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Synchronization is full measure for all α-deformations of an infinite class of continued fractions

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Abstract. We study an infinite family of one-parameter deformations, called α-continued fractions, of interval maps associated to distinct triangle Fuchsian groups. In general for such one-parameter deformations, the function giving the entropy of the map indexed by α varies in a way directly related to whether or not the orbits of the endpoints of the map synchronize. For Nakada's original α-continued fractions and for certain continued fractions introduced by Katok-Ugarcovici, both of which are associated to the classical case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, the full parameter set for which synchronization occurs has been determined.

Here, we explicitly determine the synchronization sets for each α-deformation in our infinite family. (In general, our Fuchsian groups are not subgroups of the modular group, and hence the tool of relating α-expansions back to regular continued fraction expansions is not available to us.) A curiosity here is that all of our non-synchronization sets can be described in terms of a single tree of words. In a paper in preparation, we apply the results of this present work so as to find planar extensions of each of the maps, and thereby study the entropy functions associated to each deformation. We give an indication of this in the final section here.

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1. Introduction

1.1. Main results

Associated to each of the infinite family of groups $G_{m,n}$ defined below in (1.1), we introduce interval maps $T_{m,n,\alpha}$, defined in (1.3) below, parametrized by $\alpha \in [0, 1]$. We show that for each n, the set of those α such that the $T_{3,n,\alpha}$-orbits of the endpoints of the interval of definition, denoted $\ell_0(\alpha)$ and $r_0(\alpha)$ respectively, eventually agree has full Lebesgue measure. We call such agreeing of orbits *synchronization*. We give a full description of the set of α for which synchronization occurs. The following is a simply stated implication of this detailed description.

Theorem 1.1. *For $n \geq 3$, the set of $\alpha \in [0, 1]$ such that there exists $i = i_\alpha$, $j = j_\alpha$ with $T^i_{3,n,\alpha}(r_0(\alpha)) = T^j_{3,n,\alpha}(\ell_0(\alpha))$ is of full Lebesgue measure.*

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A key phenomenon of our setting is that for all $n \geq 3$ there are two *synchronization relations*, in the sense that for n fixed there is a large subinterval of the values of α along which *all* values where synchronization occurs is announced by a basic relation in the group $G_{3,n}$ being satisfied by the elements R, L such that $T^{i-1}_{3,n,\alpha}(\, r_0(\alpha)\,) = R \cdot r_0(\alpha)$ and $T^{j-1}_{3,n,\alpha}(\, \ell_0(\alpha)\,) = L \cdot \ell_0(\alpha)$. These relations, discovered by computational investigation and justified in Propositions 5.2 and 8.2, determine (maximal) intervals along which synchronization occurs.

These *synchronization intervals* are defined by admissibility the digits of the expansion of both of $T^{i-1}_{3,n,\alpha}(\, r_0(\alpha)\,)$ and of $T^{j-1}_{3,n,\alpha}(\, \ell_0(\alpha)\,)$. While each endpoint of a synchronization interval is determined by one of the expansions no longer being admissible, the applicable synchronization relation allows us to determine the other expansion at that endpoint. This leads to the tree of words, $\mathcal{V}$, defined in Definition 4.11.

The intervals are indexed by $\mathbb{Z}_{\neq 0} \times \mathcal{V}$. (A proper subset of $\mathcal{V}$ is necessary when -1 is the indexing natural number.) That the complement in [0, 1] of the collection of the intervals of synchronization is a measure zero Cantor set is proven by use of the ergodicity result of [4] for the setting of $\alpha = 0$.

The admissibility of the two expansions defining a synchronization interval is shown by induction, with the expansion directly related to the index of the interval being straightforward, and the second expansion requiring a more delicate induction argument. A new phenomenon presents itself in the proofs of admissibility: the interval of admissibility of a candidate expansion of digits for some endpoint (in other terms, the corresponding higher rank cylinder) has an endpoint determined by the longest string of digits having a property that we name *full-branched*, see Subsubsection 4.2.3.

After initial results in Sections 2 and 3 describing the dynamics in the setting of $\alpha = 0, 1$ for all $n \geq m \geq 3$, the determination of the synchronization set of parameter α proceeds by treating three partitioning subintervals of $(0, 1)$. The setting of each of these subintervals is addressed in Sections 4, 6 and 7. We find it intriguing that a single tree of words allows the labeling of all of the synchronization intervals, despite the fact that there is more than one synchronization equation.

We expect that the case of $m > 3$ will be very similar, although the synchronization relations will involve longer words and thus some arguments will become awkwardly tedious. Our work raises the question of whether there is a simple characterization of when a one-parameter deformation of interval maps has a set of synchronization relations.

1.2. Motivation

The α-continued fractions of Nakada [17] are associated to the modular group $\mathrm{SL}_2(\mathbb{R})$, whose projective quotient is $G_{2,3}$. Nakada determined natural extensions and more for the setting of $\alpha \geq 1/2$. Kraaikamp [12] gave a more direct method for treating these values. Intermediate results occurred, with Luzzi-Marmi [15] pushing the study forward. Nakada-Natsui [18] confirmed a numeric observation of [15] by showing that, in our terminology, a certain synchronization relation implies synchronization. It was left to Tiozzo *et al.* [5] (see also [6, 7, 19]) and, independently, [14] to show that the relation accounts for all synchronization. These authors also confirmed that the entropy function, assigning to α the entropy of the interval map indexed by α, behaves nicely on the synchronization intervals and is in fact continuous in α. (Note that some authors refer to synchronization as *matching*.) The approach of [14] was to determine planar models of the natural extensions of the maps and then determine the entropy function.

Similar results on a two-parameter family of continued fraction maps related to the modular group were obtained by Katok and Ugarcovici in a series of papers, [9–11]. In particular, in [10] the synchronization of special orbits of the endpoints of the intervals of definition for their maps provide key information to determine properties of planar natural extensions. For both the Nakada α-continued fractions, and a one-parameter sub-family of the Katok-Ugarcovic family, Tiozzo and co-authors related so-called exceptional sets of the entropy function to explicit subsets of the Mandelbrot set, see [2,6,7,19].

Partial results when the underlying groups are the Hecke triangle groups (thus, the $G_{2,q}$ with $q \geq 3$) were given in [8,13].

Our goal is to study deformation families of continued fractions defined over an untreated infinite family of groups, and determine how the entropy function varies. The current work is the initial step in this, where we discover the synchronization relations and exactly determine the set of parameters α for each $G_{3,n}$, $n \geq 3$ such that synchronization occurs. In work in preparation, we apply these results to explicitly determine planar extensions and thereby study the behavior of the entropy functions. In Section 9, we sketch an example suggesting how this works.

1.3. The basics of our maps

We use the groups considered in [4]. Fix integers $n \geq m \geq 3$, and let $\mu = \mu_m = 2\cos\pi/m$, $\nu = \nu_n = 2\cos\pi/n$. Also let $t = \mu + \nu$ that is,

$$t := t_{m,n} = 2\cos\pi/m + 2\cos\pi/n.$$

Let $G_{m,n}$ be generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} \nu & 1 \\ -1 & 0 \end{pmatrix},\ C = \begin{pmatrix} -\mu & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.1}$$

and note that $C = AB$. We work projectively, hence B, C are of order n, m respectively while A is of infinite order. That is, $G_{m,n}$ is a Fuchsian triangle group of signature (m, n, ∞).

Fix $\alpha \in [0, 1]$ and define

$$\mathbb{I}_{m,n,\alpha} := \mathbb{I}_\alpha = [\,(\alpha - 1)t, \alpha t\,)\,. \tag{1.2}$$

Let

$$T_\alpha = T_{m,n,\alpha} : x \mapsto A^k C^l \cdot x, \tag{1.3}$$

where as usual, any 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ acts on real numbers by $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\cdot x = \frac{ax+b}{cx+d}$, and

- $l \in \mathbb{N}$ is minimal such that $C^l \cdot x \notin \mathbb{I}$;
- $k = -\lfloor (C^l \cdot x)/t + 1 - \alpha \rfloor$.

We consider T_α as a map on the closed interval taking values in the half-open interval $\mathbb{I}_\alpha$,

$$T_\alpha : [\,(\alpha-1)t, \alpha t\,] \to \mathbb{I}_\alpha\,.$$

When $\alpha = 0$ and $(m, n) = (3, n)$ this gives the (unaccelerated) maps treated in [4]. See Figures 1.1 and 1.2 for graphs of two of our maps.

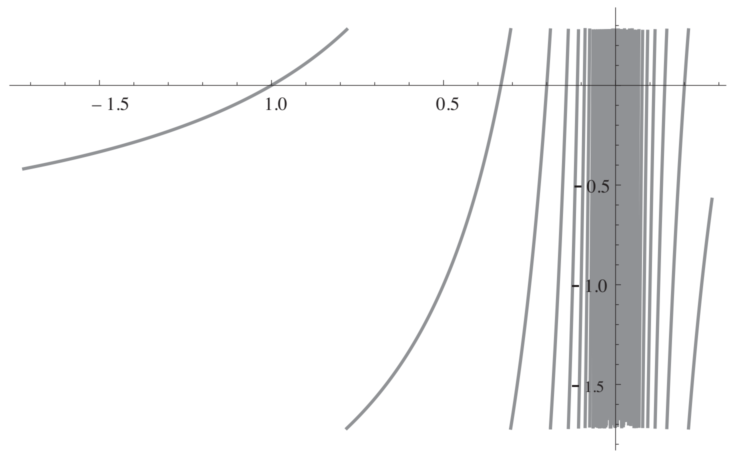


Figure 1.1. The graph of the function $x \mapsto T_{3,3,0.14}(x)$. Each branch is given by some $x \mapsto A^k C \cdot x$.

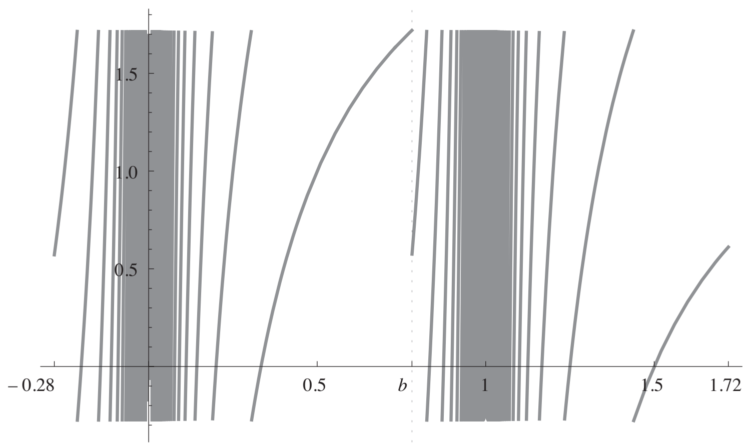


Figure 1.2. The graph of the function $x \mapsto T_{3,3,0.86}(x)$. Branches agree with $x \mapsto A^k C^2 \cdot x$ for various values of k when $x \geq b = \mathfrak{b}_\alpha$, see (1.7).

1.4. Geometric perspective, well-definedness

The reader may well ask if there always does exist an l such that $C^l \cdot x \notin \mathbb{I}_\alpha$. For the special cases of $\alpha = 0, 1$ see below; here we briefly indicate the setting for all other α. A quick study of the graph of the function $x \mapsto C \cdot x$ shows that this has horizontal asymptotes given by $y = \mu$, a pole at $x = 0$ and a zero at $x = 1/\mu$. Of course this function is increasing on each of its branches. In fact, C is an elliptic matrix (that is, its trace is of absolute value less than 2) that fixes a point in the upper half-plane of real part $\mu/2$. It thus acts as a rotation about that fixed point. Indeed, it acts as a rotation on a hyperbolic m-gon; from the words above, this m-gon has consecutive vertices $1/\mu, 0, \infty, \mu$. (Note that when $m = 3$, we have $\mu = 1/\mu = 1$.) Therefore the remaining $m - 4$-vertices lie between $1/\mu$ and μ; let us denote the set of all vertices by $v_1 = \mu, v_2, \dots, v_{m-3}, v_{m-2} = 1/\mu, v_{m-1} = 0, v_m = \infty$. Thus, C acts on the real line as $(-\infty, 0) \to (\mu, \infty) \to (v_2, \mu) \to (v_2, v_3) \to \cdots \to (1/\mu, v_{m-3}) \to (0, 1/\mu) \to (-\infty, 0)$. For $x \in \mathbb{I}_\alpha$, the map T_α is thus the composition of rotating by powers of C until $C^l \cdot x$ is no longer in $\mathbb{I}_\alpha$, and then shifting by applying the appropriate power of A to bring this image back into $\mathbb{I}_\alpha$.

Certainly the left endpoint of $\mathbb{I}_\alpha$, being negative, is sent by C to a positive real number. We now briefly indicate why this value is greater than αt. It then follows that T_α on the left endpoint is given by some $A^{-k}C$ with $k > 0$. In fact, for all negative $x \in \mathbb{I}_\alpha$, we claim that $C \cdot x > x + t$. Elementary calculus shows that the graph of $x \mapsto C \cdot x$ has a tangent line with slope 1 of equation $y = x + \mu + 2$. Since $\nu < 2$, the tangent line lies below the line $y = x + t$. Since the map has a pole at $x = 0$, it easily follows that the claim holds. The claim implies that there is a leftmost subinterval sent outside of $\mathbb{I}_\alpha$ by C; we can partition $\mathbb{I}_\alpha$ by applications of powers of C^{-1} to this leftmost subinterval (with the rightmost image subinterval restricted to its intersection with $\mathbb{I}_\alpha$). In particular, it follows that there always does indeed exist an l such that $C^l \cdot x \notin \mathbb{I}_\alpha$.

1.5. Continued fraction perspective

The main aim of this subsection is to assure the reader that our maps are indeed continued fraction-like.

Since

$$C \cdot x = \frac{-\mu x + 1}{-x} = \frac{-1}{x} + \mu,$$

we immediately find that

$$x = \frac{1}{\mu - C \cdot x}, \tag{1.4}$$

Now

$$C^2 \cdot x = C \cdot (C \cdot x) = \frac{-1}{C \cdot x} + \mu,$$

yielding that

$$C \cdot x = \frac{1}{\mu - C^2 \cdot x}. \tag{1.5}$$

Now (1.4) and (1.5) yield that

$$x = \cfrac{1}{\mu - \cfrac{1}{\mu - C^2 \cdot x}}.$$

After l times we find

$$x = \cfrac{1}{\mu - \cfrac{1}{\mu - \ddots - \cfrac{1}{\mu - C^l \cdot x}}}. \tag{1.6}$$

Since $A^k \cdot x = x + kt$, we see that $T_\alpha(x) = A^k \cdot (C^l \cdot x) = C^l \cdot x + kt$, yielding that $C^\ell \cdot x = T_\alpha(x) - kt$. Substituting this in (1.6) gives

$$x = \cfrac{1}{\mu - \cfrac{1}{\mu - \ddots - \cfrac{1}{\mu - \cfrac{1}{\mu + kt - T_\alpha(x)}}}}.$$

Continuing in this way we find a continued fraction expansion of x with partial quotients given by μ and $\mu + kt$, with $k \in \mathbb{N}$.

1.6. Digits, cylinders, admissible words, ordering

When studying the dynamics of these maps, the orbits of the interval endpoints of $\mathbb{I}_\alpha$ are of utmost importance. We define

$$\begin{aligned} \ell_0 &= (\alpha - 1)t \quad \text{and } \ \ell_i = T_\alpha^i(\ell_0), \ \text{for } i \geq 1\,, \\ r_0 &= \alpha t \qquad\quad\ \text{and } \ r_j = T_\alpha^j(r_0), \ \text{for } j \geq 1\,. \end{aligned}$$

1.6.1. *Two examples*

To aid the reader and help motivate our results, we briefly discuss two examples with $m = n = 3$.

Example 1.2. Fix $m = n = 3$ and $\alpha = 0.14$, see Figure 1.1. Here, $\ell_0 = -1.72$ and $r_0 = 0.28$. Calculation shows that $\ell_1 = A^{-1}C \cdot \ell_0$; thus, the leftmost branch of the graph shown in Figure 1.1 is given by $x \mapsto A^{-1}C \cdot x$ for $x \in [\ell_0, (A^{-1}C)^{-1} \cdot r_0\,)$. Similarly, the rightmost branch is the graph of $x \mapsto AC \cdot x$ for $x \in [(AC)^{-1} \cdot \ell_0, r_0\,]$. All of the remaining branches are surjective.

As the graph suggests, $T_{3,3,\alpha}$ is not expansive. This has no effect on the content of this work until Section 9, where we indicate future results. However, the reader may well be relieved to find that $T_{3,3,\alpha}$ is *eventually expansive*. Indeed, its third compositional power, $T^3_{3,3,\alpha}$, is expansive. We sketch a proof of this. First, for each x in the domain of $T_{3,3,\alpha}$, we have $T'_{3,3,\alpha}(x) = x^{-2}$; in particular, $T'_{3,3,\alpha}(x)$ is minimized at $x = \ell_0$. Since $0 < r_0$ is sufficiently small, it is also quite easy to verify that $(T^2_{3,3,\alpha})'(x)$ is minimized at $x = (A^{-2}C)^{-1}\ell_0$. Now let $\tilde{\ell}_{-2} = (A^{-3}CA^{-1}C)^{-1}\cdot\ell_0$. The inequality $r_0 < -\tilde{\ell}_{-2}$ makes it fairly straightforward to show that $(T^3_{3,3,\alpha})'(x)$ is minimized at $x = \tilde{\ell}_{-2}$. One finds that $(T^3_{3,3,\alpha})'(\tilde{\ell}_{-2}) > 1.75$, and thus $T^3_{3,3,\alpha}$ is expansive as claimed.

Example 1.3. Now fix $m = n = 3$ and $\alpha = 0.86$, see Figure 1.2. Here we have $\ell_0 = -0.18$ and $r_0 = 1.72$. In Figure 1.2, we have marked $C^{-1}\cdot\ell_0$ by b, and as indicated in the caption there, this is $b = \mathfrak{b}_\alpha$ see (1.7) below. Here, one finds that for each $x \in [\ell_0, \mathfrak{b}_\alpha)$, there is some k such that $T_{3,3,\alpha}$ is given by $x \mapsto A^kC\cdot x$. For $x \geq \mathfrak{b}_\alpha$, it is of the general form $x \mapsto A^kC^2\cdot x$. By the definition of our map, the map $x \mapsto C\cdot x$ relates each branch for these larger values of x to a branch for values less than $\mathfrak{b}_\alpha$.

For each $x < \mathfrak{b}_\alpha$ in the domain of $T_{3,3,\alpha}$, we have $T'_{3,3,\alpha}(x) = x^{-2}$; for $x \geq \mathfrak{b}_\alpha$, we have $T'_{3,3,\alpha}(x) = (x-1)^{-2}$. It is an easy matter to check that $T_{3,3,\alpha}$ is expansive.

1.6.2. *Cylinders, notation for digit sequences, full cylinders*

The *cylinders* for the map T_α are

$$\Delta(k,l) = \Delta_\alpha(k,l) = \Delta_{m,n,\alpha}(k,l) := \{\, x \mid T_\alpha(x) = A^kC^l\cdot x \,\}.$$

See Figure 1.3 for a representation of some explicit cylinders. Note that since each of A, C are of positive determinant, T_α is an increasing function on each of its cylinders. We call (k,l) the α-digit of x if $x \in \Delta_\alpha(k,l)$. Define

$$\underline{b}^\alpha_{[1,\infty)} = (k_1,l_1)(k_2,l_2)\ldots$$

to be the (infinite) word (also called the sequence of digits) for the orbit of $\ell_0 = (\alpha-1)t$; that is, $\ell_0 \in \Delta(k_1,l_1)$, $\ell_1 \in \Delta(k_2,l_2)$, etc. Similarly, define $\overline{b}^\alpha_{[1,\infty)}$ as the word giving the digits of the orbit of r_0. We will also need to consider subwords of these; for any $r \leq s \in \mathbb{N}$, let $\underline{b}^\alpha_{[r,s]}$ denote the subword $(k_r,l_r)\cdots(k_s,l_s)$ of $\underline{b}^\alpha_{[1,\infty)}$. We similarly denote subwords of $\overline{b}^\alpha_{[1,\infty)}$.

A cylinder $\Delta_\alpha(k,l)$ is called *full* if its image under T_α is all of $\mathbb{I}_\alpha$. Since the action by C has a pole at $x = 0$, for all $\alpha \neq 0, 1$ there are full cylinders $\Delta_\alpha(k,1)$ with $k \in \mathbb{Z}$ of arbitrarily large absolute value.

The leftmost cylinder of $\mathbb{I}_\alpha$, thus the cylinder of $\ell_0(\alpha)$, has image $[\ell_1, r_0)$ and is said to be a *right full* cylinder. (In the rare instance that $\ell_0(\alpha) = \ell_1(\alpha)$ it is of course a full cylinder.) The only cylinder with $l = 1$ and $k < 0$ that could possibly be non-full is this leftmost cylinder. Let

$$\mathfrak{b} = \mathfrak{b}_\alpha = C^{-1}\cdot\ell_0(\alpha). \tag{1.7}$$

Note that since $\ell_0(\alpha) < 0$, one has $\mathfrak{b} < 1/\mu$. If $\mathfrak{b} \notin \mathbb{I}_\alpha$, then the rightmost cylinder of $\mathbb{I}_\alpha$ has $l = 1$, $k > 0$; its image is $[\ell_0, r_1)$ and in general is only *left full*. If $\mathfrak{b} \in \mathbb{I}_\alpha$, then all cylinders of index $(k, l), l = 1, k > 0$ are full; since C acts so as to send $\Delta_\alpha(k, l+1)$ to $\Delta_\alpha(k, l)$ it follows that each the cylinder of index $(k, 2)$ (with possible exception of the rightmost cylinder) is full if and only if the cylinder of index $(k, 1)$ is. In general, continuing with analysis of this type shows that the only candidates for non-full cylinders are those of index $\{(k, 1), (k, 2), \dots, (k, m-1), (k', l')\}$ where $(k, 1)$ is the α-digit of $\ell_0(\alpha)$ and (k', l') that of $r_0(\alpha)$. As the referee kindly pointed out to us, in the special case that $k' = k$, the cylinder $\Delta_\alpha(k, l')$ will generally be neither left- nor right full, as it has image $[\ell_1, r_1)$. In all other cases, each of the non-full cylinders has image either $[\ell_0, r_1)$ or $[\ell_1, r_0)$.

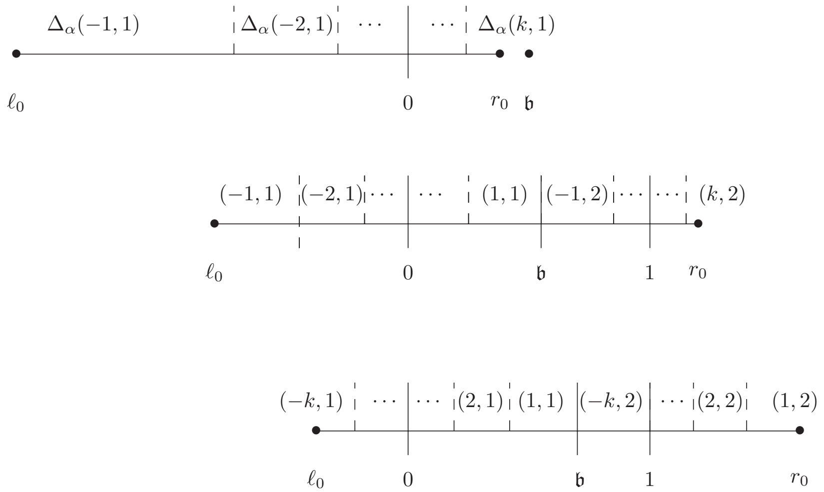


Figure 1.3. Schematic representation of cylinders for three values of α when $m = n = 3$. Top: small values of α; middle: intermediate values of α; bottom: large values of α. See Sections 4, 7 and 6 for the respective details. Here $\mathfrak{b} = \mathfrak{b}_\alpha$ as defined in (1.7).

1.6.3. *Admissibility and orders, definitions*

A word U in the letters A, C is called *admissible* for a pair α and $x \in \mathbb{I}_\alpha$ if $U = A^{k_u}C^{l_u} \cdots A^{k_1}C^{l_1}$ and for each j, with $1 \leq j \leq u$, one has $A^{k_j}C^{l_j} \cdots A^{k_1}C^{l_1} \cdot x = T_\alpha^j(x)$. Note that this is equivalent to having for each j both that (1) $A^{k_j}C^{l_j} \cdots A^{k_1}C^{l_1} \cdot x \in \mathbb{I}_\alpha$ and (2) l_j is minimal such that $C^{l_j}A^{k_{j-1}}C^{l_{j-1}} \cdots A^{k_1}C^{l_1} \notin \mathbb{I}_\alpha$. We also simply say that such a word U is admissible for α if there exists an $x \in \mathbb{I}_\alpha$ such that U is admissible for the pair α, x.

The α-alphabet is the set of possible single digits for $x \in \mathbb{I}_\alpha$, that is all (k, l) such that $\Delta_\alpha(k, l) \neq \emptyset$. The standard ordering of real numbers then induces an ordering of this alphabet: $(k, l) \prec (k', l')$ if $\Delta_\alpha(k, l)$ lies to the left of $\Delta_\alpha(k', l')$. (Confer Figures 1.3 and 1.4; in this second figure the order on each $\mathbb{I}_\alpha$ is rather from bottom to top.)

The analysis for the setting of fullness of cylinders also yields that for $k > 0$, $\Delta_\alpha(-k, 1)$ lies to the left of any $\Delta_\alpha(-k - j, 1)$ with $j > 0$, as well as to the

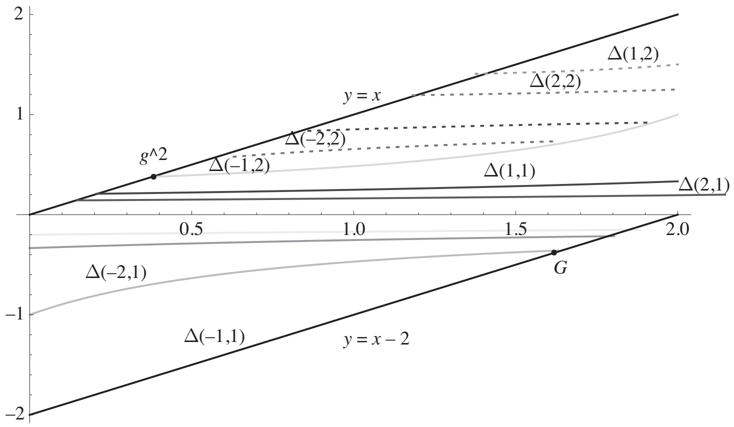


Figure 1.4. The unions of the various $\Delta_{3,3,\alpha}(k,l)$, see Subsection 1.6. Each $\mathbb{I}_\alpha$ is given as a vertical fiber (of coordinate $x = \alpha t$ with $t = t_{3,3} = 2$), from the left endpoint at the bottom up to the right endpoint at the top of the fiber. For each α such that they exist, the $\Delta_\alpha(k, 2)$ have limiting value as $|k| \to \infty$ of 1. Similarly, the $\Delta_\alpha(k, 1)$ have limiting value as $|k| \to \infty$ of 0.

left of any $\Delta_\alpha(k', 1)$, with $k' > 0$. Similarly, $\Delta_\alpha(k', 1)$ lies to the left of any $\Delta_\alpha(k' - j, 1)$, $j < k'$. Now since C acts in an order preserving manner, we find that any $\Delta_\alpha(k,l)$ lies to the left of all $\Delta_\alpha(k, l+1)$. We thus find that for each α, the ordering on the α-alphabet is a restriction of the following order.

For each α, the α-alphabet is a (strict) subset of $(\mathbb{Z} \setminus \{0\}) \times \{1, 2, \dots, m-1\}$. We define the *full order* on $(\mathbb{Z} \setminus \{0\}) \times \{1, 2, \dots, m-1\}$ by

$$\begin{aligned}(k,l) \prec (k',l') \ \text{ if and only if }\ &\text{(i) } l < l',\\ \text{or }\ &\text{(ii) } l = l' \text{ and one of } k' < k < 0,\ k < 0 < k', \\ &\qquad \text{or } 0 < k' < k.\end{aligned} \tag{1.8}$$

For $m = 3$ and $n \geq 3$, the full order is indicated in Table 1.1.

$$\begin{array}{l}
(1,2) \succ (2,2) \succ \cdots \succ (N,2) \succ \cdots \\
\hline
\succ \cdots \succ (-M,2) \succ \cdots \succ (-2,2) \succ (-1,2) \\
\hline
\succ (1,1) \succ (2,1) \succ \cdots \succ (N,1) \succ \cdots \\
\hline
\succ \cdots \succ (-M,1) \succ \cdots \succ (-2,1) \succ (-1,1).
\end{array}$$

Table 1.1. The full order. Here $m = 3, n \geq 3$.

This ordering extends to the set of all words (including infinite words) in the usual lexicographic manner.

The α-alphabet depends only on the first digits of $\ell_0(\alpha)$ and $r_0(\alpha)$, as we now prove.

Lemma 1.4. *Fix m, n, α. Then the α-alphabet depends only on the first digits of $\ell_0(\alpha)$ and $r_0(\alpha)$.*

More precisely, denote the α-digits of $\ell_0(\alpha)$ and $r_0(\alpha)$ by $(k, 1)$, (k', l'), respectively. Then the α-alphabet is

$$\{(k'+j, l') \mid j \geq 0\} \cup \bigcup_{1\leq l\leq l'} \{(k-j, l) \mid j \geq 0\} \cup \bigcup_{1\leq l<l'} \{(j, l) \mid j > 0\}.$$

Proof. Since $\ell_0(\alpha) \in \Delta_\alpha(k, 1)$, all indices corresponding to cylinders between $\Delta_\alpha(k, 1)$ and $x = 0$ are certainly in the alphabet. These indices are $(k - j, 1)$, with $j > 0$. The pre-images of these cylinders under powers of C are also cylinders of T_α, up to and including the l'^{th} power. If $l' = 1$, then all indices corresponding to cylinders between 0 and $\Delta_\alpha(k', 1)$ are in the alphabet. If $l' > 1$, then all $(j, 1)$ with $j > 0$ are present, and so are all preimages under powers of C are also cylinders of T_α, up to and including the $(l' - 1)^{\text{st}}$ power. The cylinders between the pole of $C^{l'}$ and $\Delta_\alpha(k', l')$ have their indices in the language, and we have accounted for all possible indices. □

1.6.4. *Relating admissibility and orders*

If $x \in \mathbb{I}_\alpha$, then each of $T_\alpha(x), T_\alpha^2(x), \ldots$ is also in $\mathbb{I}_\alpha$. This is directly related to the notion of admissibility. Here we use the notation of Subsubsection 1.6.2.

Lemma 1.5. *Fix m, n, α. As above, let $(k, 1)$ be the α-digit of $\ell_0(\alpha)$ and (k', l') that of $r_0(\alpha)$. Admissible words $A^{k_u}C^{l_u} \cdots A^{k_1}C^{l_1}$ are characterized by the properties:*

(i) *each (k_i, l_i) is in the α-alphabet;*

(ii) *for each $1 \leq j \leq u$,*

$$\underline{b}^\alpha_{[1,u-j+1]} \preceq (k_j, l_j) \cdots (k_u, l_u) \preceq \overline{b}^\alpha_{[1,u-j+1]};$$

and,

(iii) *whenever $k_j = k$,*

$$\underline{b}^\alpha_{[2,u-j]} \preceq (k_{j+1}, l_{j+1}) \cdots (k_u, l_u).$$

Proof. That admissible words have these properties is straightforward. Indeed, by definition admissibility implies that there is an $x \in \mathbb{I}_\alpha$ whose α-digit sequence begins $(k_1, l_1)(k_2, l_2) \cdots (k_u, l_u)$, with each digit in the α-alphabet. Thus, each $A^{k_{j-1}}C^{l_{j-1}} \cdots A^{k_1}C^{l_1} \cdot x$ has α-digit sequence beginning $(k_j, l_j) \cdots (k_u, l_u)$, and each of these is in $\mathbb{I}_\alpha$. Since the endpoints of $\mathbb{I}_\alpha$ correspond to $\underline{b}^\alpha_{[1,\infty)}$ and $\overline{b}^\alpha_{[1,\infty)}$, and the ordering in the language corresponds to the usual ordering of real numbers, the inequalities of (ii) must hold. The images of non-full cylinders are given in Subsubsection 1.6.2; if some $k_j = k$, then $\ell_1 \leq A^{k_j}C^{l_j} \cdots A^{k_1}C^{l_1} \cdot x$. Thus the inequalities of (iii) hold.

For the other direction, our word is admissible for α and x_1 if there is a sequence of x_j with $x_j \in \Delta_\alpha(k_j, l_j)$ such that $A^{k_j}C^{l_j} \cdot x_j = x_{j+1}$. When each $\Delta_\alpha(k_j, l_j)$ is a full cylinder, we can choose any $x_u \in \Delta_\alpha(k_u, l_u)$ and then repeatedly solve for $x_j \in \Delta_\alpha(k_j, l_j)$ such that $A^{k_j}C^{l_j} \cdot x_j = x_{j+1}$.

Recall that the T_α-image of a non-full cylinder $\Delta_\alpha(k_j, l_j)$ is one of: $[\ell_1, r_0)$, $[\ell_0, r_1)$, $[\ell_1, r_1)$ according to (1) $k_j = k$ and $k \neq k'$; (2) $(k_j, l_j) = (k', l')$ and $k \neq k'$; or, (3) $(k_j, l_j) = (k', l')$ and $k = k'$. Suppose that $j < u$ and let $x \in \mathbb{I}_\alpha$ be such that its α-expansion begins $(k_{j+1}, l_{j+1}) \cdots (k_u, l_u)$. When (1) holds, (ii) applies and yields that $x \geq \ell_1$ and hence x is in the corresponding image. Similarly, when (2) holds, (i) implies that $(k_{j+1}, l_{j+1}) \cdots (k_u, l_u) \preceq \overline{b}^\alpha_{[2,u-j+1]}$ and thus $x \leq r_1$ (and, we can assume strict inequality) and is in the image. Finally, when (3) holds we have $\ell_1 \leq x \leq r_1$ and again x belongs to the T_α-image of $\Delta_\alpha(k_j, l_j)$. Using this, we can show that there always exists an x_1 as in the proof of the cases where only full cylinders appear. $\square$

The following two results provide a key tool for proving admissibility of digits by induction, see for example Subsubsection 4.4.1.

Lemma 1.6. *Suppose $N \in \mathbb{N}$ and for some α', α'', with $0 \leq \alpha' < \alpha'' \leq 1$, one has $\underline{b}^{\alpha'}_{[1,N]} = \underline{b}^{\alpha''}_{[1,N]}$. Then $\underline{b}^{\alpha}_{[1,N]} = \underline{b}^{\alpha'}_{[1,N]}$ for all $\alpha \in [\alpha', \alpha'']$.*

Proof. Recall that for any $\beta \in [0, 1]$, the initial digit of $\ell_0(\beta)$ is $(k, 1)$ for some $k < 0$. Of course $A^kC \cdot \ell_0(\beta) \geq \ell_0(\beta)$. When $k < -1$, the matrix A^kC is hyperbolic and the smallest β such that the initial digit of $\ell_0(\beta)$ is $(k, 1)$ is determined by $\ell_0(\beta)$ being a fixed point of A^kC. For this β, the cylinder $\Delta_\beta(k, 1)$ is a subinterval of $\mathbb{I}_\beta$. Therefore, for $x > \beta$ and sufficiently close to β, $A^kC \cdot x > \ell_0(\beta)$. Considering the graph of the function $x \mapsto A^kC \cdot x$, this implies that $\ell_0(\beta)$ is the larger of the two fixed points of A^kC. Hence, $(k, 1)$ is admissible for all $\alpha \geq \beta$ such that $A^kC \cdot \ell_0(\alpha) < r_0(\alpha)$. The extreme value for this latter condition is given by $A^{k-1}C \cdot \ell_0(\alpha) = \ell_0(\alpha)$. Therefore, the set of α for which $(k, 1)$ is α-admissible forms an interval. (When $k = -1$, since the parabolic $A^{-1}C$ has no fixed point, this statement also holds.) Similarly, the admissibility of the initial digit of $x = r_0(\beta)$ implies that the image of this value lies below the line $y = x$; admissibility continues for all $\alpha \geq \beta$ until $r_0(\alpha)$ is a fixed point.

With α', α'' as announced, fix some $\alpha \in (\alpha', \alpha'')$. Since $N \geq 1$, we certainly have that $\ell_0(\alpha)$, $\ell_0(\alpha')$ and $\ell_0(\alpha'')$ all share a common first digit. Since also the $r_0(\beta)$ are increasing, Lemma 1.4 yields that the α-alphabet contains the α'-alphabet. In particular, each of the N digits in $\underline{b}^{\alpha'}_{[1,N]}$ is contained in the α-alphabet.

Again from the increasing nature of the endpoints ℓ_0, r_0, we find for each $j \leq N$ both $\underline{b}^{\alpha}_{[1,N-j+1]} \preceq \underline{b}^{\alpha''}_{[1,N-j+1]}$ and $\overline{b}^{\alpha'}_{[1,N-j+1]} \preceq \overline{b}^{\alpha}_{[1,N-j+1]}$. Thus by Lemma 1.5, the admissibility of $\underline{b}^{\alpha'}_{[1,N]}$ for α', α'' implies its admissibility for α.

Finally, $\underline{b}^{\alpha}_{[1,N]} = \underline{b}^{\alpha'}_{[1,N]}$ for otherwise we would contradict the increasing nature of ℓ_0. □

The following is proven *mutatis mutandis*.

Lemma 1.7. *Suppose $N \in \mathbb{N}$ and for some α', α'', with $0 \leq \alpha' < \alpha'' \leq 1$, one has $\overline{b}^{\alpha'}_{[1,N]} = \overline{b}^{\alpha''}_{[1,N]}$. Then $\overline{b}^{\alpha}_{[1,N]} = \overline{b}^{\alpha'}_{[1,N]}$ for all $\alpha \in [\alpha', \alpha'']$.*

ACKNOWLEDGEMENTS. It is a real pleasure to thank the referee for a careful reading, helping us to clarify history, and for the explicit suggestions to include examples as in Section 9, as well as to concentrate on group conjugations so as to streamline Sections 5 and 8.

2. Dynamics in the case of $\alpha = 0$ for all signatures

Calta-Schmidt [4] study the dynamics of what in our notation is $T_{3,n,\alpha}$ with $\alpha = 0$, and of its natural extension. We briefly generalize that work in this section (our $T(x)$ gives their $g(x)$ upon restricting to $m = 3$).

Fix integers m, n (both greater than 2). In the notation of (1.2), let $\mathbb{I} := \mathbb{I}_{m,n,0}$ and thus $\mathbb{I} = [-t, 0)$. We have

$$\begin{aligned} T := T_{m,n,0} : \mathbb{I} &\to \mathbb{I} \\ x &\mapsto A^{-k}C \cdot x\,, \end{aligned}$$

where $k = k(x)$ is the unique positive integer such that $T(x) \in \mathbb{I}$. Notice that $T(x) = -kt + 2\cos\pi/m - 1/x$. Let $\Delta_k := \Delta_{\alpha=0}(-k, 1)$. For $k \geq 2$ we have the full cylinders $\Delta_k = [\frac{1}{\mu-(k-1)t}, \frac{1}{\mu-kt})$; that is, T sends each surjectively onto $\mathbb{I}$. Recall $\nu = 2\cos\pi/n$. We have that $\Delta_1 = [-t, -1/\nu)$ and its image under T is the interval $[-\nu + 1/t,\ 0)$. The T-orbit of $x = -t$ is of central importance, thus let

$$\ell_j = T^j(-t) \text{ for } j = 0, 1, \ldots\,.$$

The following element of the underlying Fuchsian group is key to the study of this orbit and therefore to many arguments in this paper.

$$W = A^{-2}C\,(A^{-1}C)^{n-3}\left[A^{-2}C(A^{-1}C)^{n-2}\right]^{m-2}. \tag{2.1}$$

Since $A^{-1}C = B$ and $B^n = \mathrm{Id}$ (projectively),

$$\begin{aligned} W &= A^{-1}B\;B^{n-3}\left[A^{-1}B\,B^{n-2}\right]^{m-2} \\ &= A^{-1}B^{-2}\left[A^{-1}B^{-1}\right]^{m-2} = A^{-1}B^{-2}\left[BC^{-1}B^{-1}\right]^{m-2} \\ &= A^{-1}B^{-2}\,B\left[C^{-1}\right]^{m-2}B^{-1} = A^{-1}B^{-1}C^{2}B^{-1} \\ &= A^{-1}B^{-1}(AB)^{2}B^{-1} = A^{-1}B^{-1}A\,BA\,. \end{aligned}$$

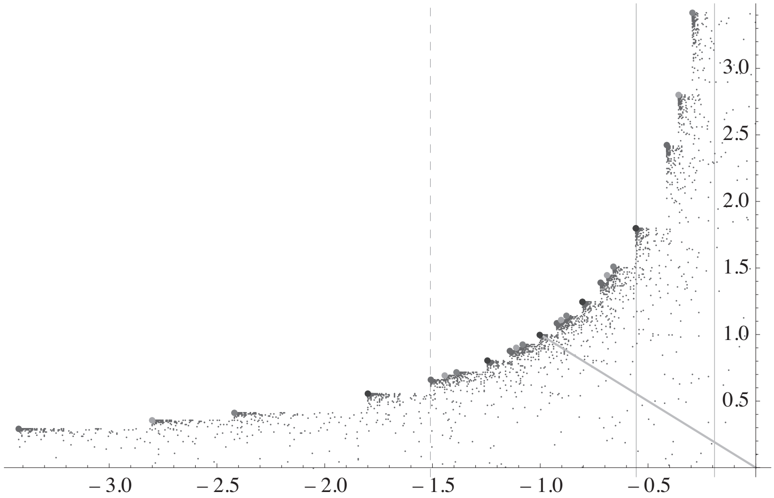


Figure 2.1. A trace of 20,000 consecutive orbit elements to show the natural extension of the determinant plus one map when $(m, n) = (5, 7)$ and $\alpha = 0$. Vertical lines: dotted at ℓ_1, and right ends of cylinders for $k = 1$ and $k = 2$. Solid anti-diagonal: $y = -x$. Bullet points placed at initial orbit of vertex of x-coordinate $\ell_0 = -t$. Order of orbit of ℓ_0 itself: $\ell_0 < \ell_6 < \ell_{12} < \ell_{18} < \ell_1 < \ell_7 < \ell_{13} < \ell_{19} < \ell_2 < \ell_8 < \ell_{14} < \ell_{20} < \ell_3 < \ell_9 < \ell_{15} < \ell_{21} < \ell_4 < \ell_{10} < \ell_{16} < \ell_{22} < \ell_5 < \ell_{11} < \ell_{17}$. (See ArXiv preprint for colored version.)

Now, since $A \cdot (-t) = 0$, and $B \cdot 0 = \infty$, while A fixes ∞, we certainly have that W fixes $x = -t$. Substituting $A^{-1}C$ for B gives the form that we will use several times below.

$$W = A^{-1}C^{-1}ACA\,. \tag{2.2}$$

We claim that the right hand side of (2.1) is the admissible word for the corresponding element in the T-orbit of $\ell_0 = -t$. That is, all ℓ_j lie in Δ_1 other than $\ell_{n-2+k(n-1)}$ for $0 \le k \le m-3$ and also $\ell_{2n-4+(m-3)(n-1)}$; all of these latter orbit entries lie in Δ_2. Furthermore, $\ell_{2n-4+(m-3)(n-1)}$ is the left endpoint of Δ_2, and thus $\ell_{2n-3+(m-3)(n-1)} = \ell_0$. To justify this, we will show:

(i) $(A^{-1}C)^{n-3} \cdot \ell_0 < -1/\nu < (A^{-1}C)^{n-2} \cdot \ell_0$;

(ii) $[A^{-2}C(A^{-1}C)^{n-2}]^{m-2} \cdot \ell_0 = -\nu$;

(iii) $A^{-2}C(A^{-1}C)^{n-2} \cdot (-\nu) = \infty$.

Note that (iii) is immediate, as the pole of $A^{-2}C(A^{-1}C)^{n-2} = A^{-1}B^{-1}$ is that of B^{-1}, and certainly $B\cdot\infty = -\nu$. Similarly, $[A^{-2}C(A^{-1}C)^{n-2}]^{m-2} = (A^{-1}B^{-1})^{m-2} = (A^{-1}B^{-1})^{-2}$, and $A^{-1}B^{-1}$ is a conjugate of C^{-1} and thus is of order m. From this, we find that

$$[A^{-2}C(A^{-1}C)^{n-2}]^{m-2} \cdot \ell_0 = (BA)^2 \cdot (-t) = BC \cdot 0 = B \cdot \infty = -\nu\,,$$

and (ii) also holds. Finally, we have $\ell_0 = -t < -\nu < -\nu + 1/t = \ell_1 = B \cdot \ell_0$ and since $B \cdot -1/\nu = 0$, $B \cdot 0 = \infty$, $B \cdot \infty = -\nu$, we find that $\ell_0 < B^3 \cdot (-1/\nu) < \ell_1$, and now (i) easily follows.

Now, $A^{-1}C = B$ has no fixed points, has its pole at $x = 0$, and defines an increasing function. Thus for $x < 0$ we have $A^{-1}C \cdot x > x$. From this, (i) implies that $\ell_j = (A^{-1}C)^j \cdot \ell_0$ for $0 \le j \le n-3$. Since $B^{-2} \cdot (-t) < 0$, this also holds for $j = n-2$. Since $A^{-2}C(A^{-1}C)^{n-2} = A^{-1}B^{-1} = C^{-1}$, the increasing function $x \mapsto A^{-2}C(A^{-1}C)^{n-2} \cdot x$ has no fixed points, and hence (iii) implies that $A^{-2}C(A^{-1}C)^{n-2} \cdot x > x$ for all $x < -\nu$. Combining this with (ii) gives that for each $j < m-2$ we have $\ell_0 < [A^{-2}C(A^{-1}C)^{n-2}]^j \cdot \ell_0 < -\nu < -\nu + 1/t = \ell_1$. This in turn gives the correctness of the various ℓ_j corresponding to the sub-words of $[A^{-2}C(A^{-1}C)^{n-2}]^{m-2}$. That the remaining factors of (2.1) correspond to the T-orbit is easily argued, especially since the fact that W fixes $x = -t$ combines with (ii) to show that $[A^{-2}C\,(A^{-1}C)^{n-3}]^{-1} \cdot \ell_0 = -\nu$.

We thus have that the ordering of the T-orbit of ℓ_0 as real numbers is as given in Table 2.1. Note that the orbit elements contained in Δ_2 are found as final entry from each column.

$$
\begin{array}{lllll}
\ell_0 & < \ell_{n-1} & < \ell_{2n-2} & & < \cdots < \ell_{(m-2)(n-1)} \\
< \ell_1 & < \ell_n & < \ell_{2n-1} & & < \cdots < \ell_{(m-2)(n-1)+1} \\
\vdots & \vdots & & & \vdots \\
< \ell_{n-3} & < \ell_{2n-2} & < \ell_{3n-3} & & < \cdots < \ell_{(m-2)(n-1)+n-3} \\
< \ell_{n-2} & < \ell_{2n-3} & < \ell_{3n-4} & < \cdots < \ell_{(m-3)(n-1)+n-2} &
\end{array}
$$

Table 2.1. The T-orbit of ℓ_0 ordered as real numbers, when $\alpha = 0$.

We let Ω be the union of $mn - m - n$ rectangles whose bases lie on the x-axis with endpoints being consecutive elements in the orbits of ℓ_0 under the real ordering, beginning with $[\ell_0, \ell_{n-1}]$, along with $[\ell_{(m-3)(n-1)+n-2}, 0]$, and whose heights we label L_i, $1 \le i \le mn - m - n$, also in accordance with the real order of the bases.

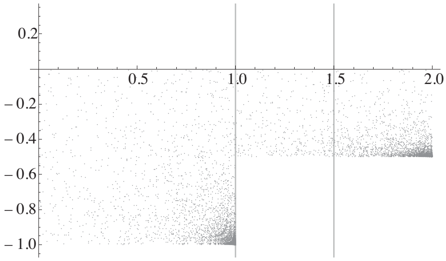
Set $L_1 = 1/t$; $L_{m-1+i} = (RA^{-1}CR^{-1}) \cdot L_i$ for any $1 \le i < (m-1)(n-2)$; and, $L_i = (RA^{-2}CR^{-1}) \cdot L_{i-1+(n-2)(m-1)}$ for $2 \le i \le m-1$. Since these relations accord with $\mathcal{T} = \mathcal{T}_{m,n,0}$, and $-t$ being fixed by W gives that $1/t$ is fixed by RWR^{-1}, we have that the left upper vertex Ω is $(-t, 1/t)$ and in fact the left upper vertex of the i^{th} rectangle is $(-1/L_i, L_i)$, thus showing that Ω has infinite μ-measure. We have seen that $\ell_{mn-m-n-1} = -1/\nu$, and hence find that the rightmost element of that orbit has value

$$
\begin{aligned}
\ell_{(m-3)(n-1)+n-2} &= [(A^{-1}C)^{n-3}A^{-2}C]^{-1} \cdot (-1/\nu) = B^{-1}AB^3 \cdot (-1/\nu) \\
&= B^{-1}A \cdot (-\nu) = -1/t\,.
\end{aligned}
$$

Therefore, the rightmost rectangle has height $L_{mn-m-n} = t$. One can show that $\mathcal{T}$ is bijective on Ω up to a set of measure zero.

Remark 2.1. In [4], an acceleration of the $\alpha = 0$ interval map (when $m = 3$) is defined. This new interval map is shown to be ergodic with respect to a finite invariant measure (that is absolutely continuous with respect to Lebesgue measure).

3. Dynamics in the case of $\alpha = 1$ for all signatures



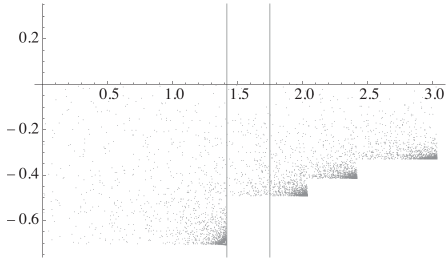


Figure 3.1. Plots showing the domains of the natural extensions for $(m, n) = (3, 3)$ and $(m, n) = (4, 5)$ when $\alpha = 1$. Vertical lines marked at $x = \mu$ and $x = \mu + 1/t$ in both cases.

This case is dominated by

$$U = AC^{m-2}(AC^{-1})^{n-2}. \tag{3.1}$$

We now let the interval be $\mathbb{I} := \mathbb{I}_{m,n,\alpha=1} = [0, t)$.

Proposition 3.1. *For all $m, n \geq 3$, the map $T_{m,n,1}$ has*

(1) *U as the admissible word for the orbit of $t = r_0(1)$;*
(2) *exactly one non-full cylinder, $\Delta(1, n-1) = [\mu + 1/t, t]$;*
(3) *full cylinders of the form $\Delta(k, l)$ for all $l \in \{1, \ldots, m-2\}$ and $k \in \mathbb{N}$, as well as $\Delta(k, m-1)$ for $k > 1$.*

Proof. We first claim that all possible exponents of C are seen, thus that for each $l \in \{1, \ldots, m-1\}$, there are non-empty cylinders $\Delta(k, l) = \Delta_1(k, l)$ and in fact for each l these are indexed by $k \in \mathbb{N}$. A first application of C sends all of $(0, 1/\mu)$ to negative values, and in particular outside of $\mathbb{I}$. As well, since $1/\mu$ is sent to 0, these images are brought back into $\mathbb{I}$ by positive powers of A. Furthermore, this shows that each of the cylinders $\Delta(k, 1), k \in \mathbb{N}$ is full. Therefore, also the cylinders $\Delta(k, l), k \in \mathbb{N}$ are full for all $1 \leq l \leq m-2$.

We turn attention to the case of $l = m-1$. In fact, there are full cylinders $\Delta(k, m-1)$ for all $k > 1$ and the sole non-full cylinder is $\Delta(1, m-1)$, as we now briefly explain. Since $CA^{-1} \cdot 0 = \mu + 1/t < t$, one has that $\Delta(1, m-1) = [\mu + 1/t, t]$. As well, since of course $A^2C \cdot (\mu + 1/t) = A \cdot 0 = t$, the cylinders $\Delta(k, m-1)$ for all $k > 1$ are all indeed full (and their union is $[\mu, \mu + 1/t]$). It remains only to consider the orbit of t, which we know begins by $t \mapsto AC^{m-1} \cdot t$. We observe that $AC^{m-1} = AC^{-1} = AB^{-1}A^{-1}$, and is thus clearly an elliptic matrix of order n. We now translate by A so as to consider the orbit of 0 under powers of B^{-1}. This elliptic matrix fixes a point of real part $-\nu/2$ and rotates a hyperbolic n-gon; one also easily verifies that $-\nu \mapsto \infty \mapsto 0 \mapsto -1/\nu$ is part of the orbit of the vertices of the n-gon. The predecessor of $-\nu$ is $B \cdot -\nu = -\nu + 1/\nu$. We translate back to $\mathbb{I}$ by A, and thus this predecessor corresponds to $\mu + 1/\nu$, a value that is visibly greater than $\mu + 1/t$. In conclusion, the $T_{m,n,1}$-orbit of t begins with $(AC^{m-1})^s \cdot t$ for $1 \leq s \leq n-2$. But, $(AC^{m-1})^{n-2} \cdot t = (CA^{-1})^2 \cdot t = CA^{-1}C \cdot 0 = C \cdot \infty = \mu$. That is, the orbit of t reaches the right endpoint of the (full) cylinder $\Delta(2, m-1)$, and thus thereafter returns to t. □

4. Orbit synchronization on the interval $\alpha < \gamma_{3,n}$, $n \geq 3$

We define $\gamma_{3,n}$ as the value of α such that

$$C^{-1} \cdot \ell_0(\gamma_{3,n}) = r_0(\gamma_{3,n}).$$

Compare with Figure 4.1, where the leftmost gray vertical line occurs at $x = 2\gamma_{3,3} = (G-1)^2$, with $G = (1+\sqrt{5})/2$. In particular, for all $\alpha \leq \gamma_{3,n}$, the point $\mathfrak{b}_\alpha$ lies outside of $\mathbb{I}_\alpha$. Since $\ell_0(\alpha) < 0$ for all $\alpha < 1$, it follows that $0 < r_0(\gamma_{3,n}) < 1$.

We define $\epsilon_{3,n}$ such that

$$A^{-1}C \cdot \ell_0(\epsilon_{3,n}) = r_0(\epsilon_{3,n}).$$

Compare with Figure 4.1, where the rightmost gray vertical line occurs at $x = 2\epsilon_{3,3} = G$.

One finds that $\ell_1(\alpha) = A^{-1}C \cdot \ell_0(\alpha)$ holds for all $0 \leq \alpha < \epsilon_{3,n}$. Elementary calculations show that for all $n \geq 3$, $r_0(\epsilon_{3,n}) \geq (1+\sqrt{5})/2$, and thus this equation holds in particular for all $\alpha < \gamma_{3,n}$.

In this section we prove the following:

Theorem 4.1. *For $m = 3$ and $n \geq 3$, the set of $\alpha \in (0, \gamma_{3,n})$ such that there exists $i = i_\alpha$, $j = j_\alpha$ with $T^i_{3,n,\alpha}(\,r_0(\alpha)\,) = T^j_{3,n,\alpha}(\,\ell_0(\alpha)\,)$ is of full measure.*

4.1. Right cylinders and (potential) synchronization intervals

Basic motivation for our approach to synchronization of the T_α-orbits of $r_0(\alpha)$ and $\ell_0(\alpha)$, with $\alpha < \gamma_{3,n}$, comes from the following. We will eventually show for this range of our parameters that synchronization depends on right and left digits being related by

$$C^{-1}AC = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}. \tag{4.1}$$

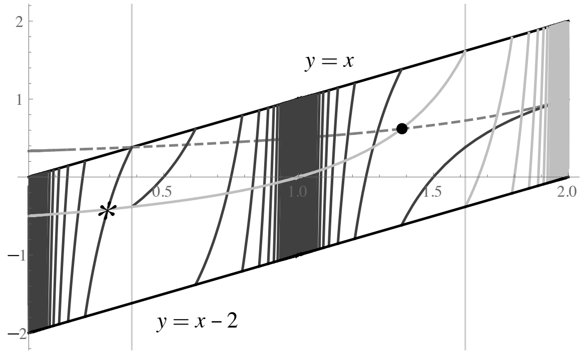


Figure 4.1. The first image of the endpoints. In light gray is the graph of $x \mapsto T_{3,3,\alpha}(x - t)$, with $x = \alpha t$; these thus give the values of $\ell_1(\alpha)$. In dark gray that of $x \mapsto T_{3,3,\alpha}(x)$; these give the values of $r_1(\alpha)$. (Here $t = t_{3,3} = 2$.) The dashed curve is the image of the left endpoint under C^{-1} (giving the values of $\mathfrak{b}_\alpha$), above this curve a next digit begins with a C^2. The asterisk marks an accidental synchronization, see Example 4.4. The large dot has x-coordinate $\delta_{3,n}t$, see Section 6.

Lemma 4.2. *Fix $m = 3, n \geq 3$, $\alpha < \gamma_{3,n}$, and $i, j \in \mathbb{N}$. Suppose that $\ell_{i-1} = C^{-1}AC \cdot r_{j-1}$. Then*

(1) $\ell_i = r_j$,
(2) $\ell_{i-1} \geq r_{j-1}$ *with equality if and only if both equal zero.*

Proof. There is some $u \in \mathbb{Z} \setminus \{0\}$, that $\ell_i = A^u C \cdot \ell_{i-1}$ (recall that since $\alpha < \gamma_{3,n}$, the exponent l is always one). This thus equals $A^u C\, C^{-1}AC \cdot r_{j-1} = A^{u+1}C \cdot r_{j-1}$. In particular $A^{u+1}C \cdot r_{j-1} \in \mathbb{I}_\alpha$. We conclude that $r_j = A^{u+1}C \cdot r_{j-1}$. From this, we find $\ell_i = r_j$. Since $C^{-1}AC$ clearly fixes zero, see (4.1), it follows that $\ell_{i-1} = r_{j-1}$ holds if either is zero; otherwise, the fact that the exponent of A is greater when passing from r_{j-1} to r_j than from ℓ_{i-1} to ℓ_i (in light of the ordering of digits, (1.8)) shows that $\ell_{i-1} > r_{j-1}$. □

Definition 4.3. We say *synchronization* occurs at α if there exist i, j such that $r_j = \ell_i$. A *synchronization interval* is an interval of α values for each which synchronization holds with the same pair of indices i, j. (We will assume that at least for one α in the interval, both i, j are minimal.)

The following example is represented in Figure 4.2.

Example 4.4. Fix $n = 3$, so $t = 2$. In Figure 4.1, the first dark gray branch to the left of $x = 2\,\gamma_{3,3}$ corresponds to $r_1(\alpha) = AC \cdot r_0(\alpha)$, equivalently to $\overline{b}^{\alpha}_{[1,\infty)} = (1, 1)\cdots$. This branch intersects the line $y = x - 2$ at a point we call 2ζ. Thus, $AC \cdot r_0(\zeta) = \ell_0(\zeta)$. Here, $\zeta = (5 - \sqrt{21})/2 = 0.20871\cdots$.

Explicit computation reveals that for sufficiently close $\alpha \geq \zeta$, the digits of the $\ell_0(\alpha)$ are given as $\underline{b}^{\alpha}_{[1,\infty)} = (-1, 1)(-2, 1)(-2, 1)(-1, 1)\cdots$. Since $m = n = 3$, (2.1) gives $W = A^{-2}CA^{-2}CA^{-1}C$. Solving for η such that $r_0(\eta) = A^{-1}CW \cdot$

$\ell_0(\eta)$, we find $\eta = (-1 + \sqrt{21})/10 = 0.35825\cdots$. One can verify that for all $\alpha \in [\zeta, \eta)$ both

$$\overline{b}^{\alpha}_{[1,\infty)} = (1, 1)\cdots \quad \text{and} \quad \underline{b}^{\alpha}_{[1,\infty)} = (-1, 1)(-2, 1)(-2, 1)(-1, 1)\cdots .$$

Since $\ell_0(\alpha) = A^{-1} \cdot r_0(\alpha)$ for any α, the digit expansion of the $\ell_0(\alpha)$ for these α shows that $A^{-1}CWA^{-1} \cdot r_0(\alpha) = \ell_4(\alpha)$. Direct calculation, or an application of part (ii) of Lemma 5.1, shows that $(C^{-1}AC)\ AC = A^{-1}CWA^{-1}$. Hence, for all $\alpha \in [\zeta, \eta)$, the hypotheses of Lemma 4.2 are fulfilled with $i = 5$ and $j = 2$. Therefore, $r_2(\alpha) = \ell_5(\alpha)$ holds, and we conclude that $[\zeta, \eta)$ is a synchronization interval.

Note that in Figure 4.1, the visible intersection of this first dark gray branch with the light gray branch (which corresponds to $\ell_1 = A^{-1}C \cdot \ell_0$) marks a point of what one could call "accidental" synchronization. That is, for $\alpha = (2 - \sqrt{2})/4 = 0.146\cdots$, we have $r_1(\alpha) = \ell_1(\alpha)$. Of course, this implies that each of these is periodic. In particular, this and indeed any accidental synchronization occurs at an algebraic value of α.

We seek synchronization intervals of the form $[\zeta, \eta)$, where the endpoints are identified by $R \cdot r_0(\zeta) = r_0(\zeta)$ and $LA \cdot \ell_0(\eta) = r_0(\eta)$, for certain $R, L \in G_{3,n}$. Our synchronization intervals form a subset of full measure; to prove this, it will be very helpful to have the digits of the ζ, η. The following is key to finding these digits.

Lemma 4.5. *Fix $m = 3, n \geq 3$, an interval $[\zeta, \eta] \subseteq (0, \gamma_{3,n})$ and $i, j \in \mathbb{N}$. Suppose that there are $R, L \in G_{3,n}$ (none of which is the identity) such that*

(a) $L = C^{-1}ACR$;
(b) $R \cdot r_0 = r_{j-1}$ *for all* $\alpha \in [\zeta, \eta]$;
(c) $LA \cdot \ell_0 = \ell_{i-1}$ *for all* $\alpha \in [\zeta, \eta)$, *while* $LA \cdot \ell_0(\eta) = r_0(\eta)$;
(d) $R \cdot r_0(\zeta) = \ell_0(\zeta)$.

Then

(i) $\ell_{i-1}(\eta) = A^{-1}LA \cdot \ell_0(\eta) = \ell_0(\eta)$;
(ii) $r_j(\eta) = A^{k+1}C \cdot r_{j-1}(\eta) = r_1(\eta)$, *where k is such that* $A^kC \cdot r_0(\eta) = r_1(\eta)$;
(iii) $A^{-2}CLA \cdot \ell_0(\zeta) = \ell_1(\zeta)$.

Proof. For any α, the identity (a) implies $LA \cdot \ell_0 = C^{-1}ACR \cdot r_0$.

Recall that for all α, $r_0(\alpha) \notin \mathbb{I}_\alpha$. Now set $\alpha = \eta$. Hypothesis (c) implies that $\ell_{i-1} = A^{-1}LA \cdot \ell_0 = \ell_0$. Now, if $r_1 = A^kC \cdot r_0$, then $r_1 = A^kCLA \cdot \ell_0$, which again by (a) gives $r_1 = A^kC\ C^{-1}ACR \cdot r_0 = A^{k+1}CR \cdot r_0$. Now (b) gives $A^{k+1}C \cdot r_{j-1} = r_1$.

Finally, (d) with $\ell_1 = A^{-1}C \cdot \ell_0$ yields $\ell_1(\zeta) = A^{-1}CR \cdot r_0(\zeta)$. Hypothesis (a) now yields that $A^{-2}CLA \cdot \ell_0(\zeta) = \ell_1(\zeta)$. □

We will be describing synchronization subintervals of $\alpha \in [0, 1]$ in terms of common initial portions of the digits of $r_0(\alpha)$.

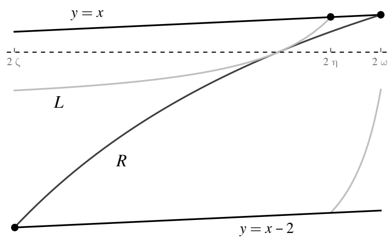


Figure 4.2. Determining a synchronization interval $[\zeta, \eta)$, compare with Example 4.4. Here, $m = n = 3$ and $t = t_{3,3} = 2$. The labels L, R mark respectively the curves $y = A^{-1}CWA^{-1} \cdot r_0(\alpha)$, $y = AC \cdot r_0(\alpha)$ where $\alpha = x/2 = x/t_{3,3}$. (In terms of Definitions 4.7 and 4.32 for $R_{k,v}$ and $L_{k,v}$ in general, these are $y = L_{1,1} \cdot r_0(\alpha)$, $y = R_{1,1} \cdot r_0(\alpha)$, respectively.) The dark is the single branch of $y = r_1(\alpha)$ while light gray colors the two branches of $y = \ell_4(\alpha)$ for $(5 - \sqrt{21})/2 < x < (3 - \sqrt{5})/2$. The x-axis is shown as a dotted line.

Definition 4.6.

(1) If the α-digits for some x are all of the form $(k_i, 1)$, it is convenient to suppress the notation indicating that the exponent of C is simply one. We refer then to *simplified digits*, and uniformly use a d instead of a b in notation referring to simplified digits. Thus the statement $\underline{d}^{\alpha}_{[1,\infty)} = k_1, k_2, \dots$ is equivalent to $\underline{b}^{\alpha}_{[1,\infty)} = (k_1, 1)(k_2, 1)\dots$ and similarly for expressions involving $\overline{d}$. Of course, sequences of simplified digits are ordered by way of the order (1.8).

(2) Given $s \in \mathbb{N}$ and integers $c_1, c_2, \dots, c_s$ and $d_1, d_2, \dots, d_{s-1}$, let $v = c_1 d_1 \cdots d_{s-1} c_s$ and for any $k \in \mathbb{N}$, define the upper (simplified) *digit sequence* of k and v as

$$\overline{d}(k, v) = k^{c_1}, (k+1)^{d_1}, \cdots, (k+1)^{d_{s-1}}, k^{c_s}.$$

(3) Define an order on the words w, v of the form above by $v \preceq w$ whenever $\overline{d}(k, v) \preceq \overline{d}(k, w)$ for some positive k. Hence, $c_1 d_1 \cdots d_{s-1} c_s \prec c'_1 d'_1 \cdots d'_{r-1} c'_r$ if there is an i such that $c_j = c'_j$, $d_j = d'_j$ for all $j < i$ and either (1) $c_i < c'_i$, or (2) $c_i = c'_i$ and $d_i > d_{i+1}$.

Definition 4.7.

(1) The length of $\overline{d}(k, v)$ is $\overline{S}(v) := |\,\overline{d}(k, v)\,| = c_s + \sum_{i=1}^{s-1} (c_i + d_i)$. Notice that $\overline{S}(v)$ is indeed independent of k.

(2) The α-*cylinder* of k, v is

$$I_{k,v} = \{\alpha \mid \overline{d}^{\alpha}_{[1,\overline{S}(k,v)]} = \overline{d}(k, v)\}.$$

That is, $I_{k,v}$ is the set of all α such that the initial simplified digits of $r_0(\alpha)$ are $k^{c_1}, (k+1)^{d_1} \cdots (k+1)^{d_{s-1}}, k^{c_s}$.

(3) The *right matrix* of k, v is

$$R_{k,v} = (A^k C)^{c_s}\ (A^{k+1}C)^{d_{s-1}}(A^k C)^{c_{s-1}} \cdots (A^{k+1}C)^{d_1}(A^k C)^{c_1}\,.$$

(4) The *potential synchronization interval* associated to k, v is $J_{k,v} = [\zeta, \eta)$, where $\zeta = \zeta_{k,v}$ and $\eta = \eta_{k,v}$ are such that

$$R_{k,v} \cdot r_0(\zeta) = \ell_0(\zeta) \quad \text{and} \quad r_0(\eta) = C^{-1}ACR_{k,v} \cdot r_0(\eta)\,.$$

Note that if $\alpha \in J_{k,v}$, then $R_{k,v} \cdot r_0(\alpha) = r_{\overline{S}(v)}(\alpha)$.

4.2. Tree of words and a partition

From Lemma 4.5, we have that

$$\overline{d}^{\eta_{k,v}}_{[1,\infty)} = k^{c_1}, (k+1)^{d_1}, \cdots, (k+1)^{d_{s-1}}, k^{c_s}, k+1, \cdots$$

and furthermore, this sequence continues with the digits of $r_1(\eta_{k,v})$. Thus, this sequence is periodic with pre-period $\overline{d}(k, v)$ and period $k+1, k^{c_1-1}, (k+1)^{d_1}, \cdots$ $\cdots, (k+1)^{d_{s-1}}, k^{c_s}$. This period is expressible in terms of the word v', which we now define.

Definition 4.8. For each $s > 1$ and each word $v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s$, define

$$v' = \begin{cases} 1(c_1 - 1)d_1 c_2 \cdots c_{s-1} d_{s-1} c_s & \text{if} \quad c_1 \neq 1 \\ (d_1 + 1)c_2 \cdots c_{s-1} d_{s-1} c_s & \text{otherwise}\,. \end{cases}$$

We interpret this also to mean that when $v = c$ with $c > 1$ then $v' = 1(c-1)$, and when $v = 1$ then $v' = 1$.

As necessary, we extend the notion $\overline{d}(k, v)$ in the natural manner to include the setting of infinite words, and also extend the notion of $R_{k,v}$ to include more general words.

Lemma 4.9. *Let $k \in \mathbb{N}$ and $v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s$. If $\eta_{k,v} \in I_{k,v}$, then $\overline{d}^{\eta_{k,v}}_{[1,\infty)} = \overline{d}(k, v(v')^\infty)$.*

Proof. For simplicity, let $\eta = \eta_{k,v}$. From Lemma 4.5 (ii), the simplified digit of $r_0(\eta)$ following $\overline{d}(k, v)$ is $k+1$, and thereafter the simplified digits begin with the sequence of $r_1(\eta)$. This can be expressed as $\overline{d}^{\eta}_{[1,\infty)} = \overline{d}(k, v)\,\overline{(k+1)\,\overline{d}(k, v)_{[2,\overline{S}]}}$.

When $v = c_1 = c$ we have $\overline{S} = c$ and we find $\overline{d}^{\eta}_{[1,\infty)} = k^c, \overline{(k+1), k^{c-1}}$ (when $c = 1$, we take this to mean $k, \overline{k+1}$.) For longer v, we must group the new occurrence of $k+1$. This grouping depends on whether $c_1 = 1$ or not. In either case, one indeed finds that $\overline{d}^{\eta_{k,v}}_{[1,\infty)} = \overline{d}(k, v(v')^\infty)$. $\square$

Definition 4.10. Set $\Theta_{-1}(c_1) = c_1 + 1$ and $\Theta_q(1) = 1q1$ for $q \geq 1$. For $c > 1$, set $\Theta_q(c) = c[1(c-1)]^q 1c$ for any $q \geq 0$. (To avoid double labeling and also to stay within our desired language, $\Theta_0(1)$ is undefined; note that $\Theta_1(1) = 111$, compare with $\Theta_0(c) = c1c$ for $c > 1$.)

We now recursively define values of the operators Θ_q. Suppose $v = \Theta_p(u) = uv''$ for some $p \geq 0$ and some suffix v''. Then define for any $q \geq 0$

$$\Theta_q(v) = v(v')^q v'' .$$

Definition 4.11. Let $\mathcal{V}$ denote the set of all words obtainable from $v = 1$ by finite sequences of applications of the various Θ_q. We call v the *parent* of each $\Theta_q(v)$, and also refer to $\Theta_q(v)$ as a *child* of v. See Figures 4.3 and 4.4 for portions of this directed tree.

$$\begin{tikzcd}
& 1 \arrow[dl, "\Theta_s"'] \arrow[d, "\Theta_q"] \\
1s1 \arrow[d, "\Theta_t"] & 1q1 \arrow[d, "\Theta_s"] \\
1s1[(s+1)1]^t s1 & 1q1[(q+1)1]^s q1 \arrow[ul, dotted, "\mathscr{D}"'] \arrow[d, "\Theta_t"] \\
& 1q1[(q+1)1]^s q1\{[(q+1)1]^{s+1} q1\}^t [(q+1)1]^s q1 \arrow[ul, dotted, "\mathscr{D}"]
\end{tikzcd}$$

Figure 4.3. Each vertex of the directed tree $\mathcal{V}$ of Definition 4.11, has countably infinite valency. A small portion of $\mathcal{V}$, with some values of $\mathscr{D}$ of Definition 4.17, is indicated. Here q, s, t are any positive integers.

$$\begin{tikzcd}
1 \arrow[r, "\Theta_{-1}"] & 2 \arrow[r, "\Theta_{-1}"] \arrow[d, "\Theta_{u-1}"'] & 3 \arrow[d, "\Theta_0"] & \\
& 2(11)^{u-1}12 & 313 \arrow[ul, dotted, "\mathscr{D}"] \arrow[r, "\Theta_0"] \arrow[d, "\Theta_u"] & 31313 \arrow[ul, dotted, "\mathscr{D}"'] \\
& & 313(1213)^u 13 \arrow[ul, dotted, "\mathscr{D}"] &
\end{tikzcd}$$

Figure 4.4. A second small portion of $\mathcal{V}$ of Definition 4.11, with some values of $\mathscr{D}$ of Definition 4.17. Here u is any positive integer. Note that the pieces of $\mathcal{V}$ depicted here and in Figure 4.3 share the vertex 1.

The following result gives the basic structure of the collection of potential synchronization intervals.

Theorem 4.12. *We have the following partition*

$$(0, \gamma_{3,n}) = \bigcup_{k=1}^{\infty} I_{k,1}\,.$$

Furthermore, for each $k \in \mathbb{N}$ and each $v \in \mathcal{V}$, the following is a partition:

$$I_{k,v} = J_{k,v} \cup \bigcup_{q=q'}^{\infty} I_{k,\Theta_q(v)}\,,$$

where $q' = 0$ unless $v = c_1$, in which case $q' = -1$.

When $n = 3$, the first statement of the theorem describes the partition given by the intervals of definition of the leftmost dark gray branches of Figure 4.1. See also Figure 4.5.

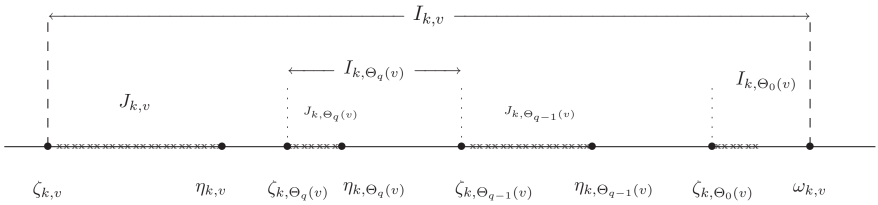


Figure 4.5. A hint of the partition of a general α-cylinder $I_{k,v}$. That $\omega_{k,\Theta_q(v)} = \zeta_{k,\Theta_{q-1}(v)}$ holds for appropriate v is part of Lemma 4.28.

4.2.1. *Palindromes*

The notation $\overleftarrow{v}$ denotes the word formed by taking the letters of v in reverse order. Thus, v is a palindrome if and only if $v = \overleftarrow{v}$.

Proposition 4.13. *Suppose $v \in \mathcal{V}$. Then:*

(1) *v can be expressed as a word in an alphabet of at most three letters;*
(2) *v is a palindrome;*
(3) *if $v = \Theta_q(u)$ for some $q \geq 1$, then $v = uxu$ for some palindrome x.*
(4) *if u is the parent of v, then there is a palindrome y such that $v' = yu$;*
(5) *with this same y, one has $v'v'' = yv$;*
(6) *if further $v \neq c_1$, then there are palindromes a, z such that $v'' = az$, $y = z'a$.*

Proof. The first statement naturally has two cases: if $c_1 = 1$ then we claim that all $c_i = 1$ and all d_j are contained in $\{d_1 + 1, d_1\}$; if $c_1 > 1$ then all $d_j = 1$ and $c_i \in \{c_1, c_1 - 1\}$. We prove this by induction. Our bases cases are: $\Theta_{-1}(c_1) = c_1 + 1$ and $\Theta_q(1) = 1q1$ for $q \geq 1$; for $c > 1$, set $\Theta_q(c) = c[1(c-1)]^q 1c$ for any $q \geq 0$. The statement clearly holds here. Thereafter, v, v' are words in these small alphabets, and v'' is a subword of v hence every $\Theta_q(v)$ has the desired property.

Statement (3) follows from (2). Statement (4) implies (5), as $(yu)v'' = y(uv'') = yv$. It remains to prove (2), (4) and (6).

Beyond easily handled cases of short v, there are naturally three cases to consider.

Case 1. Suppose $v = \Theta_0^h(c)$ for some $h \geq 1$ and some $c > 1$. Induction gives $v = \Theta_0^h(c) = c(1\,c)^h$. This is obviously a palindrome, that is (2) holds. We find

$$v' = 1(c-1)(1\,c)^h = 1(c-1)1\ \ c(1\,c)^{h-1} = 1(c-1)1\ \Theta_0^{h-1}(c) = yu,$$

where $y = 1(c-1)1$ and $u = \Theta_0^{h-1}(c)$. But, here $v = \Theta_0(u)$ is the parent of v, and hence (4) also holds in this case.

Set $z = c$ and $a = 1$. Then $y = z'a$ and $v'' = 1\,c = az$. Therefore, (6) also holds in this case.

Case 2. Suppose $v = \Theta_q(u)$ for some $q \geq 1$. If $u = c_1$, all of (2), (4) and (6) are easily verified.

Assume now that (2), (4) and (6) hold for u, in the sense that $u = \mathfrak{Z}\mathfrak{a}\mathfrak{z}$, $u'' = \mathfrak{a}\mathfrak{z}$ with u, $\mathfrak{Z}$, $\mathfrak{a}$, $\mathfrak{z}$ and $\mathfrak{z}'\mathfrak{a}$ all palindromes. Since $u = \overleftarrow{u}$, we have $u' = (\mathfrak{z}\mathfrak{a}\mathfrak{Z})' = \mathfrak{z}'\mathfrak{a}\mathfrak{Z}$. Therefore,

$$\begin{aligned} v = \Theta_q(u) &= u(u')^q u'' = u(u')^{q-1}u'(\mathfrak{a}\mathfrak{z}) = u(u')^{q-1}\mathfrak{z}'\mathfrak{a}\mathfrak{Z}(\mathfrak{a}\mathfrak{z}) \\ &= u(u')^{q-1}\mathfrak{z}'\mathfrak{a}\,u = u(\mathfrak{z}'\mathfrak{a}\mathfrak{Z})^{q-1}\mathfrak{z}'\mathfrak{a}\,u. \end{aligned}$$

Since $\mathfrak{z}'\mathfrak{a}$ and $\mathfrak{Z}$ are palindromes, we find that v is a palindrome; that is (2) holds.

Set $y = (\mathfrak{z}'\mathfrak{a}\mathfrak{Z})^q\mathfrak{z}'\mathfrak{a}$. This is clearly also a palindrome. Since

$$v' = (\mathfrak{z}'\mathfrak{a}\mathfrak{Z})^q\mathfrak{z}'\mathfrak{a}\,u$$

(4) also holds.

Now, $v = uv''$ gives $v'' = (\mathfrak{z}'\mathfrak{a}\mathfrak{Z})^{q-1}\mathfrak{z}'\mathfrak{a}\,u$. Let $a = (\mathfrak{z}'\mathfrak{a}\mathfrak{Z})^{q-1}\mathfrak{z}'\mathfrak{a}$ and $z = u$. Then $v'' = az$ and $y = z'a$. That is, (6) holds.

Case 3. Suppose that $v = \Theta_0(u)$ and $v \neq \Theta_0^h(c)$ for any $h \geq 1$ and any $c = c_1$. Assume that (2), (4) and (6) all hold for u in the sense described in the proof of the previous case.

We find

$$v = uu'' = (\mathfrak{Z}\mathfrak{a}\mathfrak{z})\mathfrak{a}\mathfrak{z}$$

and thus

$$\overleftarrow{v} = \overleftarrow{u''}\ \overleftarrow{u} = \overleftarrow{\mathfrak{a}\mathfrak{z}}\,u = \mathfrak{z}\mathfrak{a}\mathfrak{Z}\mathfrak{a}\mathfrak{z}.$$

Thus (2) holds in this case.

Now,

$$v' = u'u'' = (\mathfrak{z}'\mathfrak{a}\mathfrak{Z})\mathfrak{a}\mathfrak{z} = (\mathfrak{z}'\mathfrak{a})u.$$

Thus (4) holds; since $v'' = u''$, (6) also holds. □

Remark 4.14. We thus have for $v \in \mathcal{V}$, other than those v of the form $v = c_1$,

$$\begin{cases} v &= uv'' = uaz \\ v' &= yu = z'au. \end{cases}$$

In the case of $v = \Theta_q(u)$ with $q \geq 1$, we have $z = u$. (Thus in the previous proposition, x is a.)

The child $\Theta_q(v)$ has length less than twice the length of v only when $q \in \{-1, 0\}$. The following addresses the setting of $q = 0$.

Lemma 4.15. *Suppose that u is a child of $Z \in \mathcal{V}$. Then the palindrome $\Theta_0(u)$ is characterized by the property of u being both a prefix and a suffix, and these subwords having exactly the subword Z in common.*

Proof. Let $v = \Theta_0(u)$. Since $v'' = u''$, we can write $u = Zaz$ and $v = uaz = Zazaz$ with each of Z, u, a, z palindromes. Now, $v = (Zaz)az = zaZaz$. We have $v = za\underbrace{Zaz}_{u} = \underbrace{zaZ}_{u}az$. Thus the property does indeed characterize $\Theta_0(u)$. □

Lemma 4.16. *If $v \in \mathcal{V}$, then $v' \prec v''$.*

Proof. One directly verifies the result for any $v = c_1$ and for any $v = \Theta_q(c_1)$. The remaining possibilities can be divided into two cases.

Case 1. When $v = \Theta_q(u)$ for some $q \geq 1$, we can write $v = uau$. We find $v' = (u'a)u = a\overleftarrow{u'}u$, and $v'' = au$. Writing $u = c_1d_1 \cdots c_{s-1}d_{s-1}c_s$, from Definition 4.8 and that fact that u is a palindrome show that $\overleftarrow{u'}$ and u agree until the relationship is determined by $d_1c_1 \succ (d_1 + 1)$ or $d_1c_1 \succ d_1(c_1 - 1)\,1$, when $c_1 = 1$ or $c_1 > 1$ respectively.

Case 2. Suppose that $v = \Theta_0(u)$. Then as in Lemma 4.15 we can write $v = uaz = zau$, $v'' = az$. We have $v' = z'au = a\overleftarrow{z'}u$. As in the previous case, we find that $v' \prec v''$. □

4.2.2. *Derived words*

We will often argue by induction on the length of v. These arguments rely on the following map, $\mathscr{D}$, giving the *derived word* $\mathscr{D}(v)$ of v whenever v is of length at least three.

Definition 4.17. Let $v = c_1d_1 \cdots d_{s-1}c_s$, with $s \geq 2$.

(i) If $c_1 = c > 1$, and v is such that $d_1 = 1$ and the set of c_i, $1 \leq i \leq s$, is contained in the set of two letters $\{a = c_1, b = c_1 - 1\}$, express

$$v = (a\,1)^{e_1}(b\,1)^{f_1} \cdots (b\,1)^{f_{g-1}}(a\,1)^{e_g - 1}a.$$

(ii) if $c_1 = 1$, and v is such that the set of d_j, $1 \leq j < s$ is contained in the set of two letters $\{a = d_1, b = d_1 + 1\}$, express

$$v = (1\,a)^{e_1}(1\,b)^{f_1} \cdots (1\,b)^{f_{g-1}}(1\,a)^{e_g}1.$$

In both cases, let

$$\mathscr{D}(v) = e_1 f_1 \cdots f_{g-1} e_g\,.$$

Note that (the proof of) part (1) of Proposition 4.13 shows that $\mathscr{D}(v)$ is defined for each $v \in \mathcal{V}$ of length at least three. We call the various subwords $(c_1\,1)^{e_i}$, $(c_1 - 1\,1)^{f_j}$, $(1\,d_1)^{e_i}$, $(1\,d_1+1)^{f_j}$ *full blocks* for v.

For examples of $\mathscr{D}(v)$, the reader is encouraged to choose various values of the indices q, s, t, u of Figures 4.3 and 4.4.

Lemma 4.18. *The map $\mathscr{D}$ sends $\mathcal{V}$ to itself, preserving the parent-child relationship. That is, if $u \in \mathcal{V}$ is of length at least two, and $v = \Theta_q(u)$, then there is a q' such that $\mathscr{D}(v) = \Theta_{q'}(\,\mathscr{D}(u)\,)$. Moreover, $q' = q$ unless $v = \Theta_0^h(c)$ for some $h \geq 1$.*

Proof. Just as in the proof of Proposition 4.13, there are easily verified base cases which we leave to the reader. We treat three main cases, see Figure 4.6, as well as Figures 4.3 and 4.4.

$$\begin{tikzcd}
c \arrow[d, "\Theta_0^{h-1}"] & \\
c(1\,c)^{h-1} \arrow[r, "\mathscr{D}"] \arrow[d, "\Theta_0"] & h \arrow[d, "\Theta_{-1}"] \\
c(1\,c)^h \arrow[r, "\mathscr{D}"] & h+1
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[d] & \\
v \arrow[r] \arrow[d, "\Theta_{q\geq 1}"] & \mathscr{D}(v) \arrow[d, "\Theta_{q\geq 1}"] \\
v(v')^q v'' \arrow[r, "\mathscr{D}"] & \Theta_q(\,\mathscr{D}(v)\,)
\end{tikzcd}
\qquad
\begin{tikzcd}
u \neq c_1 \arrow[d] & \\
v \arrow[r] \arrow[d, "\Theta_0"] & \mathscr{D}(v) \arrow[d, "\Theta_0"] \\
vv'' \arrow[r, "\mathscr{D}"] & \Theta_0(\,\mathscr{D}(v)\,)
\end{tikzcd}$$

Figure 4.6. Taking derived words, $v \mapsto \mathscr{D}(v)$, respects parent-child relations. See Lemma 4.18, and compare with Figures 4.3 and 4.4.

Case 1. Suppose $v = \Theta_0^h(c)$ for some $h \geq 2$ and some $c > 1$. We have $v = c(1\,c)^h = (c\,1)^h c$. Hence, $\mathscr{D}(v) = h+1$, and thus also $\mathscr{D}(\,\Theta_0^{h-1}(c)\,) = h$. We note that $\Theta_{-1}(h) = h+1$, and of course that $v = \Theta_0(\,\Theta_0^{h-1}(c)\,)$. That is, the result holds in this case.

Case 2. Suppose $v = \Theta_q(u)$, with $q \geq 1$ for some $u \in \mathcal{V}$. Although this part of the proof is fairly straightforward, it has perforce a panoply of variables representing words; the reader may wish to consult (4.2), below, as a guide.

We can write $\mathscr{D}(v) = \mathscr{D}(\,u(u')^q u''\,) = \mathscr{D}(\,u(u')^{q-1} y\,u\,)$. Calculation, simply using the definition of $\mathscr{D}$, shows that $\mathscr{D}(\,u(u')^q u''\,)$ has prefix $\mathscr{D}(u)$. The definition also yields that the derived word of any palindrome is also a palindrome, thus here $\mathscr{D}(v)$ also has suffix $\mathscr{D}(u)$. Direct calculation shows that for any words u, x and any $p \geq 1$, $\mathscr{D}(u(u')^p x) = \mathscr{D}(u)\,\{\,[\mathscr{D}(u)]'\,\}^p X$ for some X. By Proposition 4.13, there is some palindrome Y such that $[\mathscr{D}(u)]' = Y\mathscr{D}(Z)$ where $\mathscr{D}(Z)$ is the parent of $\mathscr{D}(v)$. Therefore, $\mathscr{D}(v) = \mathscr{D}(\,u(u')^{q-1}y\,u\,) = \mathscr{D}(u)\,[Y\mathscr{D}(Z)]^{q-1}X\mathscr{D}(u)$ for some (new) X. Since $\mathscr{D}(v)$ is a palindrome, we have $X = Y$. Again by Proposition 4.13, $Y\mathscr{D}(u) = Y\mathscr{D}(Z)[\mathscr{D}(u)]'' = [\mathscr{D}(u)]'\,[\mathscr{D}(u)]''$. Therefore, $\mathscr{D}(v) = \mathscr{D}(u)\,\{\,[\mathscr{D}(u)]'\,\}^{q-1}\,[\mathscr{D}(u)]'\,[\mathscr{D}(u)]'' = \Theta_q(\,\mathscr{D}(u)\,)$.

As a summary, we have

$$\begin{aligned}\mathscr{D}(u(u')^q u'') &= \mathscr{D}(u(u')^{q-1}u'u'') = \mathscr{D}(u(u')^{q-1}yu)\\ &= \mathscr{D}(u)\,\{[\mathscr{D}(u)]'\}^{q-1}X\mathscr{D}(u) = \mathscr{D}(u)\,[Y\mathscr{D}(Z)]^{q-1}X\mathscr{D}(u)\\ &= \mathscr{D}(u)\,[Y\mathscr{D}(Z)]^{q-1}Y\mathscr{D}(u) \qquad (4.2)\\ &= \mathscr{D}(u)\,\{[\mathscr{D}(u)]'\}^{q-1}Y\mathscr{D}(Z)[\mathscr{D}(u)]''\\ &= \mathscr{D}(u)\,\{[\mathscr{D}(u)]'\}^{q}[\mathscr{D}(u)]'' = \Theta_q(\,\mathscr{D}(u)\,).\end{aligned}$$

Case 3. Suppose that $v = \Theta_0(u)$ and $v \neq \Theta_0^h(c)$ for any $h \geq 1$ and any $c = c_1$. Lemma 4.15 and the definition of $\mathscr{D}$ yield the result in this case. ☐

4.2.3. *Fullness of branches*

We aim to describe symbolically $T_{3,n,\alpha}$-orbits, and in particular to determine intervals in the parameter α where initial segments of such orbits share common digits. For any word determining sequences of digits, we must determine the endpoints of the parameter interval along which the word does describe *admissible* sequences of digits, see Figure 4.7. The following notion is key to this.

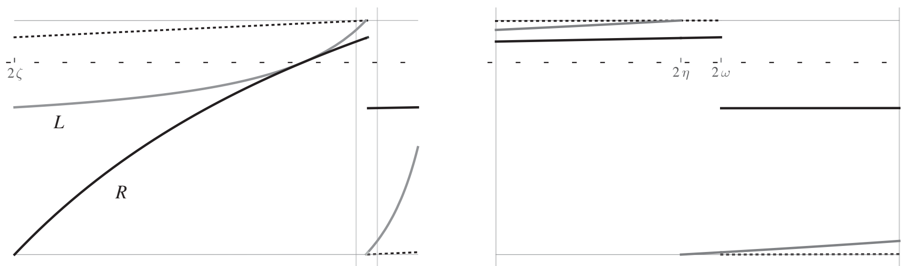


Figure 4.7. A non-full branch. Here $n = m = 3$, $v = 111$ and $k = 1$; we have that $\omega_{1,111}$ is determined by the fixed point of $R_{1,11}$. The labels L, R mark respectively the curves $y = L_{1,111} \cdot r_0(\alpha)$, $y = R_{1,111} \cdot r_0(\alpha)$ where $\alpha = x/2 = x/t_{3,3}$. Dark gray gives branches of $y = r_3(\alpha)$, while light gray colors the two branches of $y = \ell_9(\alpha)$; the branches of $y = r_2(\alpha)$ are dotted. The left portion has $2(9-\sqrt{15})/7 < x < (11-6\sqrt{2})/7$. The right "zooms in" to $0.35910 < x < 0.35915$. (This interval lies between the vertical gray lines in both portions.) The x-axis is shown as a dashed line. Thin gray lines give $y = x$ and $y = x - 2$. (This example is in fact the longest interval $I_{k,v}$ where $R_{k,v}$ is not full for any $n \geq 3$, $k \in \mathbb{N}$, $v \in \mathcal{V}$.)

Definition 4.19. Let u a word with alternating letters c_i, d_j. (We allow prefixes of words v including those that end with some d_j.)

(1) If u begins with say c_1 and ends in some d_j, then powers of u are again alternating words in the letters c_i, d_j. In the case that $u = c_1 \cdots c_j$ begins with c_1 and ends in some c_j, then we define $u^2 = c_1 \cdots (c_j + c_1) \cdots c_j$ and similarly for higher powers.
(2) We say that u is *full branched* if for any prefix $u_{[1,\ell]}$ of u, the inequality $u^\infty \preceq u_{[1,\ell]}^\infty$ holds. We denote the longest prefix of u that is full branched by $\mathfrak{f}(u)$. Note that an equivalent definition is: $\mathfrak{f}(u)$ is the longest prefix of u satisfying $(\,\mathfrak{f}(u)\,)^\infty = \min\{(\,u_{[1,\ell]}\,)^\infty \,:\, u_{[1,\ell]} \text{ is a prefix of } u\}$.
(3) We define $\omega_{k,u}$ as the α-value such that $R_{k,\mathfrak{f}(u)} \cdot r_0(\omega_{k,u}) = r_0(\omega_{k,u})$. That is, $r_0(\omega_{k,u})$ has the (T_α-inadmissible) simplified digit expansion $(\,\overline{d}(k, \mathfrak{f}(u)\,)\,)^\infty$.

Example 4.20.

(1) Recall that the leftmost dark gray branches in Figure 4.1 are branches of $r_1(\alpha)$ as a function of $r_0(\alpha)$, with $\alpha < \gamma_{3,3}$. These branches agree with portions of the graphs $y = A^k C \cdot x$ and $x = r_0(\alpha)$. For each k, the corresponding branch intersects with $y = x - t$ on the left, and $y = x$ on right. The leftmost of these two points is $x = r_0(\zeta_{k,1})$. Certainly for every α between such a set of intersection points, we have $r_0(\alpha) = k, \cdots$. On the other hand, by definition $\mathfrak{f}(1) = 1$, and thus $r_0(\omega_{k,1})$ is the fixed point of $A^k C \cdot x$. That is, $r_0(\omega_{k,1})$ is our right intersection point. Furthermore, for all $\alpha \in [\zeta_{k,1}, \omega_{k,1})$ we have $y = r_1(\alpha)$ is given by $r_1(\alpha) = A^k C \cdot r_0(\alpha)$. That is, the first simplified digit of $r_0(\alpha)$ is k. In other words, $I_{k,1} = [\zeta_{k,1}, \omega_{k,1})$.
Being a fixed point, $r_0(\omega_{k,1}) = k, k, \ldots,$ is purely periodic with period k. Note however that this is *not* a T_α-admissible expansion, as were it so then r_1 for this value of α would equal r_0. But, $r_0(\alpha) \notin \mathbb{I}_\alpha$!
(2) Now suppose $c > 1$. By definition, $\mathfrak{f}(c) = c$, that is $v = c_1$ is full branched. Therefore, for each k, $\omega_{k,c}$ is such that $r_0(\omega_{k,c})$ is the fixed point of $(A^k C)^c$. Of course, $(A^k C)^c$ has the same fixed point as $A^k C$. That is, $\omega_{k,c} = \omega_{k,1}$. Related to this, there are values of α sufficiently close to $\omega_{k,1}$ so that for each of these α, the first c simplified digits of $r_0(\alpha)$ are all equal to k. Equivalently, $r_0(\alpha), r_1(\alpha), \ldots, r_c(\alpha)$ are all in $\Delta_\alpha(k, 1)$. And, this is also to say that $\overline{d}(k, c)$ gives the first c simplified digits of $r_0(\alpha)$. The reader should easily find α in the complement of $I_{k,c}$ inside of $I_{k,1}$.
(3) Consider $v = 111$. We consider each prefix in turn. Of course $u = 1$ is full-branched. We next compare $(11)^\infty = 11 \cdot 11 \cdot 11 \cdot \cdots$ with 1^∞; by our convention for powers, we certainly find that $(11)^\infty \prec 1^\infty$. We next compare $(11)^\infty$ with $(111)^\infty = 111 \cdot 111 \cdot 111 \cdot \cdots = 121212\cdots$; certainly $(11)^\infty \prec (111)^\infty$. Therefore, $\mathfrak{f}(111) = 11$. Compare this with Figure 4.7.
(4) Consider $v = \Theta_1(313) = 313(1213)13$. Arguing just as for the previous case, we find that $(31)^\infty$ is the minimal element of $\{3^\infty, (31)^\infty, (313)^\infty\}$. Since $3131 = (31)^2$, we certainly have that $(31)^\infty = (3131)^\infty$, and thus this latter is our current candidate for the maximal length full branched prefix of v. We compare it with $(31312)^\infty$. We find that these infinite words agree in their first four letters, but in the fifth (a "c_j"-position) they differ. Confer Figure 4.8.

Since $31313 \prec 31315$, we find that $(3131)^\infty \prec (31315)^\infty$. We thus now compare $(3131)^\infty$ with $(313121)^\infty$. Confer Figure 4.8 we find that $(313121)^\infty$ is the smaller. One easily sees that is it also smaller than $(3131213)^\infty$. We now compare it with $(31312131)^\infty$. See Figure 4.8. Again, $(313121)^\infty$ is the smaller. One easily sees that also $(313121)^\infty \prec v^\infty$. Therefore, $\mathfrak{f}(v) = (313121)^\infty$.

$$\begin{array}{cccc|c|cccc} 3&1&\cdot 3&1&\cdot 3&1&\cdot 3&1&\cdots \\ 3&1&3&1&2{+}3&1&3&1&3 \end{array} \qquad \begin{array}{cccc|c|cccc} 3&1&3&1&\cdot 3&1&3&1&\cdots \\ 3&1&3&1&2&1&\cdot 3&1&3 \end{array} \qquad \begin{array}{cccccccccc|c|c} 3&1&3&1&2&1&\cdot 3&1&3&1&2&1 \\ 3&1&3&1&2&1&3&1&\cdot 3&1&3&\cdots \end{array}$$

Figure 4.8. Naive calculations for $\mathfrak{f}(\,\Theta_1(313)\,)$. From left to right, finding: $(31)^\infty \prec (31312)^\infty$; $(3131)^\infty \succ (313121)^\infty$; $(313121)^\infty \prec (31312131)^\infty$. The beginning of second copies of words are marked with a dot.

The following shows that certain phenomena illustrated in the above examples hold in general.

Lemma 4.21. *If $v \in \mathcal{V}$ is of length greater than one, then $\mathfrak{f}(v)$ has even length. In this case, for each $k \in \mathbb{N}$, a simplified digit expansion for $r_0(\omega_{k,v})$ is $\overline{d}(k, (\,\mathfrak{f}(v)\,)^\infty\,)$.*

Proof. Our convention for powers of words shows that any prefix u of odd length greater than one, thus having initial letter c_1 and final letter some c_i, has its second power including the letter $c_j + c_1$. Already u^2 is larger than the prefix $(c_1 d_1)^\infty$. The first statement thus holds.

For any word $u = c_1 d_1 \cdots d_j$, we have

$$\overline{d}(k, u^\infty\,) = [k^{c_1}, (k+1)^{d_1}, \cdots, (k+1)^{d_j}]^\infty = \overline{d}(k, u\,)^\infty.$$

Thus, since $\omega_{k,v}$ is defined to be $(\,\overline{d}(k, \mathfrak{f}(u)\,)\,)^\infty$, the result holds. □

The next result indicates the utility of the notion of full branchedness.

Lemma 4.22. *Let $v \in \mathcal{V}$ and fix $k \in \mathbb{N}$. The α-cylinder set $I_{k,v}$ is a subset of $[\zeta_{k,v}, \omega_{k,v})$.*

Proof. The set of α such that $\overline{d}^{\alpha}_{[1,\overline{S}(k,v)]} = \overline{d}(k, v)$ is contained in the interval $[\zeta, \omega)$ such that $R_{k,v} \cdot r_0(\zeta) = \ell_0(\zeta)$ and $R_{k,v} \cdot r_0(\omega) = r_0(\omega)$. (Note that this is implied by the connected nature of each of the $\Delta(k, l)$, confer Figure 1.4.) The left endpoint here is exactly $\zeta = \zeta_{k,v}$.

Now, the definition of the α-cylinder $I_{k,v}$ as the set of those α such that the digit sequence determined by k and v are α-admissible implies that $I_{k,v}$ is contained in the intersection of the corresponding α-cylinders for k and the prefixes u of v. In particular, the least right endpoint of these cylinders gives an upper bound of the right endpoint of $I_{k,v}$. But, each of these cylinders has its right endpoint bounded above by its own corresponding fixed point, $\overline{d}(k, u^\infty)$. Hence, we find that the right endpoint of $I_{k,v}$ is less than or equal to the least of these $\overline{d}(k, u^\infty)$. Since this least point is $\overline{d}(k, (\,\mathfrak{f}(v)\,)^\infty)$, we are done. □

Lemma 4.23. *Fix $m = 3$, and $n \in \mathbb{N}$. Then $\gamma_{3,n} = \omega_{1,1}$.*

Proof. Let $\gamma = \gamma_{3,n}$ By definition, $C^{-1} \cdot \ell_0(\gamma) = r_0(\gamma)$. Applying AC to both sides of this equality yields $r_0(\gamma) = AC \cdot r_0(\gamma)$. Since $\mathfrak{f}(1) = 1$, we have that $r_0(\omega_{1,1}) = AC \cdot r_0(\omega_{1,1})$ and thus $\gamma_{3,n} = \omega_{1,1}$. □

Recall that the full blocks of v are defined in Definition 4.17.

Lemma 4.24. *Suppose $v \in \mathcal{V}$ is of length greater than one. Then $\mathfrak{f}(v)$ ends with a full block of v.*

Proof. When $c_1 = 1$, we have $1\,d_1 \succ 1\,(d_1 + 1)$. When $c_1 > 1$, we have $c_1\,1 \succ (c_1 - 1)\,1$.

We treat the case of $c_1 = 1$, the other case being similar. First, if $\mathscr{D}(v) = e_1$ then one easily verifies that $\mathfrak{f}(v) = (1\,d_1)^{e_1}$. Otherwise, the argument of Lemma 4.21 showing that v of length greater than one have $\mathfrak{f}(v)$ of even length, gives here that $\mathfrak{f}(v)$ ends with some power of $[1\,(d_1+1)]$. Suppose we have a prefix of v which ends with a non-full block of this type, say $w = (1\,d_1)^{e_1}[1\,(d_1+1)]^{f_1} \cdots [1\,(d_1+1)]^{f_i - j}$. Then the square of w has the intermediate term $[1(d_1 + 1)]^{f_i - j}(1\,d_1)^{e_1}$, whereas the prefix that completes w to the end of the block $[1\,(d_1 + 1)]^{f_i}$ agrees with w^2 up to a replacement of $(1\,d_1)$ by $[1\,(d_1 + 1)]$. That is, this new prefix is smaller than w^2, and thus certainly its infinite power is smaller than w^2. The result thus holds in this case. □

Proposition 4.25. *Suppose $v \in \mathcal{V}$. Then*

$$\mathfrak{f}(\,\Theta_q(v)\,) = \begin{cases} (c\,1)^h & \textit{if } q = 0,\ v = \Theta_0^h(c) \\ v(v')^{q-1}y & \textit{if } q \geq 1, \textit{ with } y \textit{ as in Proposition } 4.13 \\ (ua)^{h+1} & \textit{if } q = 0,\ v = \Theta_0^h \circ \Theta_p(u),\ p \geq 1, \textit{ and } \Theta_p(u) = uau. \end{cases}$$

In particular, for all $v \in \mathcal{V}$, the word v is a prefix of $(\,\mathfrak{f}(v)\,)^2$.

Proof. Direct evaluation, as in Example 4.20 shows that $\mathfrak{f}(c\,1\,c) = c\,1$. Now suppose the result holds for some $h - 1 \geq 1$; the only remaining (even length) candidate prefixes that could be $\mathfrak{f}(v)$ are $(c\,1)^{h-1}$ and $(c\,1)^h$. Since these words have the same infinite powers, by definition the longer of these, that is $(c\,1)^h$, is $\mathfrak{f}(v)$.

For $q \geq 1$, base cases can be directly verified. We now use induction on the length of v, and thus assume $\mathfrak{f}(\,\Theta_q(\,\mathscr{D}(v)\,)\,) = \mathscr{D}(u)\,\{[\mathscr{D}(u)]'\}^{q-1}Y$, with Y as in (4.2). From (4.2), we then have

$$\mathscr{D}(\,v(v')^{q-1}yv\,) = \mathfrak{f}(\,\Theta_q(\,\mathscr{D}(v)\,)\,)\ \mathscr{D}(u).$$

Since the blocks of v of exponent e_i are larger than the blocks of exponent f_i, one finds that $\mathfrak{f}(\,\Theta_q(v)\,)$ can be no longer than $v(v')^{q-1}y$. Since $\mathfrak{f}(\,\Theta_q(v)\,)$ ends with a full block, $\mathfrak{f}(\,\Theta_q(v)\,)$ can also be no shorter than $v(v')^{q-1}y$. Thus, the result holds.

We can indeed assume that $\Theta_p(u) = uau$, see Remark 4.14. From the previous case, $\mathfrak{f}(\,\Theta_p(u)\,) = ua$. One easily finds that $\Theta_0^h \circ \Theta_p(u) = u(au)^{h+1}$. Thus, we seek to prove that $\mathfrak{f}(\,u(au)^{h+1}\,)$ is formed by dropping the suffix u. Here also, we can apply $\mathscr{D}$, as the verification for bases cases is straightforward.

That $(\,\mathfrak{f}(v)\,)^2$ has prefix v is easily checked in each case. □

The following result could well be placed earlier, but is not used until directly hereafter.

Lemma 4.26. *Let $k \in \mathbb{N}$ and $v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s$. If $\zeta_{k,v} \in I_{k,v}$ then $\overline{d}^{\zeta_{k,v}}_{[1,\infty)} = \overline{d}(k, v), \underline{d}^{\zeta_{k,v}}_{[1,\infty)}$. Furthermore, if $v = \overleftarrow{v}$, then $r_0(\zeta_{k,v})$ is the fixed point of $R_{k,\overleftarrow{v'}}$.*

Proof. By definition, $\zeta_{k,v}$ is such that $R_{k,v} \cdot r_0(\zeta_{k,v}) = \ell_0(\zeta_{k,v})$ and hence $r_0(\zeta_{k,v})$ is the fixed point of $AR_{k,v} = A^{k+1}C(A^kC)^{c_s-1} \quad (A^{k+1}C)^{d_{s-1}}(A^kC)^{c_{s-1}} \cdots (A^{k+1}C)^{d_1}(A^kC)^{c_1}$. When v is a palindrome, that is when $v = \overleftarrow{v}$, this matrix is indeed $R_{k,\overleftarrow{v'}}$. (Note that when $v = 1$ this matrix must be interpreted as $A^{k+1}C$.) The definition of $\zeta_{k,v}$ also shows that the sequence of upper simplified digits of $\zeta_{k,v}$ is formed by $\overline{d}(k, v)$ followed by the digits of $\ell_0(\zeta_{k,v})$. That is, $\overline{d}^{\zeta_{k,v}}_{[1,\infty)} = \overline{d}(k, v), \underline{d}^{\zeta_{k,v}}_{[1,\infty)}$. □

Corollary 4.27. *Fix $k \in \mathbb{N}$. Suppose that $v = \Theta_p(u)$ for some $u \in \mathcal{V}$ and some $p \geq 1$. Then for $h \geq 0$, we have $\omega_{k,\Theta_0^h(v)} = \omega_{k,v}$. Furthermore, $\lim_{h\to\infty} \zeta_{k,\Theta_0^h(v)} = \omega_{k,v}$.*

Proof. Proposition 4.25 implies that $(\,\mathfrak{f}(\,\Theta_0^h(v)\,)\,)^\infty = (ua)^\infty = (\,\mathfrak{f}(v)\,)^\infty$. Lemma 4.21 now implies that $\omega_{k,\Theta_0^h(v)} = \omega_{k,v}$. For each h, Lemma 4.26 shows that $\zeta_{k,\Theta_0^h(v)}$ has prefix $\overline{d}(k, \Theta_0^h(v)\,) = \overline{d}(k, u(au)^{h+1}\,)$. Therefore, the second statement holds as well. □

Lemma 4.28. *Suppose that $v \in \mathcal{V}$. Then for $q \in \mathbb{N}$ (except when $(v, q) = (1, 1)$),*

$$\mathfrak{f}(\Theta_q(v)) = \overleftarrow{(\Theta_{q-1}(v)\,)'}.$$

Furthermore, for each $k \in \mathbb{N}$, $\omega_{k,\Theta_q(v)} = \zeta_{k,\Theta_{q-1}(v)}$.

Proof. Proposition 4.25 gives $\mathfrak{f}(\Theta_q(v)) = v(v')^{q-1}y$, with notation as there. Now $(\,\Theta_{q-1}(v)\,)' = v'(v')^{q-1}v'' = (v')^{q-1}yv$. Since both $\Theta_{q-1}(v)$ and y are palindromes, $\overleftarrow{(\Theta_{q-1}(v)\,)'} = v(v')^{q-1}y$. Therefore, $\overleftarrow{(\Theta_{q-1}(v)\,)'} = \mathfrak{f}(\Theta_q(v))$.

By definition, $r_0(\omega_{k,\Theta_q(v)})$ is the fixed point of $R_{k,\mathfrak{f}(\Theta_q(v))}$. By Lemma 4.26, $r_0(\zeta_{k,\Theta_{q-1}(v)})$ is fixed by $R_{k,\overleftarrow{w}}$, where $w = (\Theta_{q-1}(v)\,)'$. Therefore, $\zeta_{k,\Theta_{q-1}(v)} = \omega_{k,\Theta_q(v)}$. □

The following illustrates how $\Theta_1(1)$ plays a role similar to the $\Theta_0(c)$, $c > 1$.

Lemma 4.29. *For each $k, c \in \mathbb{N}$,*

$$\omega_{k,c\,1\,c} = \zeta_{k,c+1}\,.$$

Proof. Since $\mathfrak{f}(c\,1\,c) = c\,1$, this follows since $A^{k+1}C(\,A^kC)^c \cdot r_0(\alpha) = r_0(\alpha)$ is equivalent to $(\,A^kC)^{c+1} \cdot r_0(\alpha) = \ell_0(\alpha)$. □

Lemma 4.30. *Suppose that $v \in \mathcal{V}$. Then $v\,(v')^\infty \prec (\,\mathfrak{f}(v)\,)^\infty$.*

Proof. The result is immediate for $v = c_1$. We treat our usual remaining cases.

Case 1. Suppose $v = \Theta_0^h(c)$ for some $h \geq 1$ and some $c > 1$. We have $v = c(1\,c)^h$, and $\mathfrak{f}(v) = (c\,1)^h$. Since $v\,(v')^\infty = c(1\,c)^h\,[\,1\,(c-1)\,1\,1]^\infty \prec (c\,1)^{h+2}$, the result holds in this case.

Case 2. $v = \Theta_p(u)$, $p \geq 1$. Write $v = uau$ in our usual decomposition. Then $v'' = au$ and $\mathfrak{f}(v) = ua$. We find

$$(\,\mathfrak{f}(v)\,)^3 = uau\,au\,a = vv''a \succ vv',$$

thus certainly $(\,\mathfrak{f}(v)\,)^\infty \succ v\,(v')^\infty$.

Case 3. Finally, suppose $v = \Theta_0^h(uau)$. Thus, $v = u(au)^{h+1}$, $v'' = au$ and $\mathfrak{f}(v) = (ua)^{h+1}$. Hence, $(\,\mathfrak{f}(v)\,)^2 = u(au)^{h+1} \cdot au \cdot a(ua)^{h-1} = vv''a(ua)^{h-1}$. Since $v'' \succ v'$, it follows that $(\,\mathfrak{f}(v)\,)^2 \succ vv'$, and thus $(\,\mathfrak{f}(v)\,)^\infty \succ v\,(v')^\infty$. □

4.3. The complement of the potential synchronization intervals is a Cantor set

Proposition 4.31. *For all $v \in \mathcal{V}$, and all $k \in \mathbb{N}$, both*

$$I_{k,v} = [\zeta_{k,v}, \omega_{k,v}) \;\; \textit{and} \;\; I_{k,v} \supset J_{k,v}\,.$$

Proof. Fix $k \in \mathbb{N}$. We argue by induction on the length of the word v. Recall that by Lemma 4.9, $\eta_{k,v} \in I_{k,v}$ implies that $\overline{d}(k, v(v')^\infty)$ gives the sequence of simplified digits of $r_0(\eta_{k,v})$.

The base cases, given by $v = c_1$, are easily verified.

Case 1. Suppose $v = \Theta_0^h(c)$ for some $h \geq 1$ and some $c > 1$. We have $v = c(1\,c)^h$. We begin by assuming the result for both words $c, c+1$. We then have $\eta_{k,c} < \zeta_{k,c+1} < \omega_{k,c+1} = \omega_{k,c}$. Therefore, there must be proper extensions of the word $c_1 = c$ that are admissible on $[\eta_{k,c}, \zeta_{k,c+1})$. By Lemma 4.9, $\overline{d}^{\,\eta_{k,c}}_{[1,\infty)} = k^c,\, \overline{(k+1), k^{c-1}}$; that is, $c\,1$ is admissible at this value of α. Now, $(A^kC)^{c+1} \cdot r_0(\alpha) = \ell_0(\alpha)$ determines the value $\alpha = \zeta_{k,c+1}$. Equivalently, $A^{k+1}C(A^kC)^c \cdot r_0(\alpha) = r_0(\alpha)$.

From this last, we conclude that $\mathfrak{f}(c\,1\,c) = c\,1$ is admissible throughout $[\eta_{k,c}, \zeta_{k,c+1})$, with arbitrarily high powers of $c\,1$ admissible for α sufficiently close but smaller than $\zeta_{k,c+1}$. (Recall that $\omega_{k,c\,1\,c} = \zeta_{k,c+1}$.) Since $\Theta_0(c) = c\,1\,c$ is a prefix of $(c\,1)^2$, we find that there is an interval with right endpoint $\omega_{k,c\,1\,c}$ on which the word $\Theta_0(c)$ is admissible. The left endpoint of this interval is characterized as the leftmost α such that the third letter is admissible; it must hence be $\zeta_{k,c\,1\,c}$. Thus, our result holds when $h = 1$. We now partition the interval $[\eta_{k,c}, \zeta_{k,c+1})$ according to the highest power of $c\,1$ that is admissible for α, from which the result follows.

Case 2. $v = \Theta_p(u)$, $p \geq 1$. Write $v = uau$ in our usual decomposition. By hypothesis, $\overline{d}(k, v)$ gives a prefix of $\overline{d}^{\alpha}_{[1,\infty)}$ for all $\alpha \in [\zeta_{k,v}, \omega_{k,v})$.

We have $\mathfrak{f}(v) = ua$, and the admissibility of $v = uau$ implies the admissibility of ua on $[\zeta_{k,v}, \omega_{k,v})$. Thus for each $\alpha \in [\zeta_{k,v}, \omega_{k,v})$, there exists a maximal $N = N(\alpha) \in \mathbb{N}$ such that $u(au)^N$ gives a prefix of $\overline{d}^{\alpha}_{[1,\infty)}$. Since $\omega_{k,v}$ corresponds to the fixed point of R_{ua}, the values $N(\alpha)$ are unbounded. Furthermore, if $N(\alpha) = \alpha$, then for all $\alpha' \in [\alpha, \omega_{k,v})$ we have $N(\alpha') \geq N(\alpha)$. Thus, (since the hypothesis implies that $N = 1$ is realized) each $N \in \mathbb{N}$ is realized as $N(\alpha)$ for some $\alpha \in I_{k,v}$, and we can partition $I_{k,v}$ by subintervals identified from the value of $N(\alpha)$.

Now, $\Theta_0(v) = vv'' = u(au)^2$, and hence $\Theta_0(v)$ is admissible on $[\zeta_{k,\Theta_0(v)}, \omega_{k,v})$. Similarly, each $\Theta_0^h(v)$ is admissible on $[\zeta_{k,\Theta_0^h(v)}, \omega_{k,v})$. By considering our ordering on words, it is clear that $\zeta_{k,\Theta_0^h(v)} < \eta_{k,\Theta_0^h(v)} < \omega_{k,v}$. Therefore, the result holds for all $\Theta_0^h(v)$.

The admissibility v on all of $I_{k,v}$ and the admissibility of $\Theta_0(v)$ on exactly $[\zeta_{k,\Theta_0(v)}, \omega_{k,v})$ implies that there must be a shortest extension of $v = uau$ which is admissible for those α immediately to the left of $\zeta_{k,\Theta_0(v)}$. Lemma 4.28 shows that $\mathfrak{f}(\,\Theta_1(v)\,)$ is this extension. The fixed point of $R_{k,\mathfrak{f}(\Theta_1(v))}$ is $r_0(\omega_{k,\Theta_1(v)})$, and we again argue that arbitrarily high powers of this word, $\mathfrak{f}(\,\Theta_1(v)\,)$, must be admissible just to the left of the corresponding value α, that is of $\omega_{k,\Theta_1(v)}$.

Since *any* $v = uau$ is always a prefix of the square of the corresponding $\mathfrak{f}(v) = ua$, we find that all of $\Theta_1(v)$ is admissible on an interval ending at $\omega_{k,\Theta_1(v)}$. By definition of $\zeta_{k,\Theta_1(v)}$ it follows that this interval is all of $[\zeta_{k,\Theta_1(v)}, \omega_{k,\Theta_1(v)})$. We iterate this argument for increasing q, to give that for each q, $\Theta_q(v)$ is admissible on exactly $[\zeta_{k,\Theta_q(v)}, \omega_{k,\Theta_q(v)})$. The definition of $\eta_{k,\Theta_q(v)}$ shows that it lies strictly between $\zeta_{k,\Theta_q(v)}$ and $\omega_{k,\Theta_q(v)})$.

Case 3. Suppose that $v = \Theta_0^h(uau)$ for some uau of Case 2. Thus, $v = u(au)^{h+1}$.

By the proof of Case 2, we can also assume our result for $\Theta_0^{h+1}(uau) = \Theta_0(v)$. In particular, the left boundary of $I_{k,\Theta_0(v)}$ does occur at $\zeta_{k,\Theta_0(v)}$. By Lemma 4.28, $\omega_{k,\Theta_1(v)} = \zeta_{k,\Theta_0(v)}$. Arguments as in the previous case yield that all of $\Theta_1(v)$ is admissible on an interval ending at $\omega_{k,\Theta_1(v)}$, and that this interval is indeed $[\zeta_{k,\Theta_1(v)}, \omega_{k,\Theta_1(v)})$. In this case also, induction on q is successful. That $J_{\Theta_q(v)} \subset I_{\Theta_q(v)}$ is here also straightforward. □

Proof of Theorem 4.12. That $(0, \gamma_{3,n}) = \cup_{k=1}^{\infty} I_{k,1}$ follows simply from the fact that for $\alpha \in (0, \gamma_{3,n})$ and $x \in \mathbb{I}_\alpha$, $T_\alpha(x) = A^k C \cdot x$ for some k. Proposition 4.31 shows that for all $v \in \mathcal{V}$, $I_{k,v} = [\zeta_{k,v}, \omega_{k,v})$ can be partitioned by $J_{k,v} = [\zeta_{k,v}, \eta_{k,v})$ and its complement. By Corollary 4.27 (and the complementary results proven in cases 1 and 3 of the proof of Proposition 4.31), we have

have that $\omega_{k,\Theta_0(v)} = \omega_{k,v}$. By Lemma 4.28, for all $q \in \mathbb{N}$, $\omega_{k,\Theta_q(v)} = \zeta_{k,\Theta_{q-1}(v)}$. Therefore, $\cup_{q=0}^{\infty} I_{k,\Theta_q(v)}$ is a subinterval of $I_{k,v} \setminus J_{k,v}$ which has $\omega_{k,v}$ as its right endpoint. Finally, the definition of $\Theta_q(v)$ combined with Lemma 4.26 shows that $\lim_{q\to\infty} \zeta_{k,\Theta_q(v)} = \eta_{k,v}$. Therefore, the left endpoint of the union is in fact the right endpoint of $J_{k,v}$ □

4.4. Potential synchronization intervals are intervals of synchronization

We now define $L_{k,v}$ exactly so that the group identity of Proposition 5.2 gives that $L_{k,v} = C^{-1}ACR_{k,v}$, and thus the main hypothesis of Lemma 4.5 will be satisfied. That synchronization does occur along $J_{k,v}$ is then only a matter of showing that $L_{k,v}A \cdot \ell_0(\alpha)$ is admissible at all $\alpha \in J_{k,v}$.

For further ease, we set

$$w = w_{3,n} = (-1)^{n-2}, -2, (-1)^{n-3}, -2.$$

Note that the length of w is $|w| = 2n - 3$. One of our first goals is to show that as α tends to zero, $\underline{d}^{\alpha}_{[1,\infty)}$ begins with ever higher powers of w. Recall from (2.2) that (for any m, n) the element $W = A^{-2}C\,(A^{-1}C)^{n-3}\left[A^{-2}C(A^{-1}C)^{n-2}\right]^{m-2}$, equals $W = A^{-1}C^{-1}ACA$. Just as this is fundamental to understanding the case of $\alpha = 0$, so is it the key to the study of left-orbits for small values of α. For ease of reference, the case of $m = 3$ is

$$W = A^{-2}C\,(A^{-1}C)^{n-3}\,A^{-2}C(A^{-1}C)^{n-2}.$$

In the particular cases of $m = 3$, [4] show that for all n the $T_{3,n,\alpha=0}$-orbit of $\ell_0(\alpha)$ is purely periodic of period w.

4.4.1. *Left digits are admissible*

For any natural number k, let

$$\mathcal{C}_k = (-1)^{n-3}, -2, w^{k-1}. \tag{4.3}$$

Accordingly, we let

$$\tilde{\mathcal{C}}_k = W^{k-1}A^{-2}C(A^{-1}C)^{n-3}.$$

Definition 4.32. Suppose that $v \in \mathcal{V}$ and $k \in \mathbb{N}$.

(1) For typographic ease, let

$$\mathcal{C} = \mathcal{C}_k, \mathcal{D} = \mathcal{C}_{k+1} \text{ and } \tilde{\mathcal{C}} = \tilde{\mathcal{C}}_k, \tilde{\mathcal{D}} = \tilde{\mathcal{C}}_{k+1}.$$

(2) The lower (simplified) digit sequence of k, v is

$$\begin{aligned}\underline{d}(k,v) &= w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}\\ &= (-1)^{n-2}, -2, \mathcal{C}^{c_1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2},\end{aligned}$$

of length

(3)

$$\begin{aligned}\underline{S}(k, v) &= |\,\underline{d}(k, v)\,| \\ &= 2n-3 + [(k-1)(2n-3) + n-2]\sum_{i=1}^{s} c_i + [k(2n-3) + n-2]\sum_{j=1}^{s-1} d_j.\end{aligned}$$

(4) The *left matrix* of k, v is

$$\begin{aligned}L_{k,v} &= (A^{-1}C)^{n-2}\, \tilde{\mathcal{C}}^{c_s}\tilde{\mathcal{D}}^{d_{s-1}} \cdots \tilde{\mathcal{D}}^{d_1}\tilde{\mathcal{C}}^{c_1-1} W^k A^{-1} \\ &= (A^{-1}C)^{n-2}\, \tilde{\mathcal{C}}^{c_s}\tilde{\mathcal{D}}^{d_{s-1}} \cdots \tilde{\mathcal{D}}^{d_1}\tilde{\mathcal{C}}^{c_1} A^{-2}C(A^{-1}C)^{n-2}A^{-1}.\end{aligned}$$

Note that Proposition 5.2, below, states that $L_{k,v} = C^{-1}AC\, R_{k,v}$.

Our aim is to show the admissibility of $\underline{d}(k, v)$ on $J_{k,v}$. In the following, we give both the left and right simplified digit sequence for each of the endpoints of $J_{k,v}$. The right sequences follow from the results above. We present them here for ease of comparison.

Lemma 4.33. *Let $v = c_1 d_1 \cdots c_s \in \mathcal{V}$ and $k \in \mathbb{N}$. Suppose that $\underline{d}(k, v)$ is admissible on $J_{k,v}$. Set $\zeta = \zeta_{k,v}$ and $\eta = \eta_{k,v}$. Then*

$$\underline{d}^{\zeta}_{[1,\infty)} = \underline{d}(k, (\overleftarrow{v'}\,)^{\infty});$$

$$\overline{d}^{\zeta}_{[1,\infty)} = \overline{d}(k, v),\ \underline{d}^{\zeta}_{[1,\infty)};$$

$$\underline{d}^{\eta}_{[1,\infty)} = \overline{w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-3}, -2};$$

$$\overline{d}^{\eta}_{[1,\infty)} = \overline{d}(k, v(v')^{\infty}).$$

Proof. Lemma 4.26 yields the expressions for $\overline{d}^{\zeta}_{[1,\infty)}$ and $\overline{d}^{\eta}_{[1,\infty)}$.

We first show that the expansions for $\ell_0(\zeta)$, $\ell_0(\eta)$ are correct assuming admissibility of $\underline{d}(k, v)$ for all $\alpha \in [\zeta, \eta)$.

Letting $L = L_{k,v}$, Lemma 4.5 gives $A^{-2}CLA\cdot\ell_0(\zeta) = \ell_1(\zeta)$. Thus, the digits of $\ell_0(\zeta)$ are periodic, with preperiod of length one. Since $w = (-1)^{n-2}, -2, (-1)^{n-3}, -2$, we find

$$\begin{aligned}\underline{d}^{\zeta}_{[1,\infty)} &= -1, \overline{(-1)^{n-3}, -2, (-1)^{n-3}, -2, w^{k-1}, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}, -2} \\ &= w^k, \overline{\mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}, -2, (-1)^{n-3}, -2, (-1)^{n-3}, -2, w^{k-1}} \\ &= w^k, \overline{\mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, w, (-1)^{n-3}, -2, w^{k-1}} \\ &= w^k, \overline{\mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s-1}, \mathcal{D}, \mathcal{C}} \\ &= w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s-1}, \mathcal{D}, \overline{\mathcal{C}^{c_1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s-1}\mathcal{D}}.\end{aligned}$$

Since v is a palindrome, this last is indeed the infinite sequence $\underline{d}(k, (\overleftarrow{v'})^{\infty})$.

Note that in the special case that $v = c_1, c > 1$, we find $\underline{d}^{\zeta_{k,c}}_{[1,\infty)} = w^k, \overline{\mathcal{C}^{c-1}, w, \mathcal{C}} = w^k, \overline{\mathcal{C}^{c-2}, \mathcal{D}, \mathcal{C}}$ and $\underline{d}^{\zeta_{k,1}}_{[1,\infty)} = w^{k+1}, \overline{(-1)^{n-3}, -2, w^k} = w^{k+1}, \overline{\mathcal{D}}$.

Since $LA \cdot \ell_0(\eta) = r_0(\eta)$, we have that $A^{-1}LA$ fixes $\ell_0(\eta)$ and hence $\underline{d}^{\eta}_{[1,\infty)}$ is indeed purely periodic, with the indicated period. □

Lemma 4.34. *Fix $j \in \mathbb{N}$ and $0 \le i < |w|$. If there is some $\alpha \le \gamma_{3,n}$ such that $\underline{d}^{\alpha}_{[1,j|w|+i]} = w^j\, w_{[1,i]}$, then $\underline{d}^{\alpha'}_{1,j|w|+i} = w^j\, w_{[1,i]}$ for all $\alpha' < \alpha$.*

Proof. (Of course, if $i = 0$, then $w_{[1,i]}$ is the empty word.) Since it is shown in [4] that all powers of W are admissible when $\alpha = 0$, there are thus branches of digits corresponding to each $W^k \cdot \ell_0$, $A^{-1}CW^k \cdot \ell_0$, $(A^{-1}C)^2W^k \cdot \ell_0, \dots, (A^{-1}C)^{n-3}W^k \cdot \ell_0$ and $A^{-2}C(A^{-1}C)^{n-3}W^k \cdot \ell_0$ that continue to the right from $\alpha = 0$. For each, by Lemma 1.6, admissibility at α thus guarantees admissibility at each $\alpha' \le \alpha$. □

Lemma 4.35. *Fix $k \in \mathbb{N}$. We have*

$$\underline{d}^{\omega_{k,1}}_{[1,\infty)} = w^k, \overline{(-1)^{n-3}, -2, w^{k-1}} = w^k, \overline{\mathcal{C}}\,.$$

The digits $\underline{d}^{\alpha}_{[1,n-3+k|w|]} = w^k, (-1)^{n-3}$ are admissible for all $\alpha \le \omega_{k,1}$.

Proof. Since $\mathfrak{f}(1) = 1$, the definition of $\omega_{k,v}$ yields $A^{k-1}CA \cdot \ell_0(\omega_{k,1}) = \ell_0(\omega_{k,1})$. Lemma 5.1 shows that $A^{k-1}CA = C^{-1}A^{-1}C(A^{-1}C)^{n-2}W^{k-1}$. For $\alpha \le \gamma_{3,n}$, we certainly have that $\ell_1(\alpha) = A^{-1}C \cdot \ell_0(\alpha)$, thus $\ell_1(\omega_{k,1}) = A^{-2}C(A^{-1}C)^{n-2}W^{k-1} \cdot \ell_0(\omega_{k,1})$. Thus, the T_α-orbit of $\ell_0(\omega_{k,1})$ is periodic, with minimal preperiod of length one. Elementary manipulations give the claimed expression for the simplified digits, assuming admissibility.

We have that the graph of the function $x \mapsto A^{-2}C(A^{-1}C)^{n-2}W^{k-1}A^{-1} \cdot x$ meets the vertical line $x = r_0(\omega_{k,1})$ at $y = \ell_1(\omega_{k,1})$. Since the $T_{\omega_{k,1}}$-cylinder $\Delta(-2, 1)$ is full, there is also a point $y \in \mathbb{I}_{\omega_{k,1}}$ where the graph of the function $x \mapsto (A^{-1}C)^{n-2}W^{k-1}A^{-1} \cdot x$ meets $x = r_0(\omega_{k,1})$. By Lemma 4.34, this implies that $(A^{-1}C)^{n-2}W^{k-1} \cdot \ell_0(\omega_{k,1})$ is admissible. It follows that $A^{-2}C(A^{-1}C)^{n-2}W^{k-1} \cdot \ell_0(\omega_{k,1})$ is also admissible. The rest of $w^k, \overline{(-1)^{n-3}, -2, w^{k-1}}$ is determined by periodicity and is thus also admissible.

The second statement now follows immediately from Lemma 4.34. □

Remark 4.36. Note that the above yields $\underline{d}^{\omega_{k+1,1}}_{[1,\infty)} = \underline{d}^{\zeta_{k,1}}_{[1,\infty)}$, in accordance with the fact that $\mathfrak{f}(1) = 1$ implies $\omega_{k+1,1} = \zeta_{k,1}$.

In the following proof and occasionally thereafter, we will have need of the following.

Definition 4.37. For any word z, let $z_{[-2]}, z_{[-1]}$ denote the excision of the last two letters, or last letter, from z respectively.

Proposition 4.38. *Suppose that $v \in \mathcal{V}$ and $k \in \mathbb{N}$. Then for all $\alpha \in J_{k,v}$,*

$$\underline{d}^{\alpha}_{[1,\underline{S})} = \underline{d}(k, v)\,.$$

Proof. For all v, we will exhibit $\alpha' < \zeta_{k,v}$ and $\alpha'' > \eta_{k,v}$ such that both $\overline{d}^{\alpha'}_{[1,\infty)}$ and $\overline{d}^{\alpha''}_{[1,\infty)}$ are known to be admissible (by induction), and share as a common prefix all but the final letter of $\underline{d}(k, v)$. By Lemma 1.6, the admissibility of these first $\underline{S}(k, v) - 1$ digits then holds on $[\alpha', \alpha'']$. Since we already know the admissibility of our right digits, Lemma 4.39, below, applies and we can conclude admissibility of all of $\underline{d}(k, v)$ on $J_{k,v}$. See Table 4.1 for a summary of the pairs α', α'' used for the various cases of v.

	v	α'	α''
Base cases	1	0	$\omega_{k,1}$
	$c > 1$	$\eta_{k,c-1}$	$\omega_{k,c} = \omega_{k,1}$
	$c\,1\,c, c \geq 1$	$\eta_{k,c}$	$\omega_{k,c\,1\,c} = \zeta_{k,c+1}$
	$\Theta_q(c), q \geq 2$	$\eta_{k,c}$	$\zeta_{k,\Theta_{q-1}(c)}$
	$\Theta_1(c), c > 1$	$\eta_{k,c}$	$\zeta_{k,\Theta_0(c)}$
Case 1	$\Theta_0^h(c)$	$\eta_{k,\Theta_0^{h-1}(c)}$	$\zeta_{k,c+1}$
Case 2	$\Theta_q \circ \Theta_p(u),\ p, q \geq 1$	$\eta_{k,\Theta_p(u)}$	$\zeta_{k,\Theta_{q-1}\circ\Theta_p(u)}$
Case 3	$\Theta_0^h \circ \Theta_p(u),\ p \geq 1$	$\eta_{k,\Theta_0^{h-1}\circ\Theta_p(u)}$	$\zeta_{k,\Theta_{p-1}(u)}$

Table 4.1. Admissibility of $\underline{d}(k, v)$ on $J_{k,v}$ is shown by finding $\alpha' < \zeta_{k,v} < \eta_{k,v} < \alpha''$ such that $\underline{d}^{\alpha}_{[1,\infty)}$ agrees with $\underline{d}(k, v)$ through to its penultimate digit for both $\alpha = \alpha', \alpha''$. See the proof of Proposition 4.38.

Base cases. Consider $v = c_1 = c$. The lower simplified digit sequence $\underline{d}^{\omega_{k,1}}_{[1,\infty)} = w^k, \overline{\mathcal{C}}$ agrees with $\underline{d}(k, c) = w^k, \mathcal{C}^{c-1}, (-1)^{n-2}$ through to its penultimate digit. When $v = 1$, Lemma 4.35 shows that these shared digits are admissible for all $\alpha \leq \omega_{k,1}$; therefore our proof template succeeds in this case. For $c > 1$, by induction $\underline{d}^{\eta_{k,c-1}}_{[1,\infty)} = \overline{w^k, \mathcal{C}^{c-2}, (-1)^{n-3}, -2}$ is admissible, and has $\underline{d}(k, c)$ as a prefix.

Suppose $v = c\,1\,c$. Here we use $\alpha' = \eta_{k,c}$, and $\alpha'' = \omega_{k,c\,1\,c}$. Since $\underline{d}^{\eta_{k,c}}_{[1,\infty)} = w^k, \overline{\mathcal{C}^{c-1}\mathcal{D}}$, it has the prefix $w^k, \mathcal{C}^{c-1}\mathcal{D}\mathcal{C}^c, w$. This in turn has $\underline{d}(k, c\,1\,c)$ as a prefix. Lemma 4.29 and Lemma 4.33 give that $\underline{d}^{\omega_{k,c\,1\,c}}_{[1,\infty)} = \underline{d}^{\zeta_{k,c+1}}_{[1,\infty)} = w^k, \overline{\mathcal{C}^{c-1}, \mathcal{D}, \mathcal{C}}$. This agrees with $\underline{d}(k, c\,1\,c)$ through to the penultimate digit of this latter. Thus the admissibility holds on $J_{k,c\,1\,c}$.

Suppose $v = \Theta_q(c), q \geq 1$ and $c > 1$. We again use $\alpha' = \eta_{k,c}$. The sequence $\underline{d}^{\eta_{k,c}}_{[1,\infty)}$ and $\mathcal{D} = \mathcal{C}, w$, here $\underline{d}^{\alpha'}_{[1,\infty)}$ has the prefix $w^k, (\mathcal{C}^{c-1}\mathcal{D})^{q+1}\mathcal{C}^c, w$. This in turn has $\underline{d}(k, \Theta_q(c))$ as a prefix. Let $\alpha'' = \zeta_{k,\Theta_{q-1}(c)}$. Then $\underline{d}^{\alpha''}_{[1,\infty)} = w^k, \overline{(\mathcal{C}^{c-1}\mathcal{D})^{q+1}\mathcal{C}}$, which has the prefix $w^k, (\mathcal{C}^{c-1}\mathcal{D})^{q+1}\mathcal{C}^c, (-1)^{n-3}, -2$. It thus agrees with $\underline{d}(k, \Theta_q(c))$ through to the penultimate digit of this latter. Thus the admissibility holds. One checks that the same form of α', α'' works for $v = \Theta_q(c), q \geq 2$ and $c = 1$.

Case 1. Suppose $v = \Theta_0^h(c), h \geq 2$ for some $c > 1$. We have $v = c(1\,c)^h$. We induce on h, setting $\alpha' = \eta_{k,\Theta_0^{h-1}(c)}$. Then, $\underline{d}^{\alpha'}_{[1,\infty)}$ has the prefix $w^k, \mathcal{C}^{c-1}, (\mathcal{D}\mathcal{C}^c)^h$, $(-1)^{n-3}, -2$, which agrees through to the penultimate digit of $\underline{d}(k, v)$. One easily checks that setting $\alpha'' = \zeta_{k,c+1}$ yields $\underline{d}^{\alpha''}_{[1,\infty)}$ of prefix $\underline{d}(k, v)$.

Case 2. Suppose that the result holds for $v = uau = \Theta_p(u)$ with $u \in \mathcal{V}$ and $p \geq 1$. We prove that the result holds for $\Theta_q(v), q \geq 1$. We take $\alpha' = \eta_{k,v}$ and $\alpha'' = \zeta_{k,\Theta_{q-1}(v)}$ if $q \geq 1$. To appropriately restrict the use of $\underline{d}(k, v)$ to prefixes u of $v \in \mathcal{V}$, note that we must in particular suppress the final digit of -1; we denote this by $\underline{d}'(k, u)$. Recall that $v' = a\overleftarrow{u'}u, v'' = au$ and that u both begins and ends with the letter c_1.

When $c_1 > 1$, from $\overleftarrow{u'} = u_{[-1]}(c_1 - 1)\,1$ we find

$$\begin{aligned}\underline{d}^{\eta_{k,v}}_{[1,\infty)} &= \underline{d}(k, v(v')^\infty) = \underline{d}(k, v(v')^q a\overleftarrow{u'}u\,(v')^\infty)\\ &= \underline{d}'(k, v(v')^q au_{[-1]})\,\mathcal{C}^{c_1-1}\mathcal{D}\,\mathcal{C}^{c_1}\cdots.\end{aligned}$$

On the other hand, still with $c_1 > 1$, we have

$$\underline{d}(k, \Theta_q(v)\,) = \underline{d}'(k, v(v')^q au_{[-1]})\,\mathcal{C}^{c_1}.$$

Recall that $\mathcal{D} = \mathcal{C}, w$, thus $\mathcal{C}^{c_1-1}\mathcal{D} = \mathcal{C}^{c_1}, w$; we see that $\underline{d}(k, \Theta_q(v)\,)$ is indeed a prefix of $\underline{d}^{\eta_{k,v}}_{[1,\infty)}$.

When $c_1 = 1$, $\underline{d}(k, \Theta_q(v)\,) = \underline{d}'(k, v(v')^q au_{[-1]})\mathcal{D}^{d_1}\mathcal{C}$ while $\underline{d}^{\eta_{k,v}}_{[1,\infty)} = \underline{d}'(k, v(v')^q au_{[-2]})\mathcal{D}^{d_1+1}\,\mathcal{C}\cdots$. Thus, since $\mathcal{C}$ is a prefix of $\mathcal{D}$, we find that here also $\underline{d}(k, \Theta_q(v)\,)$ is a prefix of $\underline{d}^{\eta_{k,v}}_{[1,\infty)}$.

Since $\Theta_{q-1}(v) = v(v')^{q-1}v''$ is a palindrome, $\overleftarrow{(\Theta_{q-1}(v)\,)'} = v(v')^{q-1}a\overleftarrow{u'}$. By Proposition 4.13, $v'v'' = a\overleftarrow{u'}\,v$, we find

$$\begin{aligned}(\,\overleftarrow{(\Theta_{q-1}(v)\,)'}\,)^\infty &= v\,[(v')^{q-1}v'v'']^\infty\\ &= v\,[\,(v')^q v''\,]^\infty.\end{aligned} \tag{4.4}$$

Thus Lemma 4.33 yields $\underline{d}^{\zeta_{k,\Theta_{q-1}(v)}}_{[1,\infty)} = \underline{d}(k, v[(v')^q v'']^\infty)$. Therefore, $\underline{d}'(k, v(v')^q v'')$, $(-1)^{n-3}, -2$ is a prefix of $\underline{d}^{\zeta_{k,\Theta_{q-1}(v)}}_{[1,\infty)}$. That is, $\underline{d}^{\zeta_{k,\Theta_{q-1}(v)}}_{[1,\infty)}$ agrees with $\underline{d}(k, \Theta_q(v))$ exactly through to its penultimate digit.

Case 3. Again suppose that $v = \Theta_p(u) = uau$ in our usual notation. Then $v'' = (u')^p u''$, and one finds that $\Theta_0^h \circ \Theta_p(u) = \Theta_p(u)\,[(u')^p u'']^h$. For these words, we use $\alpha'' = \zeta_{k,\Theta_{p-1}(u)}$. Indeed, from (4.4), $\underline{d}_{[1,\infty)}^{\zeta_{k,\Theta_{p-1}(u)}}$ agrees with $\underline{d}(k, \Theta_0^h \circ \Theta_p(u))$ exactly through to its penultimate digit.

Since $(\Theta_0^{h-1} \circ \Theta_p(u))' = u'\,[(u')^p u'']^h$, we find that

$$\Theta_0^{h-1}(v) \cdot (\Theta_0^{h-1}(v))' = \Theta_0^h(v) \cdot u'[(u')^p u'']^h. \tag{4.5}$$

Therefore, Lemma 4.33 yields that $\alpha' = \eta_{k,\Theta_0^{h-1}(v)}$ allows the proof to succeed. $\square$

Lemma 4.39. *With notation as above, suppose that for all $\alpha \in [\zeta_{k,v}, \eta_{k,v}]$, both*

(a) $\overline{d}^{\alpha}_{[1,\overline{S}]} = \overline{d}(k, v)$;

(b) $\underline{d}^{\alpha}_{[1,\underline{S}-1]}$ *agrees with the initial subword of length $\underline{S} - 1$ of $\underline{d}(k, v)$.*

Then for all $\alpha \in [\zeta_{k,v}, \eta_{k,v})$ we have $\underline{d}^{\alpha}_{[1,\underline{S}]} = \underline{d}(k, v)$.

Proof. By the first hypothesis, there is some $\chi = \chi_{k,v}$ such that $R_v \cdot r_0(\chi_v) = 0$. Since $C^{-1}ACR_{k,v} = L_{k,v}$ and it is trivially verified that $C^{-1}AC$ fixes zero, we find that $L_{k,v} \cdot \ell_0(\chi) = 0$. In particular, we find that $\underline{d}^{\chi}_{[1,\underline{S}]} = \underline{d}(k, v)$.

Now by continuity and the fact that Möbius functions are increasing functions, we can invoke the second conclusion of Lemma 4.2 on an interval around χ to find in particular that $L_{k,v} \cdot \ell_0(\alpha) > R_v \cdot r_0(\alpha)$ holds from $\alpha = \zeta_{k,v}$ until $L_{k,v} \cdot \ell_0(\alpha) = r_0(\alpha)$. But, this describes exactly the interval $[\zeta_{k,v}, \eta_{k,v})$. $\square$

4.5. There are no other points of synchronization

Suppose that $\alpha < \gamma_{3,n}$ is not in any $J_{k,v}$ and is also not equal to any $\eta_{k,v}$. There is some k such that $\alpha \in I_{k,1}$, but of course $\alpha \notin J_{k,1}$; there is thus a unique q_1 such that $\alpha \in I_{k,\Theta_{q_1}(1)}$. Again, $\alpha \notin J_{k,\Theta_{q_1}(1)}$ and thus there is a unique q_2 with $\alpha \in I_{k,\Theta_{q_2}\circ\Theta_{q_1}(1)}$. Clearly this process iterates, and we find that there is an infinite sequence of q_i such that $\alpha \in \bigcap_{j=1}^{\infty} I_{k,\Theta_{q_j}\circ\cdots\circ\Theta_{q_1}(1)}$. Recall that for any $v \in \mathcal{V}$ and any q, $\Theta_q(v)$ has v as a prefix. Therefore, the sequence of the q_i uniquely determines both $\underline{d}^{\alpha}_{[1,\infty)}$ and $\overline{d}^{\alpha}_{[1,\infty)}$. In particular, $\overline{d}^{\alpha}_{[1,\infty)}$ has digits only in $\{k, k+1\}$, while $\underline{d}^{\alpha}_{[1,\infty)}$ has digits only in $\{-1, -2\}$. Therefore, the two orbits cannot synchronize.

Note that when α is some $\eta_{k,v}$ then again $\overline{d}^{\alpha}_{[1,\infty)}$ has digits only in $\{k, k+1\}$, while $\underline{d}^{\alpha}_{[1,\infty)}$ has digits only in $\{-1, -2\}$.

4.6. The non-synchronization set is of measure zero

Although we have proven that the complement of the $J_{k,v}$ is a Cantor set, it is then still possible that it could be a so-called fat Cantor set thus one of positive measure.

The non-synchronization points have left expansions that involve only -1 and -2 as simplified digits. We will argue by way of the maps of [4] that the set of such α is of measure zero.

The T_α-orbit of $\ell_0(\alpha)$ is always in the set of points whose simplified digits are -1 or -2. In particular, this orbit certainly remains in $[-t, 0)$. But $[-t, 0)$ is the interval of definition of the map $g = g_{3,n}$ studied in [4], where $g(x) = A^k C \cdot x$ with k defined exactly so that this image lies in $[-t, 0)$. Therefore, for any α whose T_α orbit of $\ell_0(\alpha)$ remains in the set with simplified digits -1 or -2 (note that every point in such an orbit is less than zero), this T_α-orbit is the g-orbit of $\beta = \ell_0(\alpha)$. Furthermore, [4] shows that g is ergodic with respect to what is naturally called a Gauss measure, although this measure is infinite. A so-called acceleration of g, a map f on $[-t, 0)$ is then shown in [4] to be ergodic with respect to *finite* measure (which is equivalent to Lebesgue). The process of acceleration involves taking well-defined subsequences of g-orbits. We thus find that the f-orbit of β remains of small digits, and due to the Ergodic Theorem, β lies in a measure zero subset.

5. The group element identities for the setting $\alpha < \gamma_{3,n}$

We thank the referee for suggesting a rewriting of this section, with an emphasis on group element conjugations.

Recall that $\tilde{C}_k = W^{k-1} A^{-2} C (A^{-1}C)^{n-3}$, with W as in (2.1).

Lemma 5.1. *The following identities in $G_{3,n}$ hold for each $k \in \mathbb{Z}$.*

(i) $CA\, W^k\, (CA)^{-1} = A^k$;

(ii) $A^{k-1}CA = C^{-1}A^{-1}C(A^{-1}C)^{n-2}W^{k-1}$;

(iii) $(C^{-1}AC)(A^kC)^a(C^{-1}AC)^{-1} = (A^{-1}C)^{n-2}\, \tilde{C}_k^a\, (A^{-1}C)^{-(n-2)}$ *for all* $a \in \mathbb{Z}$;

Proof. It suffices to prove (i) in the case of $k = 1$, as that conjugacy relation then implies the others. By (2.2) we have $CA\, W(CA)^{-1} = CA(A^{-1}C^{-1}ACA)\,(CA)^{-1}$, and thus $CA\, W(CA)^{-1} = A$.

From (i), and the facts that $A^{-1}C = B$ has order n while C has order three, we find

$$\begin{aligned} A^{k-1}CA &= CAW^{k-1} \\ &= CA(A^{-1}C)^n W^{k-1} \\ &= C^2AC(A^{-1}C)^{n-2}W^{k-1} \\ &= C^{-1}AC(A^{-1}C)^{n-2}W^{k-1}, \end{aligned}$$

and thus (ii) holds.

It suffices to show the conjugacy relation in (iii) in the case of $a = 1$.

$$\begin{aligned}(C^{-1}AC)(A^kC)\,(C^{-1}AC)^{-1} &= (C^{-1}AC)A^{k-1}C\\ &= (C^{-1}AC)\;C^{-1}A^{-1}C(A^{-1}C)^{n-2}W^{k-1}\;A^{-1}\\ &= (A^{-1}C)^{n-2}\;(W^{k-1})\;A^{-1}\\ &= (A^{-1}C)^{n-2}\;(\tilde{\mathcal{C}}_k(A^{-1}C)^{-(n-3)}C^{-1}A^2)\;A^{-1}\\ &= (A^{-1}C)^{n-2}\tilde{\mathcal{C}}_k(A^{-1}C)^{-n+2},\end{aligned}$$

and thus (iii) holds. □

Recall that for $v = c_1d_1\cdots d_{s-1}c_s$ and $k \in \mathbb{N}$, we have

$$R_{k,v} = (A^kC)^{c_s}\;(A^{k+1}C)^{d_{s-1}}(A^kC)^{c_{s-1}}\cdots(A^{k+1}C)^{d_1}(A^kC)^{c_1}$$

and

$$L_{k,v} = (A^{-1}C)^{n-2}\;\tilde{\mathcal{C}}^{c_s}\tilde{\mathcal{D}}^{d_{s-1}}\cdots\tilde{\mathcal{D}}^{d_1}\tilde{\mathcal{C}}^{c_1}A^{-2}C(A^{-1}C)^{n-2}A^{-1},$$

where $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_k$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{C}}_{k+1}$.

Proposition 5.2. *For each $k \in \mathbb{N}$ and each $v = c_1d_1\cdots d_{s-1}c_s \in \mathcal{V}$,*

$$L_{k,v} = C^{-1}AC\;R_{k,v}\,.$$

Proof. Since in any group, the conjugate of a product is the product of the conjugates, and $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_k$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{C}}_{k+1}$, Part (iii) of the previous lemma yields

$$(C^{-1}AC)R_{k,v}\,(C^{-1}AC)^{-1} = (A^{-1}C)^{n-2}\;\tilde{\mathcal{C}}^{c_s}\tilde{\mathcal{D}}^{d_{s-1}}\cdots\tilde{\mathcal{D}}^{d_1}\tilde{\mathcal{C}}^{c_1}\;(A^{-1}C)^{-(n-2)}.$$

Since also

$$\begin{aligned}(A^{-1}C)^{-(n-2)}\;C^{-1}AC &= A^{-1}C^2\\ &= A^{-1}C^2\;(CA^{-1})^n = A^{-2}(CA^{-1})^{n-1}\\ &= A^{-2}C(CA^{-1})^{n-2}A^{-1},\end{aligned}$$

we indeed have that $C^{-1}AC\;R_{k,v} = L_{k,v}$. □

6. Synchronization for $\alpha > \epsilon_{3,n}$, $n \geq 3$

Fix $m = 3$ and $n \geq 3$. Let $\epsilon = \epsilon_{3,n}$ be such that $A^{-1}C \cdot \ell_0(\epsilon) = r_0(\epsilon)$. Then the parameter subinterval $[\epsilon, 1)$ is partitioned by subintervals indexed by $k \geq 2$ and characterized by $\ell_1(\alpha) = A^{-k}C \cdot \ell_0(\alpha)$. (When $n = 3$, one finds that $\epsilon_{3,3} = G/2$, see Figure 4.1.)

Theorem 6.1. *For $m = 3$ and $n \geq 3$, let $\epsilon = \epsilon_{3,n}$. The set of $\alpha \in (\epsilon, 1)$ such that there exists $i = i_\alpha$, $j = j_\alpha$ with $T^i_{3,n,\alpha}(r_0(\alpha)) = T^j_{3,n,\alpha}(\ell_0(\alpha))$ is of full measure.*

Synchronization for these large values of α holds in a manner closely analogous to that for small α. We will find that the intervals indexed by exactly the same set of words $\mathcal{V}$, although the indexing will depend on negative integers and be given in terms of left digits. There are differences: in particular, each potential synchronization interval is the union of what could fairly be called two distinct synchronization intervals. On one of the subintervals, the right orbit requires an extra step before synchronization.

It will naturally be important to know the initial digits of $r_0(\alpha)$ in this range. For this, define $\delta = \delta_{3,n}$ as the value of α such that $C^{-1} \cdot \ell_0(\delta) = A^{-1}C \cdot \ell_0(\delta)$. (Note that $\delta_{3,n} < \epsilon_{3,n}$, see Figure 4.1.) Thus, $CA^{-1}CA^{-1} \cdot r_0(\delta) = \ell_0(\delta)$. That is, $(AC^2)^{n-2} \cdot r_0(\delta) = \ell_0(\delta)$. By Proposition 3.1, $(AC^2)^{n-2} \cdot r_0(\alpha)$ is admissible for $\alpha = 1$ and we conclude that $(AC^2)^{n-2} \cdot r_0(\alpha)$ is admissible for all $\alpha \in (\delta, 1]$ and in particular for all $\alpha > \epsilon_{3,n}$.

6.1. Synchronization intervals have Cantor set complement

For these large α, synchronization is signaled by left and right digits being related by $C^{-1}AC^{-1}$.

Lemma 6.2. *Fix $m = 3$. Suppose that α is such for all $x \in \mathbb{I}_\alpha$, $d^\alpha(x) = (k, \ell)$ with $\ell \in \{1, 2\}$. Fix $i, j \in \mathbb{N}$. Suppose that $\ell_{i-1} = C^{-1}AC^{-1} \cdot r_{j-1}$ and $C \cdot r_{j-1} \in \mathbb{I}_\alpha$. Then*

(i) *If $r_j = AC^2 \cdot r_{j-1}$ then $\ell_i = r_{j+1}$;*
(ii) *otherwise, $\ell_i = r_j$.*

Proof. Since $C \cdot r_{j-1} \in \mathbb{I}_\alpha$, there is some u such that $r_j = A^uC^2 \cdot r_{j-1}$.

If $r_j = AC^2 \cdot r_{j-1}$ then $C \cdot \ell_{i-1} = C \cdot (C^{-1}AC^{-1} \cdot r_{j-1}) \in \mathbb{I}_\alpha$. Therefore, there is some s such that $\ell_i = A^sC^2 \cdot \ell_{i-1}$. But then $\ell_i = A^sC^2 \cdot (C^{-1}AC^{-1} \cdot r_{j-1}) = A^sCAC^2 \cdot r_{j-1} = A^sC \cdot r_j$. By definition, $A^{-s} \cdot \ell_i \notin \mathbb{I}_\alpha$, it follows that $C \cdot r_j \notin \mathbb{I}_\alpha$ and therefore we conclude that $r_{j+1} = A^sC \cdot r_j = \ell_i$.

If $r_j = A^uC^2 \cdot r_{j-1}$ with $u \neq 1$, then $C \cdot \ell_{i-1} = AC^2 \cdot r_{j-1} \notin \mathbb{I}_\alpha$. Therefore, there is some s such that $\ell_i = A^sC \cdot \ell_{i-1}$. We find that $\ell_i = A^sC \cdot (C^{-1}AC^{-1} \cdot r_{j-1}) = A^{s+1}C^2 \cdot r_{j-1}$. We conclude that $u = s + 1$ and $\ell_i = r_j$. □

The result (ii) below leads to the conclusion that synchronization intervals for large α can be described by the same set of words as for small α.

Lemma 6.3. *Fix $m = 3$, an interval $[\eta, \zeta]$ of α such that $r_1 = AC^2 \cdot r_0$ holds on this interval, and $i, j \in \mathbb{N}$. Suppose that there are matrices R, L (neither of which is the identity) such that*

(a) *$L = C^{-1}AC^2R$;*
(b) *$R \cdot r_0 = r_{j-1}$ and $L' \cdot \ell_0 = \ell_{i-2}$, for all $\alpha \in [\eta, \zeta]$;*

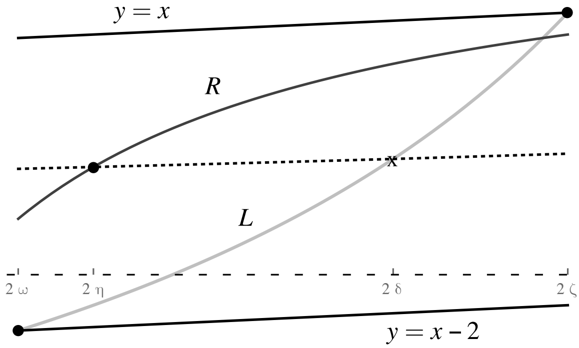


Figure 6.1. Determining the synchronization interval $[\eta, \zeta)$, when $\alpha > \epsilon_{3,n}$. Here, $m = 3, n = 3$, and $k = 2$, $v = 1$. The labels L, R mark respectively the curves $y = L_{-2,1} \cdot r_0(\alpha)$, $y = R_{-2,1} \cdot r_0(\alpha)$ where $\alpha = x/2 = x/t_{3,3}$. These are $y = \ell_1(\alpha)$ (in light gray) and $y = r_2(\alpha)$ (in dark gray) for $G < x < (-1 + \sqrt{21}/2)$. The dotted curve is the image of the left endpoint under C^{-1} (giving the values of $\mathfrak{b}_\alpha$). The x-axis is shown as a dashed line. For $\alpha > \delta = \delta_{-2,1}$, synchronization occurs after an extra step in the orbit of r_0.

(c) $LA \cdot \ell_0 = \ell_{i-1}$ *for all* $\alpha \in [\eta, \zeta)$, *while* $\ell_{i-1}(\zeta) = A^{-1}LA \cdot \ell_0(\zeta) = \ell_0(\zeta)$;
(d) $R \cdot r_0(\eta) = C^{-1} \cdot \ell_0(\eta)$.

Suppose further that $A^{-k}C \cdot \ell_0(\eta) = \ell_1(\eta)$. *Then*

(i) $r_j(\eta) = A^{-k}C^2 \cdot r_{j-1}(\eta) = \ell_1(\eta)$;
(ii) $\ell_i(\eta) = A^{-(k+1)}C \cdot \ell_{i-1}(\eta) = \ell_1(\eta)$;
(iii) $r_j(\zeta) = AC^2 \cdot r_{j-1}(\zeta)$ *and* $r_{j+1}(\zeta) = AC \cdot r_j(\zeta) = r_1(\zeta)$.

Proof. The equality $r_{j-1}(\eta) = C^{-1} \cdot \ell_0(\eta)$ implies that $r_j(\eta) = A^{-k}C^2 \cdot r_{j-1}(\eta) = \ell_1(\eta)$. At $\alpha = \eta$ we also have $r_{j-1} = CA^{-1}C \cdot \ell_{i-1}$, therefore $A^{-k}C^2 \cdot r_{j-1} = A^{-(k+1)}C \cdot \ell_{i-1}$ allows one to easily confirm the admissibility of this expression for $\ell_i(\eta)$, as well as that $\ell_i(\eta) = \ell_1(\eta)$.

Finally, $r_{j-1}(\zeta) > C^{-1} \cdot \ell_0(\zeta)$ and hence $r_j(\zeta) = A^s C^2 \cdot r_{j-1}(\zeta)$ for some s. This gives $r_j(\zeta) = A^s C^2 \cdot (CA^{-1}C \cdot r_0(\zeta)) = A^{s-1}C \cdot r_0(\zeta)$. But, $r_1 = AC^2 \cdot r_0$ holds throughout this region of large α; in particular, $C \cdot r_0(\alpha) \in \mathbb{I}_\alpha$ here. Therefore, $s = 1$ and $r_j(\zeta) = C \cdot r_0(\zeta)$. It then follows that $r_{j+1}(\zeta) = AC \cdot (C \cdot r_0(\zeta)) = r_1(\zeta)$. □

Definition 6.4. Let $\mathcal{V}$ be as in the treatment of $\alpha < \gamma_{3,n}$. For each $k \in \mathbb{N}$ and $v = c_1 d_1 \cdots d_{s-1} c_s \in \mathcal{V}$, we define the following.

(1) the lower (simplified) *digit sequence* of $-k, v$ is

$$\underline{d}(-k, v) = (-k)^{c_1}, (-k-1)^{d_1}, \cdots, (-k-1)^{d_{s-1}}, (-k)^{c_s};$$

(2) the *α-cylinder* of $-k, v$ is

$$I_{-k,v} = \{\alpha \mid \underline{d}^{\alpha}_{[1,|v|]} = \underline{d}(-k, v)\}\,;$$

(3) the *left matrix* of $-k, v$ is

$$L_{-k,v} = (A^{-k}C)^{c_s}\,(A^{-k-1}C)^{d_{s-1}}(A^{-k}C)^{c_{s-1}} \cdots (A^{-k-1}C)^{d_1}(A^{-k}C)^{c_1}A^{-1}\,;$$

(4) the *synchronization interval* associated to $-k, v$ is $J_{-k,v} = [\eta, \zeta)$ where $\eta = \eta_{k,v}$ and $\zeta = \zeta_{k,v}$ are such that

$$L_{-k,v}A \cdot \ell_0(\zeta) = r_0(\zeta) \quad \text{and} \quad CA^{-1}CL_{-k,v} \cdot r_0(\eta) = C^{-1} \cdot \ell_0(\eta)\,.$$

The following implies that the complement of the union of the $J_{k,v}$ is a Cantor set. This is the main result of this subsection.

Theorem 6.5. *We have the following partition*

$$[\epsilon_{3,n}, 1) = \bigcup_{k=2}^{\infty} I_{-k,1}\,.$$

Furthermore, for each $k \geq 2$ and each $v \in \mathcal{V}$, the following is a partition:

$$I_{-k,v} = J_{-k,v} \cup \bigcup_{q=q'}^{\infty} I_{-k,\Theta_q(v)}\,,$$

where $q' = 0$ unless $v = c_1$, in which case $q' = -1$.

Note that one extends the definition of $\underline{d}(-k, v)$ to infinite words in the obvious fashion.

Lemma 6.6. *Let $k \in \mathbb{N}$ and $v \in \mathcal{V}$. Assume that $\eta_{-k,v}, \zeta_{-k,v} \in I_{-k,v}$. Then $\underline{d}^{\eta_{-k,v}}_{[1,\infty)} = \underline{d}(-k, v(v')^{\infty})$ and $\underline{d}^{\zeta_{-k,v}}_{[1,\infty)}$ is purely periodic of period $\underline{d}(-k, \overleftarrow{v'}\,)$.*

Proof. The result for $\eta_{-k,v}$ follows from Lemma 6.3 (ii).

The definition of $\zeta_{-k,v}$ gives the second result, since $\underline{d}(-k, \overleftarrow{v'}\,) = (-k)^{c_1}, (-k-1)^{d_1}, \cdots, (-k-1)^{d_{s-1}}, (-k)^{c_s-1}, (-k-1)$. □

The following is an immediate implication of the definition of the ordering (1.8).

Lemma 6.7. *Fix $m = 3$. For any v, w words and for any $k \in \mathbb{N}$, we have*

$$v \prec w \quad \textit{if and only if} \quad \underline{d}(-k, v) \succ \underline{d}(-k, w)\,.$$

Definition 6.8. Define $\omega_{-k,v}$ such that $\underline{d}^{\omega_{-k,v}}_{[1,\infty)}$ is purely periodic of period $\underline{d}(-k, (\,\mathfrak{f}(v)\,)$, where $\mathfrak{f}(v)$ is the full branched prefix of v.

Lemma 6.9. *Let $v \in \mathcal{V}$ and fix $k \geq 2$. The α-cylinder set $I_{-k,v}$ is a subset of $(\omega_{-k,v}, \zeta_{-k,v}]$.*

Proof. The set of α such that $\underline{d}^{\alpha}_{[1,|v|]} = \underline{d}(k, v)$ is contained in the interval $(\omega, \zeta]$ such that $L_{-k,v}A \cdot \ell_0(\omega) = \ell_0(\omega)$ and $L_{-k,v}A \cdot \ell_0(\zeta) = r_0(\zeta)$. The right endpoint here is exactly $\zeta = \zeta_{-k,v}$.

Due to the order reversing relationship between words and simplified digits with $-k < 0$, the proof of Lemma 4.22 shows that $\omega_{-k,v}$ is a greatest lower bound for $I_{-k,v}$. □

Lemma 6.10. *Suppose that $v = \Theta_p(u)$ for some $u \in \mathcal{V}$ and some $p \geq 1$. Fix $k \geq 2$. Then for $q \in \mathbb{N}$, $\zeta_{-k,\Theta_{q-1}(v)} = \omega_{-k,\Theta_q(v)}$.*

Proof. From Lemma 4.28, $\mathfrak{f}(\Theta_q(v)) = \overleftarrow{(\Theta_{q-1}(v))'}$. Now Lemma 6.6 and the definition of $\omega_{-k,\Theta_q(v)}$ yield the result. □

Lemma 6.11. *For all $v \in \mathcal{V}$ and all $h \in \mathbb{N}$, $\omega_{-k,\Theta_0^h(v)} = \omega_{-k,v}$.*

Proof. Proposition 4.25 shows that $\mathfrak{f}(\,\Theta_0^h(v)\,) = \mathfrak{f}(v)$ for all $v \in \mathcal{V}$. The result thus holds. □

The following result, and its proof, are completely analogous to Proposition 4.31 where the case of small α is treated.

Proposition 6.12. *For all $v \in \mathcal{V}$, and all $k \geq 2$, both*

$$I_{-k,v} = (\omega_{k,v}, \zeta_{-k,v}] \text{ and } I_{-k,v} \supset J_{-k,v}\,.$$

Proof. Fix k. We argue by induction of the length of the word v. We have already seen that $\eta_{-k,v} \in I_{-k,v}$ implies $\underline{d}(-k, v(v')^{\infty})$ gives the sequence of simplified digits of $\ell_0(\eta_{-k,v})$. The base cases, given by $v = c_1$, are easily verified.

Case 1. Suppose $v = \Theta_0^h(c)$ for some $h \geq 1$ and some $c > 1$. We have $v = c(1\,c)^h$. The argument as for Proposition 4.31 goes through (compare with the remaining cases).

Case 2. Suppose $v = uau$ in our usual decomposition. Since $\mathfrak{f}(v) = ua$, we have $L_{-k,ua}A \cdot \ell_0(\omega_{-k,v}) = \ell_0(\omega_{-k,v})$. This equality then implies that given N, there are α sufficiently close to, and larger than, $\omega_{-k,v}$ such that $\underline{d}(-k, u(au)^N)$ is admissible for α. It follows that we can partition $I_{-k,v}$ by subintervals corresponding to the values N. Now, $\Theta_0(v) = vv'' = uauau$, and hence $\Theta_0(v)$ is admissible on $[\omega_{-k,v}, \zeta_{k,\Theta_0(v)})$. Similarly, each $\Theta_0^h(v)$ is admissible on $[\omega_{-k,v}, \zeta_{k,\Theta_0^h(v)})$. By considering our ordering on words, it is clear that $\zeta_{k,\Theta_0^h(v)} > \eta_{k,\Theta_0^h(v)} > \omega_{k,v}$.

We now proceed inductively for larger values of q. To begin, the admissibility v on all of $I_{-k,v}$ and the admissibility of $\Theta_0(v)$ on exactly $[\omega_{-k,v}, \zeta_{k,\Theta_0(v)})$ implies that there must be a shortest extension of $v = uau$ admissible for those

α immediately to the right of $\zeta_{k,\Theta_0(v)}$. Lemma 6.10 shows that $\mathfrak{f}(\,\Theta_1(v)\,)$ is this extension. Since $\underline{d}_{[1,\infty]}^{\omega_{-k,\Theta_1(v)}}$ is purely periodic of period $\underline{d}(-k,\,(\,\mathfrak{f}(\Theta_1(v))\,)$, arbitrarily high powers of this period give admissible expansions for α just to the right of $\omega_{-k,\Theta_1(v)}$. Recall that Proposition 4.25 shows that *any* $v = uau$ is always a prefix of the square of the corresponding $\mathfrak{f}(v) = ua$; we hence find that all of $\Theta_1(v)$ is admissible on an interval beginning at $\omega_{-k,\Theta_1(v)}$. We iterate this argument for increasing q, to give that for each q, $\Theta_q(v)$ is admissible on $[\zeta_{k,\Theta_q(v)}, \omega_{k,\Theta_q(v)})$.

From Lemmas 4.30 and 6.7 we can conclude that $\eta_{-k,\Theta_q(v)} > \omega_{-k,\Theta_q(v)}$. It follows that $J_{-k,\Theta_q(v)} \subset I_{-k,\Theta_q(v)}$.

Case 3. As in the proof of Proposition 4.31, the remaining case is that $v = \Theta_0^h(uau)$. Thus, $v = u(au)^{h+1}$ in our usual decomposition. The right endpoint of $I_{-k,\Theta_0^{h+1}(uau)}$ is at $\zeta_{-k,u(au)^{h+2}}$. Lemma 6.6 now yields that the left endpoint of $I_{k,\Theta_0^h(uau)} \setminus I_{k,\Theta_0^{h+1}(v)}$ is the point with purely periodic lower simplified digit expansion of period $\underline{d}(k, vu'a)$. In the proof of Proposition 4.31 we showed that $\mathfrak{f}(\,\Theta_1 \circ \Theta_0^h(u)\,) = vu'a$. Therefore, $\omega_{-k,\Theta_1(v)} = \zeta_{-k,\Theta_0^{h+1}(uau)}$ and also since $\Theta_1(v)$ is a prefix of the square of $vu'a$, it follows that $\overline{d}(-k, \Theta_1(v)\,)$ is admissible on $[\omega_{-k,\Theta_1(v)}, \zeta_{-k,\Theta_1(v)})$. That $\eta_{-k,\Theta_1(v)}$ belongs to this interval is easily shown.

Induction shows the result for $\Theta_q(v)$ when $q \geq 1$. □

Proof of Theorem 6.5. That $[\epsilon_{3,n}, 1) = \cup_{k=1}^{\infty} I_{-k,1}$ is simply a consequence of the fact that for each α in this range, there is some k such that $T_\alpha(\,\ell_0(\alpha)\,) = A^{-k}C \cdot x$.

Proposition 6.12 shows that for all $v \in \mathcal{V}$, $I_{-k,v} = [\omega_{-k,v}, \zeta_{-k,v})$ can be partitioned by $J_{-k,v} = [\eta_{-k,v}, \zeta_{-k,v})$ and its complement. Recall Lemma 6.11 states that for all h, we have $\omega_{-k,\Theta_0^h(v)} = \omega_{-k,v}$. By Lemma 6.10 (and the complementary results in the proof of Proposition 6.12), for all $q \in \mathbb{N}$, $\omega_{k,\Theta_q(v)} = \zeta_{k,\Theta_{q-1}(v)}$. Therefore, $\cup_{q=0}^{\infty} I_{-k,\Theta_q(v)}$ is a subinterval of $I_{-k,v} \setminus J_{-k,v}$ which has $\omega_{-k,v}$ as its left endpoint. Finally, the definition of $\Theta_q(v)$ combined with Lemma 6.6 shows that $\lim_{q\to\infty} \zeta_{k,\Theta_q(v)} = \eta_{k,v}$. Therefore, the right endpoint of the union is in fact the left endpoint of $J_{k,v}$. □

6.2. Synchronization holds on a set of full measure

6.2.1. *Right digits are admissible; synchronization occurs on each $J_{-k,v}$*

We now define $R_{-k,v}$ exactly so that the group identity of Proposition 8.2, below, gives that $L_{-k,v} = C^{-1}AC^2R_{-k,v}$, and thus the main hypothesis of Lemma 6.3 is satisfied. That synchronization does occur along $J_{-k,v}$ is then only a matter of showing that $R_{k,v} \cdot r_0(\alpha)$ is admissible at all $\alpha \in J_{-k,v}$.

Since $m = 3$, (3.1) gives that $U = AC(AC^2)^{n-2}$. We set

$$u = u_{3,n} = (1,2)^{n-2}, (1,1).$$

As α tends to one, $\overline{b}_{[1,\infty)}^{\alpha}$ begins with ever higher powers of u, compare with Proposition 3.1.

For typographic ease, for $k \geq 2$ let

$$\mathcal{E}_k = (1,1)u^{k-2}(1,2)^{n-3} \text{ and } \tilde{\mathcal{E}}_k = (AC^2)^{n-3}U^{k-2}AC.$$

One easily verifies that that $u^{k-1}(1,2)^{n-3} = (1,2)^{n-2}\mathcal{E}_k$.

We also let

$$\mathcal{G} = (1,2)(1,1)(1,2)^{n-3} \text{ and } \tilde{\mathcal{G}} = (AC^2)^{n-3}AC\,AC^2.$$

One easily verifies that that $\tilde{\mathcal{G}} = \tilde{\mathcal{E}}_{k+1}\tilde{\mathcal{E}}_k^{-1}$.

Definition 6.13. Suppose that $v = c_1 d_1 \cdots c_s \in \mathcal{V}$ and $k \geq 2$.

(1) for further typographic ease, let $\mathcal{E} = \mathcal{E}_k$, $\mathcal{F} = \mathcal{E}_{k+1}$ and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_k$, $\tilde{\mathcal{F}} = \tilde{\mathcal{E}}_{k+1}$;
(2) the upper digit sequence of $-k, v$ is

$$\overline{b}(-k,v) = (1,2)^{n-2}\mathcal{E}^{c_1}\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s},$$

whose length is denoted;
(3) $\overline{S}(-k,v) = |\,\overline{b}(-k,v)\,|$;
(4) the *right matrix* of $-k, v$ is

$$R_{-k,v} = \tilde{\mathcal{E}}^{c_s}\tilde{\mathcal{F}}^{d_{s-1}}\;\tilde{\mathcal{E}}^{c_{s-1}}\tilde{\mathcal{F}}^{d_{s-2}}\cdots\tilde{\mathcal{E}}^{c_2}\tilde{\mathcal{F}}^{d_1}\tilde{\mathcal{E}}^{c_1}(AC^2)^{n-2}.$$

Note that Proposition 8.2, below, implies that $L_{-k,v} = C^{-1}AC^2\,R_{-k,v}$.

We will prove admissibility of $\overline{b}(-k,v)$ on $J_{-k,v}$ by induction, similar to our proof of admissibility of $\underline{d}(k,v)$ on $J_{-k,v}$. However, the role of the various $\eta_{k,v}$ will now be played by certain points lying to the left of the corresponding $\eta_{-k,v}$, the $\beta_{-k,v,N}$ introduced in the next statement.

Lemma 6.14. *Let $v = c_1 d_1 \cdots c_s \in \mathcal{V}$ and $k \in \mathbb{N}$. Suppose that $\overline{b}(-k,v)$ is admissible on $J_{-k,v}$. Set $\zeta = \zeta_{-k,v}$ and $\eta = \eta_{-k,v}$. Then for each $N \in \mathbb{N}$, there exists $\beta_{-k,v,N}$ less than η such that*

$$\overline{b}^{\beta_{-k,v,N}}_{[1,(1+\overline{S})N\,]} = [\,\overline{b}(-k,v)(1,1)\,]^N.$$

Furthermore,

$$\overline{b}^{\zeta}_{[1,\infty)} = \begin{cases} \overline{b}(-k,v), \overline{\mathcal{G}\mathcal{E}^{c_1}} & \text{if } v = c_1 \\ \\ \overline{b}(-k, (\overleftarrow{v'}\,)^{\infty}) & \text{otherwise.} \end{cases}$$

Proof. The definition of $\eta_{-k,v}$ shows that $AC\,R_{-k,v}$ fixes $r_0(\eta_{-k,v})$. Therefore, for each $N \in \mathbb{N}$, there exists $\alpha = \alpha'_N < \eta_{-k,v}$ and sufficiently close, with $\overline{b}^{\alpha}_{[1,(1+\overline{S})N\,]} = [\,\overline{b}(-k,v)(1,1)\,]^N$.

Lemma 6.3, (iii) yields $\overline{b}^{\zeta}_{[1,\infty)} = \overline{b}(-k, v), (1,2)(1,1)\overline{b}^{\zeta}_{[2,\infty)}$. This sequence is of course periodic. The prefix $u^{k-1}(1,2)^{n-3}$ of $\overline{b}(-k, v)$ can be rewritten as $(1,2)^{n-2}\,(1,1)\,u^{k-2}\,(1,2)^{n-3}$, thus $(1,2)\,(1,1)\,\overline{b}^{\zeta}_{[2,\infty)}$ has the prefix $(1,2)(1,1)$ $(1,2)^{n-3}(1,1)u^{k-2}(1,2)^{n-3} = \mathcal{G}\,\mathcal{E}$. This prefix is followed by the complement of the prefix $u^{k-1}(1,2)^{n-3}$ of $\overline{b}(-k, v)$. The case of $v = c_1$ is now easily verified.

For general v, $\overline{b}^{\zeta}_{[1,\infty)} = \overline{b}(-k, v), \overline{\mathcal{G}\,\mathcal{E}\,\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}}$. This last equals $\overline{b}(-k, (c_1d_1\cdots d_{s-1}(c_s-1)1)^{\infty})$. Since v is a palindrome, this in turn equals $\overline{b}(-k, (c_sd_{s-1}\cdots d_1(c_1-1)1)^{\infty})$. Since $c_sd_{s-1}\cdots d_1(c_1-1)1 = \overleftarrow{v'}$, the result holds. $\square$

Lemma 6.15. *Fix $j \in \mathbb{N}$ and $0 \le i < |u|$. If there is some $\alpha \ge \epsilon_{3,n}$ such that $\overline{b}^{\alpha}_{[1,j|u|+i]} = u^j u_{[1,i]}$, then $\overline{b}^{\alpha'}_{[1,j|u|+i]} = u^j u_{[1,i]}$ for all $\alpha' > \alpha$.*

Proof. (Of course, if $i = 0$, then $w_{[1,i]}$ is the empty word.) Using Proposition 3.1 with Lemma 1.7, one has that all powers of U are admissible when $\alpha = 1$, there are thus branches of digits corresponding to each power and appropriate suffix of U that continue to the left from $\alpha = 1$. For any of these, admissibility at any given α thus guarantees admissibility at each $\alpha' \ge \alpha$. $\square$

Lemma 6.16. *Fix $k \ge 2$. We have*

$$\overline{b}^{\omega_{-k,1}}_{[1,\infty)} = u^{k-1}(1,2)^{n-3}\,\overline{\mathcal{E}}\,.$$

Furthermore, the digits $\overline{b}^{\alpha}_{[1,n-3+(k-1)|u|\,]} = u^{k-1}(1,2)^{n-3}$ are admissible for all $\alpha \ge \omega_{k,1}$.

Proof. We first give one proof for $k \ge 3$. Since $\mathfrak{f}(1) = 1$, $\zeta_{-k,1} = \omega_{-(k+1),1}$. From

$$\begin{aligned}\overline{b}^{\zeta_{-k,1}}_{[1,\infty)} &= u^{k-1}, (1,2)^{n-3}\,\overline{\mathcal{G}\mathcal{E}_k} = u^{k-1}, (1,2)^{n-3}\mathcal{G}\,\overline{\mathcal{E}_k\mathcal{G}}\\ &= u^k(1,2)^{n-3}\,\overline{\mathcal{E}_{k+1}}.\end{aligned}$$

The following works for all k. For typographical ease, let $\omega = \omega_{-k,1}$. Since $\mathfrak{f}(1) = 1$, by definition, $A^{-k}C \cdot \ell_0(\omega) = \ell_0(\omega)$. By Lemma 8.1, $CA^{-1}CA^{-k}CA^{-1} = (AC^2)^{n-3}U^{k-1}$. By Lemma 6.16, $(AC^2)^{n-3}U^{k-1} \cdot r_0(\omega)$ is an admissible expansion if it has value in $\mathbb{I}_\omega$.

For *any* α, $CA^{-1}C \cdot \ell_0(\alpha) = CA^{-1}CA^{-1} \cdot r_0(\alpha) = (AC^2)^{n-2} \cdot r_0(\alpha)$. Hence, $(AC^2)^{n-3}U^{k-1} \cdot r_0(\omega) = CA^{-1}CA^{-k}CA^{-1} \cdot \ell_0(\omega) = CA^{-1}CA^{-1} \cdot \ell_0(\omega) = (AC^2)^{n-2} \cdot r_0(\omega)$. Since $(AC^2)^{n-2}$ is a suffix of U, we have that $(AC^2)^{n-2} \cdot r_0(\omega)$ is admissible. In particular it has value in $\mathbb{I}_\omega$. It follows that $(AC^2)^{n-3}U^{k-1} \cdot r_0(\omega)$ is an admissible expansion. Furthermore, the equality $(AC^2)^{n-2} \cdot r_0(\omega) = (AC^2)^{n-3}U^{k-2}AC(AC^2)^{n-2} \cdot r_0(\omega)$ shows that $(AC^2)^{n-2} \cdot r_0(\omega)$ is fixed by $(AC^2)^{n-3}U^{k-2}AC$. Therefore,

$$\begin{aligned}\overline{b}^{\omega_{-k,1}}_{[1,\infty)} &= (1,2)^{n-2}\overline{(1,1)u^{k-2}(1,2)^{n-3}} = u^{k-1}(1,2)^{n-3}\overline{(1,1)u^{k-2}(1,2)^{n-3}}\\ &= u^{k-1}(1,2)^{n-3}\,\overline{\mathcal{E}}\,.\end{aligned}$$

Lemma 6.15 yields that the digits $\overline{b}^{\alpha}_{[1,n-3+(k-1)|u|\,]} = u^{k-1}(1,2)^{n-3}$ are admissible for all $\alpha \geq \omega_{-k,1}$. ☐

Proposition 6.17. *Suppose that $v \in \mathcal{V}$ and $k \geq 2$. Then for all $\alpha \in J_{-k,v}$,*

$$\overline{b}^{\alpha}_{[1,\overline{S}\,]} = \overline{b}(-k,v)\,.$$

Proof. By Lemma 1.7, for each v it suffices to find $\alpha' < \eta_{-k,v}$ and $\alpha'' > \zeta_{-k,v}$ such that $\overline{b}^{\alpha'}_{[1,\overline{S}\,]} = \overline{b}^{\alpha''}_{[1,\overline{S}\,]} = \overline{b}(-k,v)$. See Table 6.1 for a summary of the choices of α', α'' in the various cases.

	v	α'	α''
Base cases	1	1	$\omega_{-k,1}$
	$c>1$	$\beta_{-k,c-1,2}$	$\omega_{-k,1}$
	$c\,1\,c,\ c \geq 1$	$\beta_{-k,c,3}$	$\zeta_{-k,c+1}$
	$\Theta_q(c),\ q \geq 2$	$\beta_{-k,c,q+3}$	$\zeta_{-k,\Theta_{q-1}(c)}$
	$\Theta_1(c),\ c>1$	$\beta_{-k,c,4}$	$\zeta_{-k,\Theta_0(c)}$
Case 1	$\Theta_0^h(c)$	$\beta_{-k,\Theta_0^{h-1}(c),2}$	$\zeta_{-k,c+1}$
Case 2	$\Theta_q \circ \Theta_p(u),\ \ p,q \geq 1$	$\beta_{-k,\Theta_p(u),q+3}$	$\zeta_{-k,\Theta_{q-1}\circ\Theta_p(u)}$
Case 3	$\Theta_0^h \circ \Theta_p(u),\ p \geq 1$	$\beta_{-k,\Theta_0^{h-1}\circ\Theta_p(u),3}$	$\zeta_{-k,\Theta_{p-1}(u)}$

Table 6.1. Admissibility of $\overline{d}(-k,v)$ on $J_{-k,v}$ is shown by finding $\alpha' > \zeta_{-k,v} > \eta_{-k,v} > \alpha''$ such that $\overline{b}^{\alpha}_{[1,\infty)}$ has prefix $\overline{b}(-k,v)$ for both $\alpha = \alpha', \alpha''$. See the proof of Proposition 6.17. Compare with Table 4.1.

Since α' is always of the general form $\beta_{-k,u,N}$, we note immediately that the identity $\mathcal{F} = \mathcal{E}\mathcal{G}$ yields

$$\begin{aligned}
&[\,\overline{b}(-k,v)(1,1)\,]^N \\
&= \overline{b}(-k,v)[\mathcal{F}\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}]^{N-1}(1,1) \\
&= \overline{b}(-k,v)[\mathcal{E}\mathcal{G}\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}]^{N-1}(1,1) \\
&= \overline{b}(-k,v)\mathcal{E}[\mathcal{G}\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s+1}]^{N-2} \\
&\qquad \cdot\mathcal{E}\mathcal{G}\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}(1,1) \\
&= \overline{b}'(-k,v(v')^{N-2})X,
\end{aligned} \tag{6.1}$$

where $X = \mathcal{E}\mathcal{G}\mathcal{E}^{c_1-1}\,\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}(1,1)$ and we admit to our abuse of notation by writing $\overline{b}'$.

Base cases. Consider $v = c_1 = c$. Since $\overline{b}^{\omega_{-k,1}}_{[1,\infty)}$ has $\overline{b}(-k, 1)$ as a prefix, Lemma 6.15 yields the result when $v = 1$. From (6.1), one finds that for $c \geq 2$, $\overline{b}(-k, c)$ is a prefix of $[\overline{b}(-k, c-1)(1, 1)]^2$, which in turn is a prefix of $\overline{b}^{\beta_{-k,c-1,2}}_{[1,\infty)}$. Lemma 6.16 shows that also the right digit sequence of $\omega_{-k,1}$ has each $\overline{b}(-k, c)$ as a prefix. Therefore, the result holds for all v of length one.

When $v = c\,1\,c$, we use $\alpha' = \beta_{-k,c,3}$, and $\alpha'' = \zeta_{-k,c+1}$. Since $\overline{b}(-k, v) = u^{k-1}, (1, 2)^{n-3}\, (\mathcal{E}^c\mathcal{G})^1\, \mathcal{E}^c$, (6.1) leads to $[\,\overline{b}(-k, v)(1, 1)\,]^3$ having $\overline{b}(-k, v)$ as a prefix. Lemma 6.14 yields that $\overline{b}^{\zeta_{-k,c+1}}_{[1,\infty)}$ also has $\overline{b}(-k, v)$ as a prefix.

Suppose $v = \Theta_q(c)$, $q \geq 1$ and $c > 1$.
Thus, $\overline{b}(-k, v) = u^{k-1}, (1, 2)^{n-3}(\mathcal{E}^c\mathcal{G})^{q+1}\, \mathcal{E}^c$. We set $\alpha' = \beta_{-k,c,N}$, with $N = q + 2$. Using (6.1), one shows that $\overline{b}^{\alpha'}_{[1,(1+\overline{S})N\,]}$ has $\overline{b}(-k, v)$ as a prefix. We set $\alpha'' = \zeta_{-k,\Theta_{q-1}(c)}$. Lemma 6.14 yields that $\overline{b}^{\alpha''}_{[1,\infty)}$ has prefix $\overline{b}(-k, \Theta_{q-1}(c))\, \mathcal{G}\mathcal{E}^{c+1}$. The result thus holds in this case.

Since $c = 1$ gives $\overline{b}(-k, v) = u^{k-1}, (1, 2)^{n-3}\, \mathcal{E}\mathcal{G}\mathcal{F}^{q-1}\, \mathcal{E}$, again using (6.1) shows that the same argument succeeds.

Case 1. Suppose $v = \Theta_0^h(c)$, $h \geq 2$ for some $c > 1$. We have $v = c(1\,c)^h$ and thus $\overline{b}(-k, v) = u^{k-1}, (1, 2)^{n-3}\, \mathcal{E}^c\mathcal{G}\, (\mathcal{E}^{c+1}\mathcal{G})^{h-1}\, \mathcal{E}^c$. Lemma 6.14 easily yields that $\overline{b}^{\zeta_{-k,c+1}}_{[1,\infty)}$ has $\overline{b}(-k, v)$ as a prefix. That is, we can take $\alpha'' = \zeta_{-k,c+1}$. With $\alpha' = \beta_{-k,\Theta_0^{h-1}(c),2}$ from (6.1) one finds that $\overline{b}^{\alpha'}_{[1,\infty)}$ also has $\overline{b}(-k, v)$ as a prefix.

Case 2. Suppose that the result holds for $v = uau = \Theta_p(u)$ with $u \in \mathcal{V}$ and $p \geq 1$. We prove that the result holds for $\Theta_q(v)$, $q \geq 1$. By Lemma 6.14, for general k and v, one has $\overline{b}^{\zeta_{-k,v}}_{[1,\infty)} = \overline{b}(-k, (\overleftarrow{v'}\,)^\infty\,)$, and thus (4.4) gives $\overline{b}^{\zeta_{-k,\Theta_{q-1}(v)}}_{[1,\infty)} = \overline{b}(-k, v\,[\,(v')^q v''\,]^\infty\,)$, which clearly has $\overline{b}(-k, \Theta_q(v)\,)$ as a prefix. Therefore, we can take $\alpha'' = \zeta_{-k,\Theta_{q-1}(v)}$.

We first specialize to $c > 1$, that $v' = a\,\overleftarrow{u'}\,u = au_{[-1]}(c_1 - 1)\,1\,u$, $v'' = au$ and that u both begins and ends with the letter c_1. Since terms in general $\overline{b}(-k, v)$ corresponding to any $d_j = 1$ vanish, a final v' appearing inside of $\overline{b}'$ in (6.1) has a prefix coming from v''. From this, one finds that $[\overline{b}(-k, v)(1, 1)]^{q+3}$ has $\overline{b}(-k, \Theta_q(v)\,)$ as a prefix. That is, we can take $\alpha' = \beta_{-k,v,q+3}$.

We now consider the case of $c = 1$, where $v' = au_{[-2]}(d_1 + 1)$ and $v'' = au$. In (6.1) we can let the first $\mathcal{F}^{d_1}$ corresponding to the final power of v' revert to $\mathcal{E}\,\mathcal{G}\,\mathcal{F}^{d_1-1}$ so as to confirm that here also a final v' contributes a subword that has as a prefix the contribution of v''. Using this, $[\overline{b}(-k, v)(1, 1)\,]^{q+3}$ has $\overline{b}(-k, \Theta_q(v)\,)$ as a prefix. That is, we can take $\alpha' = \beta_{-k,v,q+3}$.

Case 3. Suppose that $\Theta_p(u) = uau$ in our usual notation. Recall that $\Theta_0^h \circ \Theta_p(u) = u\,[(u')^p u'']^{h+1}$. As in Case 2, Lemma 6.14 and (4.4) give $\overline{b}^{\zeta_{-k,\Theta_{p-1}(u)}}_{[1,\infty)} =$

$\overline{b}(-k, u\,[\,(u')^p u''\,]^\infty\,)$. Thus, we use $\alpha'' = \zeta_{-k,\Theta_{p-1}(u)}$. From (4.5) and (6.1), we can take $\alpha' = \beta_{-k,\Theta_0^{h-1}\circ\Theta_p(u),3}$. □

6.2.2. *There are no other points of synchronization*

That all $\alpha > \epsilon_{3,n}$ for which there is synchronization lie in the union of the $J_{-k,v}$ is shown as for the case of small α. We find that there is some sequence of q_i such that $\alpha \in \cap_{j=1}^{\infty} I_{k,\Theta_{q_j}\circ\cdots\circ\Theta_{q_1}(1)}$. But, this implies that $\underline{b}^{\alpha}_{[1,\infty)}$ has digits only in $\{(-k,1),(-k-1,1)\}$, while $\overline{b}^{\alpha}_{[1,\infty)}$ has digits only in $\{(-1,1),(-2,1),(1,2)\}$, with the digit $(1,2)$ appearing infinitely often. Therefore, the two orbits obviously cannot synchronize.

6.2.3. *The non-synchronization set is of measure zero*

For $\alpha > \epsilon_{3,n}$ not in any $J_{-k,v}$, there is some k such that the T_α-orbit of $\ell_0(\alpha)$ is always in the set of points whose simplified digits are $-k$ or $-k-1$. In particular, this orbit certainly remains in $[-t,0)$ and we can argue as in Subsection 4.6 to conclude that the set of these α values has measure zero.

7. Synchronization for $\gamma_{3,n} < \alpha < \epsilon_{3,n}$, $n \geq m$

Recall that $\gamma = \gamma_{3,n}$ is characterized by $C^{-1}\cdot\ell_0(\gamma) = r_0(\gamma)$. Recall also that $\ell_1 = A^{-1}C\cdot\ell_0$ for all $\alpha \in [0,\epsilon_{3,n})$. Thus, for $\gamma_{3,n} < \alpha < \epsilon_{3,n}$ we use the notation of Section 6, but now with $k = 1$ (and certain further technical adjustments as noted below).

Theorem 7.1. *Fix $m = 3$ and $n \geq m$. If $n = 3$, then there is synchronization for all $\alpha \in (\gamma_{3,3}, \epsilon_{3,3})$. If $n > 3$, then the set of $\alpha \in (\gamma_{3,n}, \epsilon_{3,n})$ for which there is not synchronization is uncountable, but of Lebesgue measure zero.*

For $n \geq 3$, define $\check{\mathcal{V}}_n \subset \mathcal{V}$ to be the trimming of $\mathcal{V}$ such that for all $v = c_1 d_1 \cdots c_s \in \check{\mathcal{V}}_n$, $c_i \leq n-2$ and furthermore such that the only word with prefix $n-2$ is $v = n-2$ itself. Define each I_{-1,c_1} to be as above, except that we insist on a left endpoint at least γ.

Theorem 7.2. *Fix $m = 3$ and $n \geq m$. We have the following equality*

$$[\gamma_{3,n}, \epsilon_{3,n}) = \bigcup_{v=1}^{n-2} I_{-1,v}\,.$$

Furthermore, for each $v \in \check{\mathcal{V}}_n \setminus \{n-2\}$, the following is a partition:

$$I_{-1,v} = J_{-1,v} \cup \bigcup_{q=q'}^{\infty} I_{-1,\Theta_q(v)}\,,$$

where $q' = 0$ unless $v = c_1$, in which case $q' = -1$. Moreover, $I_{-1,n-2} = J_{-1,n-2} = [\gamma_{3,n}, \zeta_{-1,n-2})$.

Proof. Note that for $\alpha = \zeta_{-1,1}$, we have $A^{-1}C \cdot \ell_0(\alpha) = r_0(\alpha)$, which is exactly the definition of $\alpha = \epsilon_{3,n}$. Arguing as we did for $\alpha > \epsilon_{3,n}$ shows that first equality of the theorem, giving the basic partition, holds. Similarly, that $J_{-1,v}$ with the union of the $I_{-1,\Theta_q(v)}$ partition $I_{-1,v}$ holds since the proof of Theorem 6.5 is easily checked to extend to this case.

Recall that $\eta_{-k,v}$ is such that $R_{-k,v} \cdot r_0(\eta) = C^{-1} \cdot \ell_0(\eta)$, and thus $\eta_{-1,n-2}$ is such that $r_0(\eta_{-1,n-2}) = C^{-1} \cdot \ell_0(\eta_{-1,n-2})$. That is, $\eta_{-1,n-2} = \gamma_{3,n}$. The statement $I_{-1,n-2} = J_{-1,n-2} = [\gamma_{3,n}, \zeta_{-1,n-2})$ thus follows. $\square$

Proposition 7.3. *Fix $m = 3$ and $n \geq m$. For all $v \in \check{V}_n$, synchronization occurs along $J_{-1,v}$.*

Sketch. Here also the arguments of the previous section give the proof, however we must make minor adjustments. Note that $U^{-1} = [AC(AC^2)^{n-2}]^{-1} = (AC^2)^{2-n}(AC)^{-1}$. Thus in this setting of $k = 1$, any term of the form $[(AC^2)^{n-3}U^{k-2}AC]^a$ as in the statements of Lemma 8.1 and Proposition 8.2 becomes $[(AC^2)^{-1}]^a$. Note that each such term is followed by an appearance of $(AC^2)^{n-3}$; since each exponent a will arise as either c_j or $c_j + 1$ for some c_j letter of some $v \in \check{V}$, we have $a \leq n - 2$. Since an exponent of the form $c_j + 1$ occurs only when the block is also preceeded by an occurrence of AC^2, the identity guarantees an expression that has only positive powers of AC and of AC^2. (Example 7.4 exhibits this phenomenon.) Thus our definition of $R_{-k,v}$ extends to include the case of $d = -1$. Similarly, $\mathcal{E}_1$ must now denote $(1, 2)^{-1}$ where we recognize that occurrences of $\mathcal{E}_1$ are surrounded by $(1, 2)$ occurring to a sufficiently high power so that the usual arithmetic of exponents results in a sensible word. Note that the key relation $\mathcal{E}\mathcal{G} = \mathcal{F}$ thus holds in this setting. $\square$

Finally, both that synchronization only occurs along the $J_{-1,v}$ and that this is a set of full measure follow from the arguments of the previous section.

Example 7.4. Note first that $R_{-1,n-2} = CA^{-1}C\ (A^{-1}C)^{n-2}A^{-1} = Id$ holds for all $n \geq 3$. We also have $R_{-1,1} = (AC^2)^{-1} \cdot (AC^2)^{n-2} = (AC^2)^{n-3}$. Note that these calculations agree when $n = 3$. Let $n > 3$ and let us calculate one longer right matrix:

$$\begin{aligned} R_{-1,111} &= \tilde{\mathcal{E}}\tilde{\mathcal{F}}\tilde{\mathcal{E}}(AC^2)^{n-2} = (AC^2)^{-1} \cdot (AC^2)^{n-3}AC \cdot (AC^2)^{-1} \cdot (AC^2)^{n-2} \\ &= (AC^2)^{n-4}AC(AC^2)^{n-3}. \end{aligned}$$

8. Group element identities for the setting $\alpha > \gamma_{3,n}$

Here also, we thank the referee for suggesting an emphasis on group element conjugation. Recall that $U = AC(AC^2)^{n-2}$.

Lemma 8.1. *Suppose that $m = 3$, and $k, a \in \mathbb{N}$. Then*

$$CA^{-1}C(A^{-k}C)^a(CA^{-1}C)^{-1} = [(AC^2)^{n-3}U^{k-2}AC]^a\,.$$

Proof. To prove this, it suffices to show that the conjugacy relation holds when $a = 1$. As a base case, we consider $k = 1$. The left hand side reduces to CA^{-1}. But, $CA^{-1} = (AC^2)^{-1} = (AC^2)^{n-3}[AC\,(AC^2)^{n-2}]^{-1}AC$. This case thus holds.

Now suppose that the identity holds for $a = 1$ and some value of k. We find

$$\begin{aligned}
CA^{-1}C(A^{-k-1}C)C^{-1}AC^{-1} &= CA^{-1}CA^{-k}C^{-1}\\
&= CA^{-1}C(A^{-k}C)(CA^{-1}C)^{-1}\ \ (CA^{-1}C\,C^{-2})\\
&= (AC^2)^{n-3}U^{k-2}AC\ \ CA^{-1}C^{-1}\\
&= (AC^2)^{n-3}U^{k-2}AC\ \ CA^{-1}C^{2}\\
&= (AC^2)^{n-3}U^{k-2}AC\ \ (AC^2)^{n-2}AC\\
&= (AC^2)^{n-3}U^{k-1}AC.
\end{aligned}$$

The result thus holds. □

Recall that for $k \in \mathbb{N}$ and $v = c_1d_1\ldots, c_{s-1}d_{s-1}c_s \in \mathcal{V}$

$$L_{-k,v} = (A^{-k}C)^{c_s}\ (A^{-k-1}C)^{d_{s-1}}(A^{-k}C)^{c_{s-1}}\cdots(A^{-k-1}C)^{d_1}(A^{-k}C)^{c_1}A^{-1}\ \ \text{and}$$
$$R_{-k,v} = \tilde{\mathcal{E}}^{c_s}\tilde{\mathcal{F}}^{d_{s-1}}\ \tilde{\mathcal{E}}^{c_{s-1}}\tilde{\mathcal{F}}^{d_{s-2}}\ldots\tilde{\mathcal{E}}^{c_2}\tilde{\mathcal{F}}^{d_1}\tilde{\mathcal{E}}^{c_1}(AC^2)^{n-2},$$

where $\tilde{\mathcal{E}} = (AC^2)^{n-3}U^{k-2}AC$ and $\tilde{\mathcal{F}} = (AC^2)^{n-3}U^{k-1}AC$.

Proposition 8.2. *Fix $k \in \mathbb{N}$ and $v = c_1d_1\ldots, c_{s-1}d_{s-1}c_s \in \mathcal{V}$, then*

$$CA^{-1}C\ L_{-k,v} = R_{-k,v}\,.$$

Proof. Since in any group, the conjugate of a product is the product of the conjugates, the previous lemma yields

$$(CA^{-1}C)\ L_{-k,v}A\ (CA^{-1}C)^{-1} = R_{-k,v}(AC^2)^{-n+2}.$$

Since $(AC^2)^{-n+2}\ CA^{-1}CA^{-1} = \mathrm{Id}$, the result follows. □

9. Planar extensions

As we prove in upcoming work, the detailed results of this current paper are key in describing planar extensions of the interval maps. We thank the referee for suggesting that we include in the present work an indication of these results.

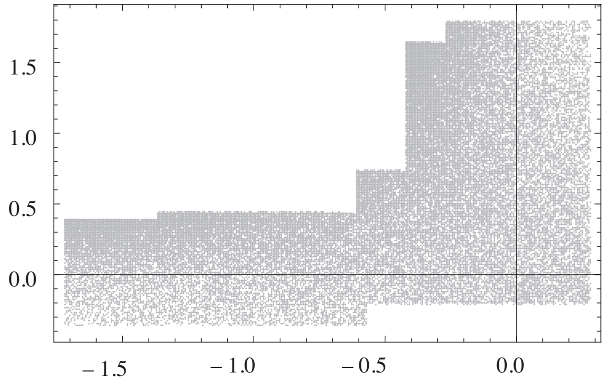


Figure 9.1. Sixty thousand points in the $\mathcal{T}_{3,3,0.14}$-orbit of $(\pi/10^3, 0.1)$.

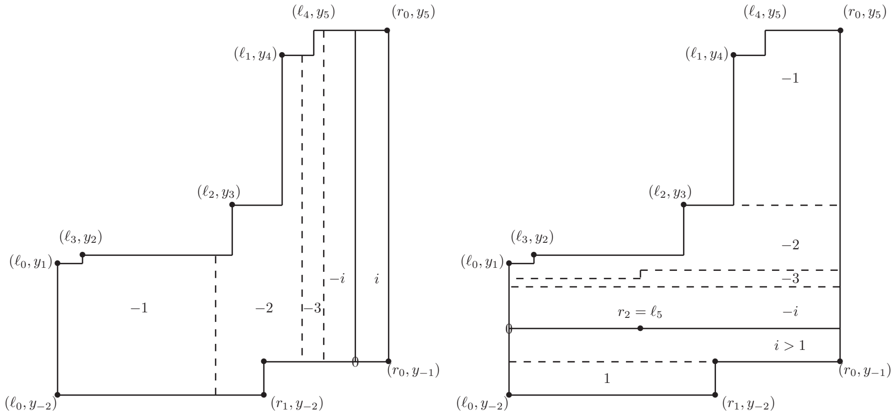


Figure 9.2. Planar extension of the one-dimensional system indexed by $m = n = 3, \alpha = 0.14$ (see Figure 1.1), given as domain (left) and image (right). On the left side, an index j such as $j = -1$ denotes the region fibering above $\Delta_\alpha(j, 1)$, the indices $-i, i$ indicate unions of similar regions; the image of each such region is marked with the corresponding symbol. The various ℓ_i, r_j denote the images of the endpoints under T_α, as usual. The value $\alpha = 0.14$ is an interior point of $J_{1,1}$. The two bottom heights of the region are $y_{-2} = -r_0(\eta_{1,1})$ and $y_{-1} = -r_1(\eta_{1,1})$. The top heights of the region $y_1, \ldots, y_5$ are similarly given as negative one times entries in the $T_{\zeta_{1,1}}$-orbit of $\ell_0(\zeta_{1,1})$. Each α in the interior of $J_{1,1}$ has a planar extension similarly described, so that the areas of the regions vary continuously.

9.1. Two-dimensional maps and measure

The standard number theoretic planar map associated to a Möbius transformation M is defined by

$$\mathcal{T}_M(x, y) := \Big(M \cdot x, N \cdot y\Big) := \Big(M \cdot x, RMR^{-1} \cdot y\Big)$$

where $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and there are of course niceties of domain of definition. Thus, $\mathcal{T}_M(x, y) = (\, M\cdot x, -1/(M\cdot(-1/y))\,)$. An elementary Jacobian matrix calculation verifies that the measure μ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx\, dy}{(1 + xy)^2}$$

is (locally) $\mathcal{T}_M$-invariant.

For simplicity of notation, fixing m, n, we set $T_\alpha := T_{m,n,\alpha}$ and $\mathbb{I}_\alpha := [\,(\alpha - 1)t, \alpha t\,)$. Each T_α is piecewise Möbius, so that there is some partition of its domain into subintervals, $\mathbb{I}_\alpha = \cup_\beta K_\beta$, such that $T_\alpha(x) = M_\beta \cdot x$ for all $x \in K_\beta$. We thus set

$$\mathcal{T}_\alpha(x, y) = \mathcal{T}_{3,n,\alpha}(x, y) = \Big(M_\beta \cdot x,\, R M_\beta R^{-1} \cdot y\Big) \text{ for } x \in K_\beta,\; y \in \mathbb{R}\setminus\{N^{-1}\cdot\infty\}.$$

See Figure 9.1 for a portion of the orbit of a point under a particular $\mathcal{T}_{3,n,\alpha}$.

9.2. Planar extensions of the interval maps

In upcoming work we prove results of the following form. See Figure 9.2; a similar result holds for larger values of α as well, see Figure 9.3 for a hint of this.

Theorem 9.1. *Fix $n \geq m = 3$, $k \in \mathbb{N}$ and $v \in \mathcal{V}$. Let α be in the interior of $J_{k,v}$. Then*

1. *$T_{3,n,\alpha}(x, y)$ is ergodic;*
2. *there is a union of finitely many rectangles, $\Omega_{3,n,\alpha}$, upon which the map $\mathcal{T}_{3,n,\alpha}(x, y)$ is bijective, up to μ-measure zero;*
3. *the set of heights of these rectangles is constant along the interior of $J_{k,v}$, while the widths are determined by the T_α-orbits of the various $r_0(\alpha)$ and $\ell_0(\alpha)$ and thus vary continuously with α;*
4. *the μ-measure of $\Omega_{3,n,\alpha}$ varies continuously with α;*
5. *the entropy $h(\,T_{3,n,\alpha}\,)$ varies continuously with α.*

9.3. An example

We sketch the above result in the setting of the synchronization interval containing $\alpha = 0.14$ when $m = n = 3$, see Example 1.2, as well as Figures 1.1, 9.1 and 9.2. One verifies that $\mathcal{T}_{3,3,\alpha}$ is bijective modulo zero on the planar region, $\Omega = \Omega_\alpha$, represented by (say the left hand side of) Figure 9.2. Therefore, $d\mu = (1 + xy)^{-2} dxdy$ is an invariant measure for this map here, and one verifies that the μ-measure of Ω is finite, which we can then normalize to be a probability measure. The resulting probability measure on $\mathbb{I}_\alpha$, the marginal measure which is simply the push-forward under the vertical projection, hence obtained by “integrating along fibers", is invariant for T. Call this measure ν. We then have that ν is equivalent to Lebesgue measure, and also is bounded above and below with respect to it.

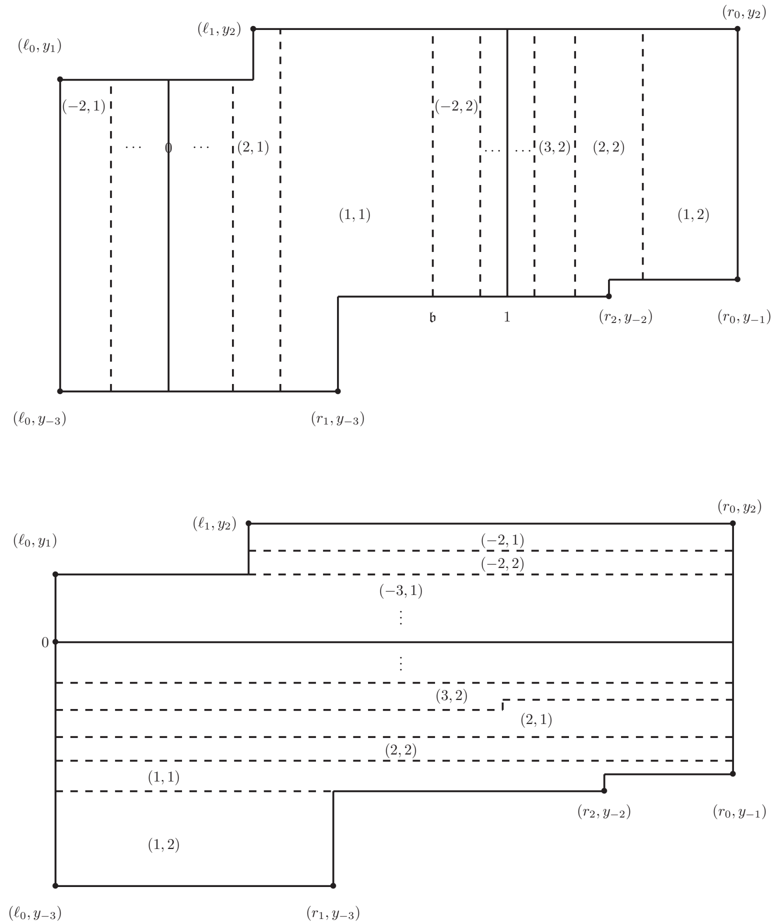


Figure 9.3. Planar extension of the one-dimensional system indexed by $m = n = 3, \alpha = 0.86$, given as domain and image. Labelling similar to that of Figure 9.2. Here α is an interior point of $J_{-2,1}$.

9.3.1. *Ergodicity of T_α*

We now sketch how to adjust Luzzi-Marmi's proof in [15] showing that the Nakada α-continued fraction maps have an exact (and hence ergodic) invariant measure, so as to show that ν has this property for our T. Recall that they prove that the full cylinders generate the Borel sets, and then verify a form of bounded distortion for the invariant measure, and then recall that together these are sufficient for a result of Rohlin to give exactness.

Luzzi and Marmi first show that the α-continued fraction maps have bounded distortion with respect to Lebesgue measure on full cylinders. Unlike the Nakada maps, our T itself is not expanding, so we invoke reasoning of Adler, given in his afterword to Bowen's [3]. This reasoning shows that it is sufficient for T to be eventually expansive to obtain a bound for all n on $\log |(T^n)'(x)/(T^n)'(y)|$ whenever x, y are in any fixed cylinder of rank n; bounded distortion (with respect to Lebesgue measure) of full cylinders then follows, as in [15] (see their subsection 5.1). Since ν is bounded above and below with respect to Lebesgue measure, bounded distortion of the type necessary for the Rohlin result holds.

It only remains to show that the full cylinders for T generate; here again the approach of [15] must be adjusted for our setting. Let E_n be the collection of non-full cylinders of rank n such that T^j sends the cylinder to a non-full cylinder for every $1 \leq j \leq n$. We wish to show that the Lebesgue measure of E_n goes to zero with n. The only non-full cylinders of rank one are $\Delta_\alpha(-1, 1)$ and $\Delta_\alpha(1, 1)$. The T-image of each of these does not meet itself. Hence, each cylinder lying in E_n is indexed by a sequence of length n of simplified digits which are alternatingly 1 and -1. There are thus two such cylinders. Applying T^2 to each of these cylinders consists of applying $ACA^{-1}C$ and $A^{-1}CAC$, respectively. In the first case, the cylinder lies to the right of $(ACA^{-1}C)^{-1} \cdot \ell_0$ and calculation shows that the derivative of T^2 at this endpoint is greater than 32.7; similarly, in the second case T^2 has derivative at least its value at $(A^{-1}CAC)^{-1} \cdot r_0$, which calculation shows is greater than 23.4. It follows that the measure of the various E_{2m} decreases to zero. Since each cylinder of E_{2m+1} lies in a cylinder of E_{2m}, the measure of E_n goes to zero as $n \to \infty$, and hence the full cylinders do indeed generate. Therefore, T is ergodic with respect to ν.

9.3.2. *Quilting shows ergodicity for α' in the interior of $J_{1,1}$*

The ergodicity of T with respect to ν implies that of $\mathcal{T}_\alpha = \mathcal{T}_{3,3,\alpha}$ with respect to μ normalized to be a probability measure on Ω_α. With our $\alpha = 0.14$ when $m = n = 3$, we have $\alpha \in J_{1,1}$. One can show that a variant of *quilting* as used in [13] succeeds for relating planar extensions Ω_α and $\Omega_{\alpha'}$ for α, α' both in the interior of some $J_{k,v}$. Recall that quilting is a matter of beginning with say Ω_α and solving for $\Omega_{\alpha'}$ (on which $\mathcal{T}_{\alpha'}$ is bijective up to μ-measure zero). This is done by deleting all forward $\mathcal{T}_\alpha$-images of that part of Ω_α fibering over $\mathbb{I}_\alpha \setminus \mathbb{I}_{\alpha'}$, while adding in an appropriate translate of this initial deleted region, as well all forward images under $\mathcal{T}_{\alpha'}$. Synchronization makes this an essentially finite procedure. In particular, any set of positive measure contained in $\Omega_{\alpha'}$ is partitioned into a piece upon which $\mathcal{T}_{\alpha'}$ agrees with $\mathcal{T}_\alpha$ and a piece that $\mathcal{T}_{\alpha'}$ eventually brings to the set where these two maps agree. Ergodicity of $\mathcal{T}_\alpha$ thus implies that of $\mathcal{T}_{\alpha'}$.

9.3.3. *Continuity of entropy*

The quilting also shows that $\Omega_{\alpha'}$ is the union of rectangles in the manner announced in Theorem 9.1. For clarity's sake, we denote the invariant measure for T_α by ν_α. Using Rohlin's entropy formula, we have

$$h(T_\alpha) = \int_{\mathbb{I}_\alpha} \log \mid T_\alpha'(x) \mid d\nu_\alpha = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} -2 \log \mid x \mid \, d\mu.$$

From the above, we see that the entropy, just as $\mu(\Omega_\alpha)$, varies continuously with α in the interior of $J_{1,1}$. When $\alpha = 0.14$, numeric integration over the explicit region as given in Figure 9.2 gives approximate values of $\mu(\Omega_{3,3,0.14}) \sim 3.30023$ and $h(T_{3,3,0.14}) \sim 1.99372$. The product of these two values should be compared with the volume of the unit tangent bundle of the hyperbolic surface (orbifold) $G_{3,3}\backslash\mathbb{H}$, which is $2\pi^2/3 \sim 6.57974$. Indeed, similar calculation indicates that $\mu(\Omega_{3,3,\alpha})h(T_{3,3,\alpha})$ equals this volume for all α in the interior of $J_{1,1}$. Of course, as in [14] or [11], one can expect that an application of Abramov's formula will show that $h(T_{3,3,\alpha})\mu(\Omega_{3,3,\alpha}) = 2\pi^2/3$ for any $\alpha \in J_{k,v}$ for any k, v. Furthermore, as in [1], one can expect that each of these maps is a factor of a Poincaré section for the geodesic flow on the unit tangent bundle of $G_{3,3}\backslash\mathbb{H}$. We discuss such matters in appropriate generality in our upcoming work.

References

[1] P. ARNOUX and T. A. SCHMIDT, *Cross sections for geodesic flows and α-continued fractions*, Nonlinearity **26** (2013), 711–726.

[2] C. BONANNO, C. CARMINATI, S. ISOLA and G. TIOZZO, *Dynamics of continued fractions and kneading sequences of unimodal maps*, Discrete Contin. Dyn. Syst. **33** (2013), 1313–1332.

[3] R. BOWEN, *Invariant measures for Markov maps of the interval*, With an afterword by Roy L. Adler and additional comments by Caroline Series, Comm. Math. Phys. **69** (1979), 1–17.

[4] K. CALTA and T. A. SCHMIDT, *Continued fractions for a class of triangle groups*, J. Austral. Math. Soc. **93** (2012) 21–42.

[5] C. CARMINATI and G. TIOZZO, *A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations*, Ergodic Theory Dynam. Systems **32** (2012), 1249–1269.

[6] C. CARMINATI and G. TIOZZO, *Tuning and plateaux for the entropy of α-continued fractions*, Nonlinearity **26** (2013), 1049–1070.

[7] C. CARMINATI, S. ISOLA and G. TIOZZO, *Continued fractions with $SL(2,\mathbb{Z})$-branches: combinatorics and entropy*, Trans. Amer. Math. Soc. **370** (2018), 4927–4973.

[8] K. DAJANI, C. KRAAIKAMP and W. STEINER, *Metrical theory for α-Rosen fractions*, J. Eur. Math. Soc. (JEMS) **11** (2009), 1259–1283.

[9] S. KATOK and I. UGARCOVICI, *Theory of (a,b)-continued fraction transformations and applications*, Electron. Res. Announc. Math. Sci. **17** (2010), 20–33.

[10] S. KATOK and I. UGARCOVICI, *Structure of attractors for (a,b)-continued fraction transformations*, J. Mod. Dyn. **4** (2010), 637–691.

[11] S. KATOK and I. UGARCOVICI, *Applications of (a,b)-continued fraction transformations* Ergodic Theory Dynam. Systems **32** (2012), 755–777.

[12] C. KRAAIKAMP, *A new class of continued fraction expansions*, Acta Arith. **57** (1991), 1–39.

[13] C. KRAAIKAMP, T. A. SCHMIDT and I. SMEETS, *Natural extensions for α-Rosen continued fractions*, J. Math. Soc. Japan **62** (2010), 649–671.

[14] C. KRAAIKAMP, T. A. SCHMIDT and W. STEINER, *Natural extensions and entropy of α-continued fractions*, Nonlinearity **25** (2012), 2207–2243.

[15] L. LUZZI and S. MARMI, *On the entropy of Japanese continued fractions*, Discrete Contin. Dyn. Syst. **20** (2008), 673–711.

[16] P. MOUSSA, A. CASSA and S. MARMI, *Continued fractions and Bruno functions*, J. Comput. Appl. Math. **105** (1995), 403–415.

[17] H. NAKADA, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426.

[18] H. NAKADA and R. NATSUI, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity **21** (2008), 1207–1225.
[19] G. TIOZZO, "Entropy, Dimension and Combinatorial Moduli for One-dimensional Dynamical Systems", Ph.D. Thesis, Harvard University, Ann Arbor, MI, 2013.

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