^0(\partial M)}. \quad (\text{B.17})$$

Claim 4 follows from the estimates (B.7), (B.9), (B.13), (B.16), and (B.17).

We define the function $G_{x_0} \in C^{1,\theta}(M \setminus \{x_0\})$ by

$$G_{x_0}(y) = K(x_0, y) + \sum_{k=1}^n \int_M \Gamma_i(x_0, z) K(z, y) dv_g(z) + u_{x_0}(y).$$

One can check that the formula (B.3) holds.

Claim 5. We have $G_{x_0} \in C^\infty(M \setminus \{x_0\})$ and (B.2).

In order to prove Claim 5, we rewrite (B.6) as

$$\begin{aligned} & \int_M K(x, y) L_g \phi(y) dv_g(y) + \int_{\partial M} K(x, y) B_g \phi(y) d\sigma_g(y) \\ &= \int_M L_{g,y} K(x, y) \phi(y) dv_g(y) - \phi(x) \\ & \quad - \int_{\partial M} \frac{n-2}{2(n-1)} H_g(y) K(x, y) \phi(y) d\sigma_g(y). \end{aligned} \quad (\text{B.18})$$

Thus,

$$\begin{aligned}
 & \int_M \left\{ \int_M \Gamma_j(x, z) K(z, y) dv_g(z) \right\} L_g \phi(y) dv_g(y) \\
 & + \int_{\partial M} \left\{ \int_M \Gamma_j(x, z) K(z, y) dv_g(z) \right\} B_g \phi(y) d\sigma_g(y) \\
 & = \int_M \Gamma_j(x, z) \left\{ \int_M K(z, y) L_g \phi(y) dv_g(y) + \int_{\partial M} K(z, y) B_g \phi(y) d\sigma_g(y) \right\} dv_g(z) \\
 & = \int_M \Gamma_j(x, z) \int_M L_{g,y} K(z, y) \phi(y) dv_g(y) dv_g(z) \\
 & \quad - \int_M \Gamma_j(x, z) \left\{ \int_{\partial M} \frac{n-2}{2(n-1)} H_g(y) K(z, y) \phi(y) d\sigma_g(y) + \phi(z) \right\} dv_g(z) \\
 & = \int_M \left\{ \int_M \Gamma_j(x, z) L_{g,y} K(z, y) dv_g(z) - \Gamma_j(x, y) \right\} \phi(y) dv_g(y) \\
 & \quad - \int_{\partial M} \left\{ \int_M \Gamma_j(x, z) K(z, y) dv_g(z) \right\} \frac{n-2}{2(n-1)} H_g(y) \phi(y) d\sigma_g(y),
 \end{aligned}$$

where we used (B.18) in the second equality. Hence, we proved that the equations

$$\begin{cases} L_{g,y} \int_M \Gamma_j(x, z) K(z, y) dv_g(z) \\ \quad = \Gamma_{j+1}(x, y) - \Gamma_j(x, y) & \text{in } M \\ B_{g,y} \int_M \Gamma_j(x, z) K(z, y) dv_g(z) \\ \quad = -\frac{n-2}{2(n-1)} H_g(y) \int_M \Gamma_j(x, z) K(z, y) dv_g(z) & \text{on } \partial M, \end{cases}$$

hold in the sense of distributions. Then it is easy to check that the equations (B.2) hold in the sense of distributions. Since $G_{x_0} \in C^{1,\theta}(M \setminus \{x_0\})$, elliptic regularity arguments imply that $G_{x_0} \in C^\infty(M \setminus \{x_0\})$. This proves Claim 5.

The property (P1) follows from (B.11) and Claim 4. In order to prove (P2), (P3) and (P4), we use (B.7), (B.8), (B.14) and Claim 4.

Claim 6. The function G_{x_0} is positive on $M \setminus \{x_0\}$.

Let us prove Claim 6. Let

$$G_{x_0}^- = \begin{cases} -G_{x_0} & \text{if } G_{x_0} < 0 \\ 0 & \text{if } G_{x_0} \geq 0. \end{cases}$$

Since $G_{x_0}^-$ has support in $M \setminus \{x_0\}$, one has

$$\begin{aligned}
 0 & = - \int_M G_{x_0}^- L_g G_{x_0} dv_g - \int_{\partial M} G_{x_0}^- B_g G_{x_0} d\sigma_g \\
 & = \int_M \left(|\nabla_g G_{x_0}^-|^2 + \frac{n-2}{4(n-1)} R_g (G_{x_0}^-)^2 \right) dv_g + \int_{\partial M} \frac{n-2}{2(n-1)} H_g (G_{x_0}^-)^2 d\sigma_g.
 \end{aligned}$$

By the hypothesis $Q(M) > 0$, we have $G_{x_0}^- \equiv 0$ which implies $G_{x_0} \geq 0$.

We now change the metric by a conformal positive factor $u \in C^\infty(M)$ such that $\tilde{g} = u^{\frac{4}{n-2}}g$ satisfies $R_{\tilde{g}} > 0$ in M and $H_{\tilde{g}} \equiv 0$ on ∂M (see [16]). Observing the conformal properties (2.2) and (2.3), we see that $\tilde{G} = u^{-1}G_{x_0} \geq 0$ satisfies $L_{\tilde{g}}\tilde{G} = 0$ in $M \setminus \{x_0\}$ and $B_{\tilde{g}}\tilde{G} = 0$ on $\partial M \setminus \{x_0\}$. Then the strong maximum principle implies $\tilde{G} > 0$, proving Claim 6.

This finishes the proof of Proposition B.2. $\square$

Let (M, g_0) be a Riemannian manifold with $Q(M) > 0$ and $H_{g_0} \equiv 0$. Let $g_{x_0} = f_{x_0}^{\frac{4}{n-2}}g_0$ be a conformal metric satisfying

$$|f_{x_0}(x) - 1| \leq C(M, g_0)d_{g_0}(x, x_0).$$

Notation. For a Riemannian metric g we set $M_{t,g} = \{x \in M : d_g(x, \partial M) < t\}$ and $\partial_{t,g}M = \{x \in M : d_g(x, \partial M) = t\}$.

Proposition B.4. *If ρ_0 is sufficiently small and $x_0 \in M_{\rho_0, g_{x_0}} \setminus \partial M$, then there exists a positive $G_{x_0} \in C^\infty(M \setminus \{x_0\})$ satisfying*

$$\begin{cases} L_{g_{x_0}}G_{x_0} = 0 & \text{in } M \setminus \{x_0\} \\ B_{g_{x_0}}G_{x_0} = 0 & \text{on } \partial M, \end{cases} \quad (\text{B.19})$$

and there exists $C = C(M, g_0, \rho_0)$ such that

$$\begin{aligned} & |G_{x_0}(y) - |\phi_0(y)|^{2-n}| \\ & \leq \begin{cases} C|\phi_0(y)|^{3-n} + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{1-n} & n \geq 4 \\ C(1 + |\log(|\phi_0(y)|)|) + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{1-n} & n = 3, \end{cases} \end{aligned} \quad (\text{B.20})$$

$$|\nabla_{g_{x_0}}(G_{x_0}(y) - |\phi_0(y)|^{2-n})| \leq C|\phi_0(y)|^{1-n} + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{-n}, \quad (\text{B.21})$$

where $\phi_0(y) = (y_1, \dots, y_n)$ are g_{x_0} -normal coordinates centered at x_0 .

Proof. We will use the notation $d(x) = d_{g_0}(x, \partial M)$. Let us define the coordinate system $\psi_0(y) = (y_1, \dots, y_n)$ on M_{ρ_0, g_0} where $(y_1, \dots, y_{n-1})$ are normal coordinates of y_{x_0} on $\partial_{d(x_0), g_0}M$ centered at x_0 , with respect to the metric induced by g_0 , and $y_n = d(y) - d(x_0)$. Here, $y_{x_0} \in \partial_{d(x_0), g_0}M$ is such that $d_{g_0}(y, y_{x_0}) = d_{g_0}(y, \partial_{d(x_0), g_0}M)$. This differs from ψ_{x_0} defined above by a translation in the last coordinate.

According to Proposition B.2, multiplying it by some constant, one can construct a function G_0 , satisfying

$$\begin{cases} L_{g_0} G_0 = 0 & \text{in } M \setminus \{x_0\} \\ B_{g_0} G_0 = 0 & \text{on } \partial M, \end{cases}$$

$$\left| G_0(y) - \frac{1}{2} (|(y_1, \dots, y_n)|^{2-n} + |(y_1, \dots, y_{n-1}, y_n + 2d(x_0))|^{2-n}) \right|$$

$$\leq \begin{cases} C d_{g_0}(y, x_0)^{3-n} & n \geq 4 \\ C(1 + |\log d_{g_0}(y, x_0)|) & n = 3, \end{cases}$$

and

$$\left| \nabla_{g_0} (G_0(y) - \frac{1}{2} (|(y_1, \dots, y_n)|^{2-n} + |(y_1, \dots, y_{n-1}, y_n + 2d(x_0))|^{2-n})) \right|$$

$$\leq C d_{g_0}(y, x_0)^{2-n}.$$

for some $C = C(M, g_0, \rho_0)$. Using $|(y_1, \dots, y_{n-1}, y_n + 2d(x_0))| \geq |(y_1, \dots, y_n)|$ and Lemma B.1 we have

$$\left| |(y_1, \dots, y_n)|^{2-n} - |(y_1, \dots, y_{n-1}, y_n + 2d(x_0))|^{2-n} \right| \leq C d(x_0) |(y_1, \dots, y_n)|^{1-n},$$

$$\left| \nabla |(y_1, \dots, y_n)|^{2-n} - \nabla |(y_1, \dots, y_{n-1}, y_n + 2d(x_0))|^{2-n} \right| \leq C d(x_0) |(y_1, \dots, y_n)|^{-n}.$$

Then

$$\begin{aligned} & |G_0(y) - |\psi_0(y)|^{2-n}| \\ & \leq \begin{cases} C d_{g_0}(y, x_0)^{3-n} + C d(x_0) d_{g_0}(y, x_0)^{1-n} & n \geq 4 \\ C(1 + |\log d_{g_0}(y, x_0)|) + C d(x_0) d_{g_0}(y, x_0)^{1-n} & n = 3, \end{cases} \end{aligned} \quad (\text{B.22})$$

$$|\nabla_{g_0} (G_0(y) - |\psi_0(y)|^{2-n})| \leq C d_{g_0}(y, x_0)^{2-n} + C d(x_0) d_{g_0}(y, x_0)^{-n}. \quad (\text{B.23})$$

Now we change this to the conformal metric g_{x_0} . Let $\phi_0(y) = (y_1, \dots, y_n)$ be g_{x_0} -conformal normal coordinates centered at x_0 . By the definition of ϕ_0 and ψ_0 one can check that $\xi = \phi_0 \circ \psi_0^{-1}$ satisfies $\xi(0) = 0$ and $d\xi(0) = id_{\mathbb{R}^n}$. Since M is compact, one can find $C = C(M, g_0)$ uniform in x_0 such that

$$|\xi(y_1, \dots, y_n) - (y_1, \dots, y_n)| \leq C |(y_1, \dots, y_n)|^2. \quad (\text{B.24})$$

The function $G_{x_0} = f_{x_0}^{-1} G_0$ satisfies (B.19), so we shall prove (B.20) and (B.21). Observe that

$$|G_{x_0}(y) - G_0(y)| \leq C d_{g_0}(y, x_0) |G_{x_0}(y)| \leq C d_{g_0}(y, x_0)^{3-n}. \quad (\text{B.25})$$

Combining (B.22), (B.24) and (B.25), one gets (B.20) from the following steps:

$$\begin{aligned}
 |G_{x_0}(y) - |\phi_0(y)|^{2-n}| &\leq |G_{x_0}(y) - G_0(y)| + |G_0(y) - |\psi_0(y)|^{2-n}| \\
 &\quad + ||\psi_0(y)|^{2-n} - |\xi \circ \psi_0(y)|^{2-n}| \\
 &\leq Cd_{g_0}(y, x_0)^{3-n} + Cd(x_0)d_{g_0}(y, x_0)^{1-n} + C|\psi_0(y)|^{3-n} \\
 &\leq Cd_{g_0}(y, x_0)^{3-n} + Cd_{g_{x_0}}(x_0, \partial M)(x_0)d_{g_0}(y, x_0)^{1-n}
 \end{aligned}$$

for $n \geq 4$ and with obvious modifications for $n = 3$. Similarly, using (B.23), (B.24) and (B.25), one gets (B.21). $\square$

References

- [1] S. ALMARAZ, *Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow*, J. Differential Equations **259** (2015), 2626–2694.
- [2] S. ALMARAZ, E. Barbosa and L. de Lima, *A positive mass theorem for asymptotically flat manifolds with a non-compact boundary*, Comm. Anal. Geom. **24** (2016), 673–715.
- [3] A. AMBROSETTI, Y. Li and A. Malchiodi, *On the Yamabe problem and the scalar curvature problem under boundary conditons*, Math. Ann. **322** (2002), 667–699.
- [4] T. AUBIN, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269–296.
- [5] T. AUBIN, “Some Nonlinear Problems in Riemannian Geometry”, Springer monographs in mathematics, Springer-Verlag, Berlin, 1998.
- [6] R. BARTNIK, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), 661–693.
- [7] S. BRENDLE, *A generalization of the Yamabe flow for manifolds with boundary*, Asian J. Math. **6** (2002), 625–644.
- [8] S. BRENDLE, *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. **69** (2005), 217–278.
- [9] S. BRENDLE, *Convergence of the Yamabe flow in dimension 6 and higher*, Invent. Math. **170** (2007), 541–576.
- [10] S. BRENDLE and S. CHEN, *An existence theorem for the Yamabe problem on manifolds with boundary*, J. Eur. Math. Soc. (JEMS) **16** (2014), 991–1016.
- [11] L. CAFFARELLI, B. GIDAS and J. SPRUCK, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.
- [12] P. CHERRIER, *Problèmes de Neumann non linéaires sur les variétés Riemanniennes*, J. Funct. Anal. **57** (1984), 154–206.
- [13] B. CHOW, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. **45** (1992), 1003–1014.
- [14] O. DRUET, E. HEBEY and F. ROBERT, “Blow-up Theory for Elliptic PDEs in Riemannian Geometry”, Math. Notes, Princeton University Press, Princeton, 2004.
- [15] J. ESCOBAR, *The Yamabe problem on manifolds with boundary*, J. Differential Geom. **35** (1992), 21–84.
- [16] J. ESCOBAR, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. **136** (1992), 1–50.
- [17] D. GILBARG and N. TRUDINGER, “Elliptic Partial Differential Equations of Second Order”, Springer monographs in mathematics, Springer-Verlag, Berlin Heidelberg, 2001.

- [18] G. GIRAUD, *Sur la problème de Dirichlet généralisé*, Ann. Sci. Éc. Norm. Supér. **46** (1929), 131–145.
- [19] Z. HAN and Y. LI, *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J. **99** (1999), 489–542.
- [20] J. LEE and T. PARKER, *The Yamabe problem*. Bull. Amer. Math. Soc. **17** (1987), 37–91.
- [21] Y. LI and M. ZHU, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), 383–417.
- [22] F. MARQUES, *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. **54** (2005), 1599–1620.
- [23] M. MAYER and C. NDIAYE, *Proof of the remaining cases of the boundary Yamabe problem*, preprint 2015, <http://arxiv.org/abs/1505.06114>.
- [24] D. PIEROTTI and S. TERRACINI, *On a Neumann problem with critical exponent and critical nonlinearity on the boundary*, Comm. Partial Differential Equations **20** (1995), 1155–1187.
- [25] R. SCHOEN, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1994), 479–495.
- [26] R. SCHOEN, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*. In: “Topics in Calculus of Variations”, Lect. Notes in Mathematics, Vol. 1365, Springer-Verlag, New York, 1989, 120–154.
- [27] R. SCHOEN and S.-T. YAU, *On the proof of the positive mass conjecture in General Relativity*, Comm. Math. Phys., **65** (1979), 45–76.
- [28] H. SCHWETLICK and M. STRUWE, *Convergence of the Yamabe flow for large energies*, J. Reine Angew. Math. **562** (2003), 59–100.
- [29] L. SIMON, *Asymptotics for a class of non-linear evolution equations with applications to geometric problems*, Ann. of Math. **118** (1983), 525–571.
- [30] M. STRUWE, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), 511–517.
- [31] N. TRUDINGER, *Remarks concerning the conformal deformation of a Riemannian structure on compact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (3) (1968), 265–274.
- [32] E. WITTEN, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.
- [33] H. YAMABE, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. **12** (1960), 21–37.
- [34] R. YE, *Global existence and convergence of the Yamabe flow*, J. Differential Geom. **39** (1994), 35–50.

Instituto de Matemática
 Universidade Federal Fluminense (UFF)
 Rua Prof. Marcos Waldemar de Freitas S/N
 Niterói, RJ
 24210-201, Brazil
 almaraz@vm.uff.br

Johns Hopkins University
 Mathematics Department
 3400 N. Charles St.
 222 Krieger Hall
 Baltimore, MD 21218, USA
 lsun@math.jhu.edu