

## Hölder continuity of $\omega$ -minimizers of functionals with generalized Orlicz growth

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**Abstract.** We show local Hölder continuity of quasiminimizers of functionals with non-standard (Musielak–Orlicz) growth. Compared with previous results, we cover more general minimizing functionals and need fewer assumptions. We prove Harnack’s inequality and a Morrey type estimate for quasiminimizers. Combining this with Ekeland’s variational principle, we obtain local Hölder continuity for  $\omega$ -minimizers.

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### 1. Introduction

Generalized Orlicz spaces have recently attracted increasing intensity (*cf.* Section 3). The results have also been applied to the study of differential equations with non-standard growth (*e.g.*, [16,37,40,41,45]). In [43], the first two authors and Toivanen gave the first proof of Harnack’s inequality for solutions under generalized Orlicz growth. We start this paper giving a more sophisticated proof of this inequality, with better dependence of the constants on the structure of the equation. In contrast to the earlier result, this improved Harnack inequality can be applied to prove the Hölder continuity of  $\omega$ -minimizers, which is the second part of this paper.

In the fields of partial differential equations and the calculus of variations, there has been much research on non-standard growth problems (*e.g.*, [1,2,12,52,53]), such as the non-autonomous minimization problem

$$\min_{v \in W^{1,1}} \int_{\Omega} F(x, \nabla v) \, dx,$$

where  $F$  satisfies  $(p, q)$ -growth conditions, that is,  $|\xi|^p - 1 \lesssim F(x, \xi) \lesssim |\xi|^q + 1$ ,  $q > p$ . Zhikov [69,70] considered special cases as models of anisotropic materials

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and the so-called Lavrentiev phenomenon. In [70], he proposed model problems including

$$F(x, \xi) \approx |\xi|^{p(x)}, \quad 1 < \inf p \leq \sup p < \infty,$$

and

$$F(x, \xi) \approx |\xi|^p + a(x)|\xi|^q, \quad 1 < p \leq q < \infty, \quad a \geq 0. \quad (1.1)$$

For the first, so-called *variable exponent* case, the exponent of  $|\xi|$  is a function of the  $x$ -variable which is usually assumed to be log-Hölder continuous, and it describes various phenomena, for example electrorheological fluids [65] and image restoration [17,42], with growth continuously changing with respect to the position. The second, so-called *double phase* case describes for instance composite materials or mixtures. Here, a discontinuous phase transition occurs on the border between constituent materials. In a series of papers, Baroni, Colombo and Mingione [7, 9, 19–21] have studied regularity properties of minimizers of these problems, see also [11, 13, 30–32, 60, 68]. Cupini, Passarelli di Napoli and co-authors [18, 24] have considered the variant of the double phase functional

$$F(x, \xi) = (|\xi| - 1)_+^p + a(x)(|\xi| - 1)_+^q \quad (1.2)$$

with  $(s)_+ := \max\{s, 0\}$ , which is degenerate for small positive values of the gradient (see also [39, Section 7.2] on how this functional fits into the generalized Orlicz framework). Furthermore, minimizers of borderline functionals like

$$F(x, \xi) = |\xi|^{p(x)} \log(e + |\xi|) \quad \text{and} \quad F(x, \xi) = |\xi|^p + a(x)|\xi|^p \log(e + |\xi|)$$

have been recently studied, see for instance [8, 11, 33, 57–59]. We stress that *all of these special cases are covered by the results in this paper* (cf. [39, Section 7.2]). In many cases the results of this paper are new even in the special cases.

The notion of an  $\omega$ -minimizer, sometimes called *almost minimizer*, was introduced by Anzellotti [5], and an analogous notion was originally given by Almgren [4] in the context of geometric measure theory. It was motivated by the fact that minimizers of constrained problems can turn out to be  $\omega$ -minimizers of unconstrained problems. For instance, minimizers of energy functionals with volume constraints or obstacles are  $\omega$ -minimizers, where the function  $\omega$  is determined by the properties of the constraint [5, 26]. In this regard, the notion of an  $\omega$ -minimizer is useful and has been widely studied in the calculus of variations.

Regularity theory for minimizers has been extended to  $\omega$ -minimizers under suitable decay conditions on the function  $\omega$  in for instance [5, 26, 36, 48], see also [54] for a survey. In particular, Hölder continuity of  $\omega$ -minimizers was established by Dolcini–Esposito–Fusco [27] in the standard  $p$ -growth case and later by Esposito–Mingione [28] in more general cases. Recently, it was also proved in double phase and Orlicz growth cases by Ok [60].<sup>1</sup>

<sup>1</sup> This paper contains some problems in the proofs. With the assistance of Jihoon Ok, we have managed to circumvent these problems.

We prove an extension of these results to the generalized Orlicz growth case. Our energy functional is given by

$$\mathcal{F}(u, \Omega) := \int_{\Omega} F(x, u, \nabla u) \, dx, \tag{1.3}$$

where  $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies

$$\nu \varphi(x, |z|) \leq F(x, t, z) \leq N (\varphi(x, |z|) + \Lambda) \tag{1.4}$$

for some  $0 < \nu \leq N$  and  $\Lambda \geq 0$ . The exact definitions of the conditions in the following result are given in the next section; roughly, (A0) restricts us to unweighted situations, (A1) and (A1- $n$ ) are subtle continuity conditions and (aInc) and (aDec) $^\infty$  exclude  $L^1$ - and  $L^\infty$ -type behavior, respectively.

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $\varphi \in \Phi_w(\Omega)$  satisfy (A0), (aInc) and (aDec) $^\infty$ . Let  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  be an  $\omega$ -minimizer of  $\mathcal{F}$  and  $(t, z) \rightarrow F(x, t, z)$  be continuous. Assume that  $\varphi$  satisfies (A1), or that  $u$  is bounded and  $\varphi$  satisfies (A1- $n$ ). Then  $u$  is locally Hölder continuous.*

The proof of this result is based on the variational technique described in [27, 35]. The key idea is to find a quasiminimizer  $w \in u + W_0^{1,\varphi}(Q_r)$  of the functional

$$\int_{Q_r} \varphi(x, |\nabla w|) + \Lambda \, dx,$$

which is comparable to our original  $\omega$ -minimizer  $u$  of  $\mathcal{F}$ , by applying Ekeland’s variational principle with estimates depending on the constant  $\Lambda$ . From Harnack’s inequality (Theorem 4.1), it can be proved that the gradient of the quasiminimizer  $w$  satisfies Morrey-type decay estimates (Section 7). A challenge compared to the classical case is that the constant in Harnack’s inequality depends on  $\Lambda_u$  and hence on  $u$ . However, we show that the natural bound  $\Lambda_u \leq |Q_r|^{-1}$  is sufficient to control the constant. Therefore, using the Morrey-type decay estimates, we can derive similar decay estimates of  $\nabla u$ , which implies Hölder continuity of  $u$  (Section 8). A further challenge worth mentioning is that moving between  $\omega$ -minimizers of  $\varphi$  and  $\varphi + \Lambda$  is not possible, so for this case we need to work directly with the condition (aDec) $^\infty$ .

For the case (A1- $n$ ) (with  $u$  bounded) we need to consider an alternative notion of minimizer called *weak quasiminimizer* (cf. Definition 4.2), since we cannot otherwise guarantee boundedness of the quasiminimizer  $w$  discovered by the Ekeland variational principle. This technique is adapted from [60].

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### 2. Generalized $\Phi$ -functions

By  $\Omega \subset \mathbb{R}^n$  we denote a bounded domain, *i.e.*, an open and connected set. By  $p' := \frac{p}{p-1}$  we denote the Hölder conjugate exponent of  $p \in [1, \infty]$ . The notation  $f \lesssim g$  means that there exists a constant  $C > 0$  such that  $f \leq Cg$ . The notation  $f \approx g$  means that  $f \lesssim g \lesssim f$  whereas  $f \simeq g$  means that  $f(t/C) \leq g(t) \leq f(Ct)$  for some constant  $C \geq 1$ . By  $c$  we denote a generic constant whose value may change between appearances. A function  $f$  is *almost increasing* if there exists  $L \geq 1$  such that  $f(s) \leq Lf(t)$  for all  $s \leq t$  (more precisely,  $L$ -almost increasing). *Almost decreasing* is defined analogously. By *increasing* we mean that the inequality holds for  $L = 1$  (some call this non-decreasing), similarly for *decreasing*.

**Definition 2.1.** We say that  $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$  is a  $\Phi$ -prefunction if the following hold:

- (i) For every  $t \in [0, \infty)$  the function  $x \mapsto \varphi(x, t)$  is measurable;
- (ii) For every  $x \in \Omega$  the function  $t \mapsto \varphi(x, t)$  is increasing;
- (iii)  $\lim_{t \rightarrow 0^+} \varphi(x, t) = \varphi(x, 0) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$  for every  $x \in \Omega$ .

A  $\Phi$ -prefunction is a *weak  $\Phi$ -function*, denoted by  $\varphi \in \Phi_w(\Omega)$ , if the following hold:

- (iv) The function  $t \mapsto \frac{\varphi(x,t)}{t}$  is  $L$ -almost increasing in  $(0, \infty)$  for every  $x \in \Omega$ ;
- (v) The function  $t \mapsto \varphi(x, t)$  is left-continuous for every  $x \in \Omega$ .

Since our weak  $\Phi$ -functions are not bijections, they are not strictly speaking invertible. However, by  $\varphi^{-1}(x, \cdot) : [0, \infty) \rightarrow [0, \infty]$  we denote the *left-inverse* of  $\varphi$ :

$$\varphi^{-1}(x, \tau) := \inf\{t \geq 0 : \varphi(x, t) \geq \tau\}.$$

If  $\varphi$  is strictly increasing, then this is just the normal inverse function, but that is not a convenient assumption for us. Let  $\varphi \in \Phi_w(\Omega)$ . We say that  $\varphi$  satisfies

- (A0) If there exists  $\beta \in (0, 1]$  such that  $\beta \leq \varphi^{-1}(x, 1) \leq \frac{1}{\beta}$  for a.e.  $x \in \Omega$ , or equivalently there exists  $\beta \in (0, 1]$  such that  $\varphi(x, \beta) \leq 1 \leq \varphi(x, \frac{1}{\beta})$  for a.e.  $x \in \Omega$  (see [39, Corollary 3.7.4]);
- (A1) If there exists  $\beta \in (0, 1)$  such that, for every ball  $B$  and a.e.  $x, y \in B \cap \Omega$ ,

$$\beta\varphi^{-1}(x, t) \leq \varphi^{-1}(y, t) \quad \text{when } t \in \left[1, \frac{1}{|\beta|}\right];$$

- (A1- $n$ ) If there exists  $\beta \in (0, 1)$  such that, for every ball  $B$  and a.e.  $x, y \in B \cap \Omega$ ,

$$\varphi(x, \beta t) \leq \varphi(y, t) \quad \text{when } t \in \left[1, \frac{1}{|\beta|^{1/n}}\right];$$

- (aInc) $_p$  If  $t \mapsto \frac{\varphi(x,t)}{t^p}$  is  $L$ -almost increasing in  $(0, \infty)$  for some  $L \geq 1$  and a.e.  $x \in \Omega$ ;
- (aDec) $_q$  If  $t \mapsto \frac{\varphi(x,t)}{t^q}$  is  $L$ -almost decreasing in  $(0, \infty)$  for some  $L \geq 1$  and a.e.  $x \in \Omega$ ;
- (aDec) $_q^\infty$  If  $t \mapsto \frac{\varphi(x,t)+1}{t^q}$  is  $L$ -almost decreasing in  $(0, \infty)$  for some  $L \geq 1$  and a.e.  $x \in \Omega$ .

Moreover we say that  $\varphi$  satisfies (aInc), (aDec) or (aDec) $^\infty$  if it satisfies (aInc) $_p$ , (aDec) $_q$  or (aDec) $_q^\infty$ , respectively, for some  $p > 1$  or  $q < \infty$ . The condition (aDec) $_q^\infty$  intuitively means that  $t \mapsto \frac{\varphi(x,t)}{t^q}$  is almost increasing for  $t > T$  for some constant  $T > 0$ .

If  $\varphi$  satisfies (aDec), then

$$\varphi^{-1}(x, \varphi(x, t)) \approx \varphi(x, \varphi^{-1}(x, t)) \approx t. \tag{2.1}$$

The growth of the inverse is closely tied to that of the function:  $\varphi$  satisfies (aInc) $_p$  or (aDec) $_q$  if and only if  $\varphi^{-1}$  satisfies (aDec) $_{\frac{1}{p}}$  or (aInc) $_{\frac{1}{q}}$ , respectively. For the proofs of these facts, see [39, Section 2.3].

By [39, Proposition 4.1.5], (A1) implies that there exists  $\beta \in (0, 1)$  such that

$$\varphi(x, \beta t) \leq \varphi(y, t) \quad \text{when} \quad \varphi(y, t) \in \left[1, \frac{1}{|B|}\right]$$

for almost every  $x, y \in B \cap \Omega$  and every ball  $B$  with  $|B| \leq 1$ . Furthermore, if  $\varphi \in \Phi_w$ , then  $\varphi(\cdot, 1) \approx 1$  implies (A0), and if  $\varphi$  satisfies (aDec), then (A0) and  $\varphi(\cdot, 1) \approx 1$  are equivalent. In addition, when (aDec) holds we can multiply by constants in the range:  $[a, \frac{1}{|B|}]$ ,  $a, b > 0$ .

The next lemma shows how we can use a trick to upgrade (aDec) $^\infty$  to (aDec) while preserving many other properties.

**Lemma 2.2.** *Let  $\varphi \in \Phi_w(\Omega)$  and define  $\psi(x, t) := \varphi(x, t) + t$ . Then  $\psi \in \Phi_w(\Omega)$ . Moreover,*

- (a) *If  $\varphi$  satisfies (A0), then  $\varphi \leq \psi \lesssim \varphi + 1$  and  $\psi$  satisfies (A0);*
- (b) *If  $\varphi$  satisfies (aDec) $_q^\infty$  and (A0), then  $\psi$  satisfies (aDec) $_q$ ;*
- (c) *If  $\varphi$  satisfies (A1), then  $\psi$  satisfies (A1);*
- (d) *If  $\varphi$  satisfies (A1-n), then  $\psi$  satisfies (A1-n).*

*Proof.* Checking the properties in Definition 2.1, we find that  $\psi \in \Phi_w(\Omega)$ .

(a) The inequality  $\varphi \leq \psi$  is immediate. Let  $\varphi$  satisfy (A0) and assume first that  $t > \frac{1}{\beta}$ . Then we obtain by (A0) and (aInc) $_1$  that

$$\psi(x, t) = \varphi(x, t) + t \leq \varphi(x, t) + \varphi(x, \frac{1}{\beta})t \leq \varphi(x, t) + \frac{t}{\beta}\varphi(x, t) = \left(1 + \frac{t}{\beta}\right)\varphi(x, t).$$

If  $t \leq \frac{1}{\beta}$ , then  $\psi(x, t) \leq \varphi(x, t) + \frac{1}{\beta} \leq (1 + \frac{1}{\beta})(\varphi(x, t) + 1)$ . From the inequalities it follows that  $\psi(x, \beta) \lesssim \varphi(x, \beta) + 1 \leq 2$  and  $\psi(x, \frac{1}{\beta}) \geq 1$ , and hence (A0) follows.

(b) Let us then assume that  $\varphi$  satisfies  $(\text{aDec})_q^\infty$  and (A0). If  $t > s \geq \beta$ , then by  $(\text{aDec})_q$

$$\frac{\psi(x, t)}{t^q} \lesssim \frac{\varphi(x, t) + 1}{t^q} \lesssim \frac{\varphi(x, s) + 1}{s^q} \lesssim \frac{\psi(x, s)}{s^q}.$$

Let then  $0 < t \leq \beta$ . By  $(\text{aInc})_1$  and (A0),  $\psi(x, t) \approx t$ , so  $(\text{aDec})_q$  is clear in this range. The case  $s \leq \beta \leq t$  follows by combining the previous cases.

(c) From the definition of left-inverse we directly see that  $\psi^{-1}(x, t) \approx \min\{\varphi^{-1}(x, t), t\}$ . Thus we obtain by (A1) of  $\varphi$  for  $t \in [1, \frac{1}{|B|}]$  that

$$\psi^{-1}(x, t) \approx \min\{\varphi^{-1}(x, t), t\} \lesssim \min\{\varphi^{-1}(y, t), t\} \approx \psi^{-1}(y, t).$$

(d) Let  $t \in [1, \frac{1}{|B|^{1/n}}]$ . By (A1- $n$ ) of  $\varphi$  we obtain

$$\psi(x, \beta t) = \varphi(x, \beta t) + \beta t \leq \varphi(y, t) + t = \psi(y, t). \quad \square$$

The Krylov–Safonov lemma used in the proof of Harnack’s inequality works only for cubes, whereas (A1) and (A1- $n$ )-conditions have been defined with balls. However, a given cube  $Q$  can be covered by a finite number, depending only on  $n$ , of balls  $B_i$  with  $|B_i| = |Q|$ , and so the (A1) or (A1- $n$ ) inequalities can be obtained in  $Q$  by considering a chain of balls.

When  $\varphi_B^-(t) \leq \frac{1}{|B|}$  we will often use that (A0), (A1) and (aDec) imply

$$\varphi_B^+(t) \lesssim \varphi_B^-(t) + 1.$$

Let us here give the details. If  $\varphi_B^-(t) \in [1, \frac{1}{|B|}]$ , then the inequality holds (without the +1) by (A1) and (aDec). If  $\varphi_B^-(t) \leq 1$ , we instead use  $\varphi_B^+(t) \leq \varphi_B^+(1) \lesssim 1$  by (A0) and (aDec). Using same arguments we obtain the corresponding estimate for (A1- $n$ ).

### 3. Generalized Orlicz spaces

Generalized Orlicz spaces, also called Musielak–Orlicz spaces, have been actively studied over a long time. The basic example of a generalized Orlicz space was introduced by Orlicz [61] in 1931, and a major synthesis is due to Musielak [55] in 1983. Recent monographs on generalized Orlicz spaces are due to Yang, Liang and Ky [66], Lang and Mendez [49], and the first two authors [39] focusing on Hardy-type spaces, functional analysis, and harmonic analysis, respectively; see also the survey article [15]. Generalized Orlicz spaces include as a special case classical Orlicz spaces that are well-known and have been extensively studied, see, *e.g.*, the monograph [64] and references therein.

From this observation, we can roughly understand generalized Orlicz spaces as variable versions of Orlicz spaces with respect to the space variable  $x$ . The special

case of variable exponent spaces  $L^{p(\cdot)}$  has been studied intensively over the last 20 years [22, 25, 62]. The reason that variable exponent research thrived while little harmonic analysis was done in generalized Orlicz spaces was the belief that many classical results can be obtained in the former setting but not the latter. A spate of recent articles (e.g. [3, 14, 23, 38, 44, 47, 50, 51, 56, 63, 67]) has proved this belief to be unfounded.

Throughout the paper we write  $\varphi_B^+(t) := \sup_{x \in B} \varphi(x, t)$  and  $\varphi_B^-(t) := \inf_{x \in B} \varphi(x, t)$  and abbreviate  $\varphi^\pm := \varphi_\Omega^\pm$ . Especially  $\varphi_B^-$  will be used countless times, since it enables us to apply the following Jensen-type inequalities. The function  $\varphi_B^-$  need not to be left-continuous, see [39, Example 4.3.3] and hence it is not necessary a weak  $\Phi$ -function. However since it satisfies (aInc) it is equivalent with a convex  $\Phi$ -function (independent of  $x$ ) by [39, Lemma 2.2.1]. This is used in the next lemma, where  $\varphi$  is independent of  $x$ , e.g.,  $\varphi_B^-$ . We denote by  $L^0(\Omega)$  the set of measurable functions in  $\Omega$ . By  $f_D$  and  $f_D f dx$  we denote the integral average of  $f$  over  $D$ .

**Lemma 3.1.** *Let  $\varphi$  be a  $\Phi$ -prefunction which satisfies (aInc) $_p$  and (aDec) $_q$ ,  $D \subset \mathbb{R}^n$  be measurable with  $|D| \in (0, \infty)$  and  $f \in L^0(D)$ . Then*

$$\left( \int_D |f|^p dx \right)^{\frac{1}{p}} \lesssim \varphi^{-1} \left( \int_D \varphi(|f|) dx \right) \lesssim \left( \int_D |f|^q dx \right)^{\frac{1}{q}}.$$

*Proof.* Let  $\psi(t) := \varphi(t^{1/p})$ . Then  $\psi$  satisfies (aInc) $_1$  and so there exists a convex  $\xi \in \Phi_w$  with  $\psi \simeq \xi$  with constant  $\beta$  [39, Lemma 2.2.1]. Since  $\xi$  is convex, Jensen’s inequality implies that

$$\varphi \left( \beta^{\frac{2}{p}} \left( \int_D |f|^p dx \right)^{\frac{1}{p}} \right) \leq \xi \left( \beta \int_D |f|^p dx \right) \leq \int_D \xi(\beta |f|^p) dx \leq \int_D \varphi(|f|) dx.$$

Note that this inequality does not require (aDec). The first inequality of the claim follows from this by (2.1).

We know that  $\varphi^{-1}$  is increasing [39, Lemma 2.3.9] and thus so is  $(\varphi^{-1})^q$ . Since  $\varphi$  satisfies (aDec) $_q$ ,  $\varphi^{-1}$  satisfies (aInc) $_{1/q}$  by [39, Proposition 2.3.7] and  $(\varphi^{-1})^q$  satisfies (aInc) $_1$ . Thus  $(\varphi^{-1})^q$  is a  $\Phi$ -prefunction. Hence by [39, Lemma 2.2.1] there exists a convex  $\xi \in \Phi_w$  such that  $\xi \simeq (\varphi^{-1})^q$ . We obtain by Jensen’s inequality

$$\begin{aligned} \varphi^{-1} \left( \int_D \varphi(|f|) dx \right) &\approx \xi \left( \int_D \varphi(|f|) dx \right)^{\frac{1}{q}} \leq \left( \int_D \xi(\varphi(|f|)) dx \right)^{\frac{1}{q}} \\ &\approx \left( \int_D |f|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad \square$$

Let  $\varphi \in \Phi_w(\Omega)$ . The *generalized Orlicz space* (also known as the Musielak–Orlicz space) is defined as the set

$$L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \lim_{\lambda \rightarrow 0^+} \varrho_\varphi(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^\varphi(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_\varphi \left( \frac{f}{\lambda} \right) \leq 1 \right\},$$

where  $\varrho_\varphi(f)$  is the modular of  $f \in L^0(\Omega)$  defined by

$$\varrho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|) dx.$$

In many places, we make the following set of assumptions. However, this will be explicitly specified, as some results work also under fewer assumptions. Furthermore, all constants in our estimates depend only on the parameters in the assumptions and the dimension  $n$ , unless something else is explicitly states. Specifically, these parameters are the constants  $\beta$  and  $L$ , the exponents  $p$  and  $q$ , the minimizing parameters  $Q$  and  $\omega$  (Definition 4.2) and the structure constants  $\nu$  and  $N$  (from (1.4)). However, the dependence on  $\Lambda$  and the size of the cube  $r$  will be made explicit, since we will need the cases  $\Lambda \rightarrow \infty$  and  $r \rightarrow 0$ .

**Assumption 3.2.** The function  $\varphi \in \Phi_w(Q_r)$  satisfies  $(aInc)_p$ ,  $(aDec)_q$ , (A0) and one of the following holds for the function  $u \in W^{1,\varphi}(Q_r)$

- (1)  $\varphi$  satisfies (A1) and  $\varrho_\varphi(\nabla u) \leq 1$ ;
- (2)  $\varphi$  satisfies (A1- $n$ ) and  $u \in L^\infty(\Omega)$ .

In the second case of the assumption, constants depend also on  $\|u\|_\infty$ . Note that the assumptions could be more symmetrical by assuming  $\varrho_\varphi(\nabla u) < \infty$  in (1), in which case the constants would depend on  $\varrho_\varphi(\nabla u)$ , or, alternatively,  $\|u\|_\infty \leq 1$  in (2). However, it seems that the current versions are more natural to use.

A function  $u \in L^\varphi(\Omega)$  belongs to the Orlicz–Sobolev space  $W^{1,\varphi}(\Omega)$  if its weak partial derivatives  $\partial_1 u, \dots, \partial_n u$  exist and belong to  $L^\varphi(\Omega)$ . Furthermore,  $W_0^{1,\varphi}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  in  $W^{1,\varphi}(\Omega)$ .

We will need the Sobolev–Poincaré inequality numerous times in this article, with either zero boundary values or with average zero. For the calculus of variations, inequalities in modular form, with an error term, are more useful than inequalities concerning norms (such as the ones in [38]). Furthermore, it is useful to have a constant exponent improvement  $s > 1$  in the integrability regardless of growth. Note that the exponent  $s$  can be on the right-hand side or on the left-hand side, see [39, Proposition 6.3.12 and Corollary 6.3.15]. In this paper we need the following versions.

**Theorem 3.3 (Sobolev–Poincaré inequality).** *Let  $B_r \subset \mathbb{R}^n$  be a ball or a cube with diameter  $2r$ . Let  $\varphi \in \Phi_w(B_r)$  satisfy Assumption 3.2. For  $1 \leq s < \frac{n}{n-1}$ ,*

$$\left( \int_{B_r} \varphi \left( x, \frac{|u|}{r} \right)^s dx \right)^{\frac{1}{s}} \lesssim \int_{B_r} \varphi(x, |\nabla u|) dx + \frac{|\{\nabla u \neq 0\} \cap B_r|}{|B_r|} \tag{3.1}$$



for any  $u \in W_0^{1,1}(B_r)$ . If additionally  $1 \leq s \leq p$ , then

$$\int_{B_r} \varphi\left(x, \frac{|u - u_{B_r}|}{r}\right) dx \lesssim \left(\int_{B_r} \varphi(x, |\nabla u|)^{\frac{1}{s}} dx\right)^s + 1 \tag{3.2}$$

for any  $u \in W^{1,1}(B_r)$ ; in the case (A1), we need that  $\|\nabla u\|_{\varphi^{1/s}} \leq M$ , and the implicit constant depends on  $M$ . The average  $u_{B_r}$  can be replaced by  $u_B$  for some ball or cube  $B \subset B_r$  with  $|B| > \mu|B_r|$ , in which case the constant depends also on  $\mu$ .

The case (A1) is covered by the Sobolev–Poincaré inequality in [39, Proposition 6.3.12 and Corollary 6.3.15], whereas the case of (A1- $n$ ) is new.

*Proof.* We consider only bounded  $u$  and (A1- $n$ ). By [39, Lemma 2.2.1] there exists a convex  $\xi \in \Phi_w$  such that  $\xi \simeq \varphi^-$ . We apply (3.2) to  $\xi$ , which satisfies (A1) since it is independent of  $x$ :

$$\begin{aligned} \int_{B_r} \varphi^-\left(\frac{|u - u_{B_r}|}{r}\right) dx &\lesssim \int_{B_r} \xi\left(\frac{|u - u_{B_r}|}{r}\right) dx \\ &\lesssim \left(\int_{B_r} \xi(|\nabla u|)^{\frac{1}{s}} dx\right)^s \lesssim \left(\int_{B_r} \varphi^- (|\nabla u|)^{\frac{1}{s}} dx\right)^s \\ &\leq \left(\int_{B_r} \varphi(x, |\nabla u|)^{\frac{1}{s}} dx\right)^s. \end{aligned}$$

Furthermore, in this case the inequality  $\|\nabla u\|_{\varphi} < 1$  is not needed, since (A1) holds not only in  $[1, \frac{1}{|B|}]$  but in  $[0, \infty)$ .

On the left-hand side we use (A1- $n$ ), (A0) and (aDec) to estimate

$$\varphi\left(x, \frac{|u - u_{B_r}|}{r}\right) \lesssim \varphi^-\left(\beta \frac{|u - u_{B_r}|}{r}\right) + 1,$$

which concludes the proof in this case. Note that in this case the constant depends on  $\|u\|_{\infty}$ . The other inequality can be proved similarly from (3.1).  $\square$

### 4. Quasiminimizers

In a paper with Toivanen [43], the first two authors recently obtained the first results on regularity of quasiminimizers in the generalized Orlicz growth case. We showed a Harnack inequality and local Hölder continuity under assumptions (A0), (A1), (A1- $n$ ), (aInc) and (aDec). The first aim of this paper is to improve and extend these results in several ways. For further extensions, building on the current paper, see [10].

The first main contribution of the current paper is to extend [9] to the generalized Orlicz setting, and prove Hölder continuity assuming either (A1) or (A1- $n$ ) and bounded  $u$ ; In our earlier generalized Orlicz case result [43], we needed to assume both (A1) and (A1- $n$ ). In addition, we here extend our previous results from [43] in two ways. Of greater importance is the inclusion of  $+\Lambda$  on the right-hand side: it allows us to move between  $(\text{aDec})^\infty$  and  $(\text{aDec})$  and is crucial for applying quasiminimizer-results to prove regularity of  $\omega$ -minimizers. The  $(\text{aDec})^\infty$  assumption is a growth condition for large values of the gradient, a necessary change to handle (1.2) which does not satisfy a growth condition at the origin. A minor extension is that we allow  $F$  to depend on  $u$  and  $\nabla u$ , whereas the previous paper only allowed dependence on  $|\nabla u|$ .

For quasiminimizers, our main result is the following Harnack inequality, which implies local Hölder continuity by well-known arguments. Note that the (A1) and (A1- $n$ ) assumptions are essentially sharp, in view of the examples from the double phase case, cf. [6].

**Theorem 4.1 (Harnack’s inequality).** *Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $\varphi \in \Phi_w(\Omega)$  satisfy (A0), (aInc) and  $(\text{aDec})^\infty$ . Let  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  be a non-negative local quasiminimizer of  $\mathcal{F}$ . Assume that  $\varphi$  satisfies (A1), or that  $u$  is bounded and  $\varphi$  satisfies (A1- $n$ ). If  $Q_{2r} \subset \Omega$ , then*

$$\text{ess sup}_{x \in Q_r} u(x) \lesssim \text{ess inf}_{x \in Q_r} u(x) + (\varphi_Q^-)^{-1}(\Lambda + 1)r$$

*provided  $(\Lambda + 1)|Q_{2r}| \leq 1$  and  $\int_{Q_{2r}} \varphi(x, |\nabla u|) dx \leq 1$ . The implicit constant depends only on the parameters from the assumptions, the dimension  $n$ , and, in the case (A1- $n$ ), on  $\|u\|_\infty$ ; it is independent of  $r$  and  $\Lambda$ .*

The proof of this result (which continues through Sections 5 and 6) follows a different philosophy compared to our earlier paper [43]: previously, much effort was directed at avoiding additional error terms which do not appear in the standard case, whereas now we focus on handling the error terms which appear. The reason is that the “ $+\Lambda$ ” in (1.4) as well as  $(\text{aDec})^\infty$  lead inevitably to similar additive error terms, so they must in any case be taken care of. These more streamlined proofs are made possible by new tools developed in the monograph [39]. It is especially worth mentioning the generalized Orlicz version of the Sobolev–Poincaré inequality (Theorem 3.3) and the improved reverse Hölder inequality (Lemma 4.5). While the proofs follow the well-known approach of De Giorgi, we found that they are very dependent on well set-up formulations (much more so that the variable exponent case): for instance the placement of  $\tau$  on the left-hand side of (5.1) and the estimate of  $\frac{|A_j|}{|Q_j|}$  in the proof of Proposition 5.2. The main difficulty with the generalized Orlicz case is to move at suitable points in the proofs between  $\varphi(x, t)$  and  $\varphi_Q^-(t)$ . This is accomplished via the Sobolev–Poincaré inequality or the Caccioppoli estimate. The former leads in the proof of Proposition 5.2 to an additional term on the right-hand side, which can, however, be absorbed in the other terms in

the specific cases needed for Harnack’s inequality. Additional complications arise in several places because the Sobolev–Poincaré inequality holds only for functions with  $\|\nabla u\|_{L^\varphi(Q)} \leq 1$ .

Recall that we define, for measurable  $A \subset \Omega$ ,

$$\mathcal{F}(u, A) := \int_A F(x, u, \nabla u) \, dx.$$

By  $Q_r$  we mean a cube with side length  $r$  and faces parallel to the coordinate axes. Since we consider cubes, we speak of cubical minimizers, although spherical minimizers is a more common term for essentially the same thing. The results can also be adapted to spherical minimizers and  $\omega$ -minimizers defined in balls.

**Definition 4.2.** A function  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  is called

- (i) A *local quasiminimizer* of  $\mathcal{F}$  if there exists  $Q \geq 1$  such that

$$\mathcal{F}(u, \Omega' \cap \{u \neq v\}) \leq Q \mathcal{F}(v, \Omega' \cap \{u \neq v\})$$

for every open  $\Omega' \Subset \Omega$  and every  $v \in u + W_0^{1,1}(\Omega')$ ;

- (ii) A *weak quasiminimizer with bound  $M > 0$*  of  $\mathcal{F}$  if there exists  $Q \geq 1$  such that

$$\mathcal{F}(u, \Omega' \cap \{u \neq v\}) \leq Q \mathcal{F}(v, \Omega' \cap \{u \neq v\})$$

for every open  $\Omega' \Subset \Omega$  and every  $v \in u + W_0^{1,1}(\Omega')$  with  $\|v\|_{L^\infty(\Omega)} \leq M$ ;

- (iii) An  *$\omega$ -minimizer* of  $\mathcal{F}$  if there exists a non-decreasing concave function  $\omega : [0, \infty) \rightarrow [0, \infty)$  satisfying  $\omega(0) = 0$  such that

$$\mathcal{F}(u, Q_r) \leq (1 + \omega(r)) \mathcal{F}(v, Q_r)$$

for every  $v \in u + W_0^{1,1}(Q_r)$  with  $Q_r \Subset \Omega$ ;

- (iv) A *cubical quasiminimizer* of  $\mathcal{F}$  if there exists  $Q \geq 1$  such that

$$\mathcal{F}(u, Q_r) \leq Q \mathcal{F}(v, Q_r)$$

for every  $v \in u + W_0^{1,1}(Q_r)$  with  $Q_r \Subset \Omega$ .

Every minimizer (*i.e.*, 1-quasiminimizer) is both a quasiminimizer and an  $\omega$ -minimizer; and each of these is also a cubical quasiminimizer. In addition, it is clear that a quasiminimizer is a weak quasiminimizer with any bound  $M > 0$ . Note that there is no *a priori* relationship between quasiminimizers and  $\omega$ -minimizers:  $\omega$ -minimizers satisfy a stricter inequality but for a restricted range of sets.

We observe that if  $u$  is a quasiminimizer of  $\mathcal{F}$ , then it is also a quasiminimizer of  $\varphi + \Lambda$ . An analogous result holds for weak quasiminimizers and cubical minimizers, but not  $\omega$ -minimizers.

To deal with quasiminimizers of  $\mathcal{F}$  we need to generalize the results of [43] which only deal with quasiminimizers of  $\varphi$ . It is crucial to track the dependence of

constants on  $\Lambda$ , since in Section 8  $\Lambda$  depends on the  $\omega$ -minimizer  $u$  and may blow up in small balls.

We record the following iteration lemma, which will be needed in what follows.

**Lemma 4.3 (Lemma 4.2 in [43]).** *Let  $Z$  be a bounded non-negative function in the interval  $[r, R] \subset \mathbb{R}$  and let  $X$  satisfy (aDec) on  $[0, \infty)$ . Assume that there exists  $\theta \in [0, 1)$  such that*

$$Z(t) \leq X\left(\frac{1}{\sigma - \tau}\right) + \theta Z(\sigma)$$

for all  $r \leq \tau < \sigma \leq R$ . Then

$$Z(r) \lesssim X\left(\frac{1}{R-r}\right),$$

where the implicit constant depends only on the (aDec) constants and  $\theta$  but not on  $\|Z\|_\infty$ .

Note that  $\|Z\|_\infty$  does not impact the implicit constant in the previous result. This will be important for us later on.

Cubical quasiminimizers need not be bounded in general (cf. [35, Example 6.5, page 188]), but they do have the following higher integrability property.

**Lemma 4.4 (Reverse Hölder inequality).** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy Assumption 3.2 and suppose  $u$  is a cubical quasiminimizer of  $\mathcal{F}$ . For any  $Q_r \subset \Omega$  with  $|Q_r| \leq 1$ , there exists  $s_0 > 0$  such that*

$$\left(\int_{Q_{\frac{r}{2}}} \varphi(x, |\nabla u|)^{1+s_0} dx\right)^{\frac{1}{1+s_0}} \lesssim \int_{Q_r} \varphi(x, |\nabla u|) dx + \Lambda + 1. \tag{4.1}$$

*Proof.* Consider concentric cubes  $Q_\sigma \subset Q_\tau \subset Q_r$  for  $0 < \sigma < \tau \leq r$ . Let  $\eta \in C_0^\infty(Q_\tau)$  be a cut-off function such that  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in  $Q_\sigma$  and  $|\nabla \eta| \leq \frac{2}{\tau - \sigma}$ . We use  $v := u - \eta(u - u_{Q_r})$  as a test function in Definition 4.2 (iv) in order to get

$$\begin{aligned} v \int_{Q_\sigma} \varphi(x, |\nabla u|) dx &\leq \int_{Q_\tau} F(x, u, \nabla u) dx \\ &\leq Q \int_{Q_\tau} F(x, v, \nabla v) dx \leq QN \int_{Q_\tau} \varphi(x, |\nabla v|) + \Lambda dx. \end{aligned} \tag{4.2}$$

We note that  $|\nabla v| \leq (1 - \eta)|\nabla u| + |\nabla \eta||u - u_{Q_r}| \leq 2 \max\{(1 - \eta)|\nabla u|, |\nabla \eta||u - u_{Q_r}|\}$ . By  $|\nabla \eta| \leq \frac{2}{\tau - \sigma}$  and (aDec), we have that

$$\varphi(x, |\nabla v|) \leq 2^q L\varphi(x, (1 - \eta)|\nabla u|) + 4^q L\varphi\left(x, \frac{|u - u_{Q_r}|}{\tau - \sigma}\right).$$

Denote  $c_1 := 2^q LQN$ . Combining this inequality with (4.2), we get that

$$\begin{aligned} \nu \int_{Q_\sigma} \varphi(x, |\nabla u|) dx &\leq c_1 \int_{Q_\tau} \varphi(x, (1 - \eta)|\nabla u|) dx \\ &\quad + c \int_{Q_\tau} \varphi\left(x, \frac{|u - u_{Q_r}|}{\tau - \sigma}\right) dx + c\Lambda r^n \\ &\leq c_1 \int_{Q_\tau \setminus Q_\sigma} \varphi(x, |\nabla u|) dx + c \int_{Q_\tau} \varphi\left(x, \frac{|u - u_{Q_r}|}{\tau - \sigma}\right) dx + c\Lambda r^n, \end{aligned}$$

where the second inequality follows since  $\varphi(x, (1 - \eta)|\nabla u|) = \varphi(x, 0) = 0$  in  $Q_\sigma$ .

Now we use the hole-filling trick and add  $c_1 \int_{Q_\sigma} \varphi(x, |\nabla u|) dx$  to both sides of the previous inequality and divide by  $c_1 + \nu$ . Then it follows that

$$\begin{aligned} \int_{Q_\sigma} \varphi(x, |\nabla u|) dx &\leq \frac{c_1}{c_1 + \nu} \int_{Q_\tau} \varphi(x, |\nabla u|) dx \\ &\quad + c \int_{Q_\tau} \varphi\left(x, \frac{|u - u_{Q_r}|}{\tau - \sigma}\right) dx + c\Lambda r^n. \end{aligned}$$

By the iteration lemma (Lemma 4.3) for the first step and the Sobolev–Poincaré inequality (Theorem 3.3) for the second, we conclude that

$$\begin{aligned} \int_{Q_{\frac{r}{2}}} \varphi(x, |\nabla u|) dx &\lesssim \int_{Q_r} \varphi\left(x, \frac{|u - u_{Q_r}|}{r}\right) dx + \Lambda \\ &\lesssim \left( \int_{Q_r} \varphi(x, |\nabla u|)^{\frac{1}{s}} dx \right)^s + \Lambda + 1; \end{aligned} \tag{4.3}$$

note that the Sobolev–Poincaré inequality can be used since

$$\int_{Q_r} \varphi(x, |\nabla u|)^{\frac{1}{s}} dx \leq \int_{Q_r} \varphi(x, |\nabla u|) + 1 dx \leq 2.$$

Hence, by Gehring’s lemma (see [35, Theorem 6.6 and Corollary 6.1, pages 203–204]), the desired reverse Hölder inequality holds.  $\square$

The reverse Hölder inequality has the following “self-improving” property.

**Lemma 4.5 (Lemma 3.8, [45]).** *If  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  satisfies (4.1), then for every  $s \in [0, 1]$*

$$\left( \int_{Q_r} \varphi(x, |\nabla u|)^{1+s_0} dx \right)^{\frac{1}{1+s_0}} \lesssim \left( \int_{Q_{2r}} \varphi(x, |\nabla u|)^s dx \right)^{\frac{1}{s}} + \Lambda + 1,$$

where the implicit constant depends on  $s$  and the constant in (4.1). If  $\varphi$  satisfies (aDec), then this implies that

$$\begin{aligned} \int_{Q_r} \varphi(x, |\nabla u|) dx &\leq \left( \int_{Q_r} \varphi(x, |\nabla u|)^{1+s_0} dx \right)^{\frac{1}{1+s_0}} \\ &\lesssim \varphi_{Q_{2r}}^+ \left( \int_{Q_{2r}} |\nabla u| dx \right) + \Lambda + 1. \end{aligned}$$

Let us write

$$A(k, r) := Q_r \cap \{u > k\}.$$

**Lemma 4.6 (Caccioppoli inequality).** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (aDec) and let  $u$  be a local quasiminimizer of  $\mathcal{F}$ . Then for all  $k \geq 0$  and  $0 < r < R < \infty$  with  $Q_R \subset \Omega$  we have*

$$\int_{A(k,r)} \varphi(x, |\nabla(u - k)_+|) dx \lesssim \int_{A(k,R)} \varphi\left(x, \frac{(u - k)_+}{R - r}\right) dx + |A(k, R)|\Lambda. \tag{4.4}$$

*Proof.* Let  $r \leq \sigma < \tau \leq R$  and  $k \geq 0$ . Let  $\eta \in C_0^\infty(Q_\tau)$  be such that  $0 \leq \eta \leq 1$ ,  $\eta = 1$  in  $Q_\sigma$ , and  $|\nabla \eta| \leq \frac{2}{\tau - \sigma}$ . Denote  $v := u - \eta(u - k)_+$ . Since  $u$  is a local quasiminimizer of  $\mathcal{F}$  with constant  $Q$  and  $\text{spt}(u - v) \subset Q_\tau$ ,

$$v \int_{\{u \neq v\} \cap Q_\tau} \varphi(x, |\nabla u|) dx \leq QN \int_{\{u \neq v\} \cap Q_\tau} \varphi(x, |\nabla v|) + \Lambda dx.$$

Since  $A(k, \sigma) \subset \{u \neq v\} \cap Q_\tau \subset A(k, \tau) \subset A(k, R)$ , this implies that

$$\int_{A(k,\sigma)} \varphi(x, |\nabla u|) dx \lesssim \int_{A(k,\tau)} \varphi(x, |\nabla v|) dx + |A(k, R)|\Lambda.$$

The integrals are handled by the hole-filling trick and the iteration lemma as in Lemma 4.4 (see [43, Lemma 4.3] for exact details), while the second term on the right-hand side appears directly on the right-hand side of the claim.  $\square$

### 5. Estimating the essential supremum

We now start our proof of Harnack’s inequality. As is usual with De Giorgi’s method, we first derive bounds for the essential supremum of the function. In the next section, these will be used to bound also the infimum, which combined give the Harnack inequality. Recall that  $A(k, r) = Q_r \cap \{u > k\}$ .

In this paper we state our results in a modular format so as to make them easier to extend later. For instance, in the next result we assume the Caccioppoli inequality instead of assuming that  $u$  is a quasiminimizer. If the Caccioppoli inequality is extended to a larger class, then the next result need not be reproved (cf. Remark 6.4).

**Lemma 5.1.** *Let  $\varphi \in \Phi_w(\Omega)$  and  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  satisfy Assumption 3.2. Suppose that  $u$  satisfies the Caccioppoli inequality (4.4). Let  $k \geq 0$  and  $0 < \sigma < \tau \leq R$  with  $Q_R \subset \Omega$  and  $(\Lambda + 1)|Q_R| \leq 1$ . Then*

$$\begin{aligned} & \int_{Q_\sigma} \varphi\left(x, \frac{(u - k)_+}{\tau}\right) dx \\ & \lesssim \left(\frac{\tau}{\tau - \sigma}\right)^{\frac{q}{p}} \left(\frac{|A(k, \tau)|}{|Q_\tau|}\right)^{\frac{1}{2n}} \left(\int_{Q_\tau} \varphi\left(x, \frac{(u - k)_+}{\tau - \sigma}\right) dx + |A(k, \tau)|(\Lambda + 1)\right). \end{aligned} \tag{5.1}$$

*Proof.* We first observe that the claim is trivial if  $|A(k, \tau)| \geq \frac{1}{2}|Q_\tau|$ , so we may assume that this is not the case. Let  $\tau' := \frac{\sigma + \tau}{2}$  and  $\eta \in C_0^\infty(Q_{\tau'})$  be a cut-off function such that  $0 \leq \eta \leq 1$ ,  $\eta = 1$  in  $Q_\sigma$ , and  $|\nabla \eta| \leq \frac{4}{\tau - \sigma}$ . Denote  $v := (u - k)_+ \eta$ .

By the product rule,  $|\nabla v| \leq |\nabla(u - k)_+| + (u - k)_+ |\nabla \eta|$ . Since  $|\nabla \eta| \leq \frac{4}{\tau - \sigma}$ , we obtain by (aDec) and the Caccioppoli inequality (4.4) that

$$\begin{aligned} \int_{Q_{\tau'}} \varphi(x, |\nabla v|) dx & \lesssim \int_{Q_{\tau'}} \varphi(x, |\nabla(u - k)_+|) + \varphi\left(x, \frac{(u - k)_+}{\tau - \sigma}\right) dx \\ & \lesssim \int_{Q_\tau} \varphi\left(x, \frac{(u - k)_+}{\tau - \sigma}\right) dx + |A(k, \tau)|\Lambda. \end{aligned} \tag{5.2}$$

As an intermediate step, we next show in the case (A1) how this inequality implies that  $\varrho_\varphi(c_{\tau,\sigma} |\nabla v|) \leq 1$  for a suitable constant.

In the case of (A1), we denote  $w := (u - k)_+$  and note that  $w = 0$  in  $A := Q_\tau \setminus A(k, \tau)$ . Since  $|A| > \frac{1}{2}|Q_\tau|$ , we obtain by the  $W^{1,1}$ -Poincaré inequality, Lemma 3.1 and  $\varrho_\varphi(|\nabla w|) \leq 1$  that

$$\begin{aligned} |w_A - w_{Q_\tau}| & \leq \frac{|Q_\tau|}{|A|} \int_{Q_\tau} |w - w_{Q_\tau}| dx \lesssim \tau \int_{Q_\tau} |\nabla w| dx \\ & \lesssim \tau (\varphi_{Q_\tau}^-)^{-1} \left( \int_{Q_\tau} \varphi_{Q_\tau}^- (|\nabla w|) dx \right) \leq \tau (\varphi_{Q_\tau}^-)^{-1} \left( \frac{1}{|Q_\tau|} \right). \end{aligned}$$

Since  $w = 0$  in  $A$ , we obtain by this, (A0) and (A1) that

$$w = |w - w_A| \leq |w - w_{Q_\tau}| + |w_A - w_{Q_\tau}| \lesssim |w - w_{Q_\tau}| + \tau (\varphi_{Q_\tau}^+)^{-1} \left( \frac{1}{|Q_\tau|} \right) + \tau.$$

By the Sobolev–Poincaré inequality (Theorem 3.3 with  $s = 1$ ), (aDec), (A0),  $\varrho_\varphi(|\nabla u|) \leq 1$  and  $|Q_r| \leq 1$ , we conclude that

$$\begin{aligned} \int_{Q_\tau} \varphi\left(x, \frac{w}{\tau}\right) dx & \lesssim \int_{Q_\tau} \varphi\left(x, \frac{|w(x) - w_{Q_\tau}|}{\tau} + (\varphi_{Q_\tau}^+)^{-1} \left( \frac{1}{|Q_\tau|} \right) + 1\right) dx \\ & \lesssim \int_{Q_\tau} \varphi(x, |\nabla u|) + 1 + \frac{1}{|Q_\tau|} dx \leq 3. \end{aligned} \tag{5.3}$$

Furthermore, (aDec) implies that

$$\int_{Q_\tau} \varphi \left( x, \frac{(u-k)_+}{\tau-\sigma} \right) dx \lesssim \left( \frac{\tau}{\tau-\sigma} \right)^q \int_{Q_\tau} \varphi \left( x, \frac{w}{\tau} \right) dx \lesssim \left( \frac{\tau}{\tau-\sigma} \right)^q.$$

By (aInc)<sub>p</sub>, (5.2) and this imply that  $\varrho_\varphi(c_{\tau,\sigma}|\nabla v|) \leq 1$ , where  $c_{\tau,\sigma} := c \left( \frac{\tau-\sigma}{\tau} \right)^{q/p} \leq 1$ . We set  $c_{\tau,\sigma} := 1$  for the case (A1-*n*); then in both cases we can apply the Sobolev–Poincaré inequality (Theorem 3.3) to the function  $c_{\tau,\sigma}v$ .

We now start the main line of the proof. By Hölder’s inequality and (aDec), we obtain

$$\begin{aligned} \int_{Q_\sigma} \varphi \left( x, \frac{(u-k)_+}{\tau} \right) dx &\leq \int_{Q_{\tau'}} \varphi \left( x, \frac{v}{\tau} \right) dx \leq |A(k, \tau)|^{\frac{s-1}{s}} \left( \int_{Q_{\tau'}} \varphi \left( x, \frac{v}{\tau} \right)^s dx \right)^{\frac{1}{s}} \\ &\leq c_{\tau,\sigma}^{-q} |A(k, \tau)|^{\frac{s-1}{s}} \left( \int_{Q_{\tau'}} \varphi \left( x, \frac{c_{\tau,\sigma}v}{\tau} \right)^s dx \right)^{\frac{1}{s}} \end{aligned}$$

for  $s := (2n)'$ . Note that  $s < n'$ ,  $\varrho_\varphi(c_{\tau,\sigma}|\nabla v|) \leq 1$  and that  $c_{\tau,\sigma}v \in W_0^{1,\varphi}(Q_{\tau'})$ . Thus the Sobolev–Poincaré inequality (Theorem 3.3) for the function  $c_{\tau,\sigma}v$  yields that

$$|Q_\tau|^{\frac{s-1}{s}} \left( \int_{Q_{\tau'}} \varphi \left( x, \frac{c_{\tau,\sigma}v}{\tau} \right)^s dx \right)^{\frac{1}{s}} \leq \int_{Q_{\tau'}} \varphi(x, |\nabla v|) dx + |A(k, \tau)|;$$

here we also used that  $\nabla v = 0$  a.e. outside  $A(k, \tau)$  and  $c_{\tau,\sigma} \leq 1$ . Combining the two inequalities, noting that  $\frac{s-1}{s} = \frac{1}{2n}$  and using (aDec) for the first step, and (5.2) for the second step, we find that

$$\begin{aligned} \int_{Q_\sigma} \varphi \left( x, \frac{(u-k)_+}{\tau} \right) dx &\lesssim c_{\tau,\sigma}^{-q} \left( \frac{|A(k, \tau)|}{|Q_\tau|} \right)^{\frac{1}{2n}} \left( \int_{Q_{\tau'}} \varphi(x, |\nabla v|) dx + |A(k, \tau)| \right) \\ &\lesssim c_{\tau,\sigma}^{-q} \left( \frac{|A(k, \tau)|}{|Q_\tau|} \right)^{\frac{1}{2n}} \left( \int_{Q_\tau} \varphi \left( x, \frac{(u-k)_+}{\tau-\sigma} \right) dx + |A(k, \tau)|(\Lambda + 1) \right). \end{aligned}$$

□

Compared to classical estimates, the next proposition contains an extra term  $|u_{Q_r}|$ . Note that it involves the function  $u$ , not just  $u_+$ , which makes it more difficult to manage. However, we show that it can be handled in the cases needed to prove Harnack’s inequality. Recall that  $q > 1$  is the exponent from (aDec)<sub>q</sub> in Assumption 3.2. For brevity, we will use the following notation for the rest of the paper

$$\lambda_r := (\varphi_{Q_r}^-)^{-1}(\Lambda + 1)r.$$



**Proposition 5.2.** *Let  $\varphi \in \Phi_w(\Omega)$  and  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  satisfy Assumption 3.2 with  $(\Lambda + 1)|Q_r| \leq 1$ . Suppose that  $u$  satisfies (5.1) and  $\theta \in [\frac{1}{2}, 1)$ . Then  $u_+$  is bounded and*

$$\text{ess sup}_{Q_{\theta r}} u_+ \lesssim (1 - \theta)^{-4nq^2} \left( \left[ \int_{Q_r} (u_+)^q dx \right]^{\frac{1}{q}} + |u_{Q_{r/2}}| \right) + \lambda_r \tag{5.4}$$

for any  $Q_{2r} \subset \Omega$ . The term  $|u_{Q_r}|$  can be omitted if  $|\{u_+ = 0\} \cap Q_r| \geq \frac{1}{2}|Q_r|$  or if  $u$  is non-negative.

*Proof.* For  $k > 0$  to be chosen and any natural number  $j$ , we set

$$\alpha := \frac{1}{2n}, \quad k_j := rk \left( 1 - \frac{1}{2^j} \right), \quad \sigma_j := r \left( \theta + \frac{1 - \theta}{2^j} \right), \quad A_j := A(k_{j+1}, \sigma_j),$$

$$Q_j := Q_{\sigma_j} \quad \text{and} \quad Y_j := \int_{Q_j} \varphi \left( x, \frac{(u - k_j)_+}{r} \right) dx.$$

Note that  $\sigma_j - \sigma_{j+1} = \frac{r(1-\theta)}{2^{j+1}}$ . Using (5.1) with  $k = k_{j+1}$ ,  $\sigma = \sigma_{j+1}$  and  $\tau = \sigma_j$  for the middle step, and (aDec) for the others, we find that

$$Y_{j+1} \lesssim \int_{Q_{j+1}} \varphi \left( x, \frac{(u - k_{j+1})_+}{r(\theta + (1 - \theta)2^{-j-1})} \right) dx$$

$$\lesssim 2^{\frac{q^2 j}{p}} (1 - \theta)^{-\frac{q^2}{p}} \left( \frac{|A_j|}{|Q_j|} \right)^\alpha \left( \int_{Q_j} \varphi \left( x, \frac{(u - k_{j+1})_+}{r(1 - \theta)2^{-j-1}} \right) dx + \frac{|A_j|}{|Q_j|} (\Lambda + 1) \right)$$

$$\lesssim 2^{2q^2 j} (1 - \theta)^{-q^2} \left( \frac{|A_j|}{|Q_j|} \right)^\alpha \left( 2^{qj} (1 - \theta)^{-q} Y_j + \frac{|A_j|}{|Q_j|} (\Lambda + 1) \right),$$

where we also used  $k_j \leq k_{j+1}$  in the last step. Furthermore, we observe that  $u - k_j \geq k_{j+1} - k_j = \frac{rk}{2^{j+1}}$  in  $A_j$ . It follows by (aDec) that

$$\frac{|A_j|}{|Q_j|} \leq \int_{Q_j} \frac{1}{\varphi(x, k)} \varphi \left( x, 2^{j+1} \frac{(u - k_j)_+}{r} \right) dx \lesssim 2^{qj} \varphi_{Q_r}^-(k)^{-1} Y_j.$$

Now our inequality implies that

$$Y_{j+1} \leq c 2^{q^2 j} (1 - \theta)^{-q^2} (2^{qj} \varphi_{Q_r}^-(k)^{-1} Y_j)^\alpha [2^{qj} (1 - \theta)^{-q} Y_j + 2^{qj} \varphi_{Q_r}^-(k)^{-1} Y_j (\Lambda + 1)].$$

We will choose  $k$  such that  $\varphi_{Q_r}^-(k)^{-1} (\Lambda + 1) \leq 1$ . Then the inequality implies that

$$Y_{j+1} \leq c_1 2^{3q^2 j} (1 - \theta)^{-2q^2} \varphi_{Q_r}^-(k)^{-\alpha} Y_j^{1+\alpha}.$$

By the well-known iteration lemma [35, Lemma 7.1, page 220] it follows that  $Y_j \rightarrow 0$  as  $j \rightarrow \infty$ , provided that  $Y_0 \leq c_1^{-1/\alpha} 2^{-3q^2/\alpha^2} (1 - \theta)^{2q^2/\alpha} \varphi_{Q_r}^-(k)$ . Thus we need to ensure that

$$Y_0 = \int_{Q_r} \varphi \left( x, \frac{u_+}{r} \right) dx \leq c(\alpha, q)(1 - \theta)^{4nq^2} \varphi_{Q_r}^-(k),$$

which holds under the choice

$$\varphi_{Q_r}^-(k) = \frac{\theta_q}{c(\alpha, q)} \int_{Q_r} \varphi\left(x, \frac{u_+}{r}\right) dx + \Lambda + 1, \quad \theta_q := (1 - \theta)^{-4nq^2};$$

such  $k$  exists due to the (aDec) assumption. The latter terms are added to ensure that  $\varphi_{Q_r}^-(k)^{-1}(\Lambda + 1) \leq 1$ , as required above.

Since  $k_j \rightarrow rk$  and  $\sigma_j \rightarrow \theta r$  as  $j \rightarrow \infty$ , it follows by Fatou’s lemma that

$$\int_{Q_{\theta r}} \varphi\left(x, \frac{(u - rk)_+}{r}\right) dx \leq \liminf_{j \rightarrow \infty} Y_j = 0.$$

This implies that  $u \leq rk$  a.e. in  $Q_{\theta r}$ . Thus  $u$  is locally bounded and

$$\operatorname{ess\,sup}_{Q_{\theta r}} \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) \leq \varphi_{Q_r}^-(k) = \frac{\theta_q}{c(\alpha, q)} \int_{Q_r} \varphi\left(x, \frac{u_+}{r}\right) dx + \Lambda + 1. \quad (5.5)$$

Assume first that  $u_{Q_{r/2}} = 0$ . In the case (A1), we use (5.5) in the cubes  $Q_r$  and  $Q_{2r}$  (in which case there is no dependence on  $\theta$  in the constant), the Sobolev–Poincaré inequality (Theorem 3.3) with  $s = 1$  and  $u_{Q_{r/2}} = 0$ , and (aDec) to conclude that

$$\begin{aligned} \operatorname{ess\,sup}_{Q_{\theta r}} \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) &\lesssim \operatorname{ess\,sup}_{Q_r} \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) \lesssim \int_{Q_{2r}} \varphi\left(x, \frac{u_+}{r}\right) dx + \Lambda + 1 \\ &\lesssim \int_{Q_{2r}} \varphi(x, |\nabla u|) dx + \Lambda + 1 \lesssim \frac{1}{|Q_r|}, \end{aligned}$$

where in the last step we use  $(\Lambda + 1)|Q_r| \leq 1$ . Instead of  $u_{Q_{r/2}} = 0$  we could assume  $|\{u_+ = 0\} \cap Q_r| \geq \frac{1}{2}|Q_r|$  since then (5.3) implies that

$$\int_{Q_{2r}} \varphi\left(x, \frac{u_+}{r}\right) dx \lesssim \frac{1}{|Q_r|}.$$

In either case, it follows by (aDec), (A0) and (A1) that

$$\varphi\left(x, \frac{u_+}{r}\right) \lesssim \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) + 1 \quad \text{for a.e. } x \in Q_r.$$

In the case (A1- $n$ ), the same inequality follows from (aDec), (A0) and (A1- $n$ ), with constant depending also on  $\|u\|_\infty$ . Here the assumption  $u_{Q_{r/2}} = 0$  is not needed at all.

Now we return to (5.5) with  $\varphi_{Q_r}^-$  in the integral by the estimate in the previous paragraph. By (aInc) $_p$  we have

$$\begin{aligned} \operatorname{ess\,sup}_{Q_{\theta r}} \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) &\lesssim \frac{\theta_q}{c(\alpha, q)} \int_{Q_r} \varphi_{Q_r}^-\left(\frac{u_+}{r}\right) dx + \Lambda + 1 \\ &\lesssim \int_{Q_r} \varphi_{Q_r}^-\left(\theta_q^{\frac{1}{p}} \frac{u_+}{r}\right) dx + \Lambda + 1. \end{aligned}$$

Since  $\varphi_{Q_r}^-$  is a  $\Phi$ -prefunction that satisfies  $(\text{aDec})_q$ , we obtain by Lemma 3.1 and (2.1) that

$$\text{ess sup}_{Q_{\theta r}} \frac{u_+}{r} \lesssim \theta_q^{\frac{1}{p}} \left[ \int_{Q_r} \left( \frac{u_+}{r} \right)^q dx \right]^{\frac{1}{q}} + \frac{1}{r} \lambda_r.$$

The claim follows for this case when we multiply the previous inequality by  $r$ .

We have established the claim in the case  $u_{Q_{r/2}} = 0$ . Thus, in the general case,

$$\text{ess sup}_{Q_{\theta r}} u_+ - |u_{Q_{r/2}}| \leq \text{ess sup}_{Q_{\theta r}} (u - u_{Q_{r/2}})_+ \lesssim \theta_q^{\frac{1}{p}} \left[ \int_{Q_r} (u - u_{Q_{r/2}})_+^q dx \right]^{\frac{1}{q}} + \lambda_r.$$

Furthermore,

$$\left[ \int_{Q_r} (u - u_{Q_{r/2}})_+^q dx \right]^{\frac{1}{q}} \leq \left[ \int_{Q_r} (u_+ + |u_{Q_{r/2}}|)^q dx \right]^{\frac{1}{q}} \approx \left[ \int_{Q_r} u_+^q dx \right]^{\frac{1}{q}} + |u_{Q_{r/2}}|$$

so we have completed the proof in the general case. If  $u$  is non-negative, then  $u_+ = u$  and Hölder’s inequality allows us to absorb the extra term in the  $q$ -average as follows:

$$|u_{Q_{r/2}}| = \int_{Q_{r/2}} u dx \lesssim \left[ \int_{Q_r} u^q dx \right]^{\frac{1}{q}}.$$

□

Next we show that the exponent can be decreased arbitrarily close to zero when there is no extra term  $|u_{Q_{r/2}}|$ .

**Corollary 5.3.** *Suppose that  $u \in L^\infty(Q_r)$  satisfies (5.4) without the term  $|u_{Q_{\sigma/2}}|$  for  $Q_\sigma \subset Q_r$  with  $\sigma \in [\frac{r}{2}, r]$ . Then*

$$\text{ess sup}_{Q_{r/2}} u_+ \lesssim \left( \int_{Q_r} u_+^h dx \right)^{\frac{1}{h}} + \lambda_r,$$

for any  $h \in (0, \infty)$ . The implicit constant depends on  $h$  and the constant in (5.4).

*Proof.* The case  $h \geq q$  follows directly by Hölder’s inequality, so we consider only  $h \in (0, q)$ . Let  $\frac{r}{2} \leq \sigma < \tau \leq r$  and denote  $Z(\sigma) := \text{ess sup}_{Q_\sigma} u$ . By (5.4),

$$Z(\sigma) \lesssim (1 - \frac{\sigma}{\tau})^{-4nq^2} \left( \int_{Q_\tau} u_+^q dx \right)^{\frac{1}{q}} + \lambda_\tau \leq \left( \frac{r}{\tau - \sigma} \right)^{4nq^2} \left( \int_{Q_\tau} u_+^q dx \right)^{\frac{1}{q}} + \lambda_r.$$

Since  $\tau \in (\frac{r}{2}, r)$ , we find that

$$\left( \int_{Q_\tau} u_+^q dx \right)^{\frac{1}{q}} \lesssim \left( \int_{Q_r} u_+^h Z(\tau)^{q-h} dx \right)^{\frac{1}{q}} \approx \left( \int_{Q_r} u_+^h dx \right)^{\frac{1}{q}} Z(\tau)^{\frac{q-h}{q}}.$$

Next we use Young’s inequality with exponents  $\frac{q}{h}$  and  $\frac{q}{q-h} =: \frac{1}{\theta}$  and obtain

$$\begin{aligned} Z(\sigma) &\leq c\left(\frac{r}{\tau - \sigma}\right)^{4nq^2} \left(\int_{Q_r} u_+^h dx\right)^{\frac{1}{q}} Z(\tau)^\theta + c\lambda_r \\ &\leq \underbrace{\frac{ch}{q}\left(\frac{r}{\tau - \sigma}\right)^{\frac{4nq^3}{h}} \left(\int_{Q_r} u_+^h dx\right)^{\frac{1}{h}}}_{=: X(\frac{1}{\tau - \sigma})} + c\lambda_r + \theta Z(\tau). \end{aligned}$$

Thus  $Z(\sigma) \leq X(\frac{1}{\tau - \sigma}) + \theta Z(\tau)$  for all  $\frac{r}{2} \leq \sigma < \tau \leq r$ . Since  $Z$  is bounded in  $[\frac{r}{2}, r]$  and  $X$  satisfies  $(\text{aDec})_{4nq^3/h}$ , Lemma 4.3 yields  $Z(\frac{r}{2}) \lesssim X(\frac{2}{r})$ , which is the claim.  $\square$

### 6. Estimating the essential infimum

Let us denote  $D_l := \{u < l\} \cap Q_r$ . Suppose that  $u$  is a quasiminimizer of  $\mathcal{F}$  and  $l \in \mathbb{R}$ . Then  $l - u$  is a quasiminimizer of

$$\int_{\Omega} G(x, u, \nabla u) dx \quad \text{with} \quad G(x, t, z) := F(x, l - t, -z).$$

Furthermore,  $G$  satisfies (1.4) with the same constants as  $F$ . Thus by the Caccioppoli estimate (Lemma 4.6) and Lemma 5.1 the function  $l - u$  satisfies (5.1). Furthermore, the assumption in the next lemma implies that

$$|\{(l - u)_+ = 0\} \cap Q_r| = |Q_r \setminus D_l| \geq \left(1 - \frac{1}{2^q c_1^q}\right) |Q_r| \geq \frac{1}{2} |Q_r|,$$

so one of the conditions in Proposition 5.2 for omitting the term  $|u_{Q_r}|$  is satisfied. Thus the implication of the next lemma holds in particular for local quasiminimizers.

**Lemma 6.1.** *Let  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  be non-negative and  $l > 0$ . If  $l - u$  satisfies (5.4) for  $\theta = \frac{1}{2}$  without the term  $|u_{Q_{r/2}}|$  with constant  $c_1$ , then*

$$|D_l| \leq \frac{1}{2^q c_1^q} |Q_r| \quad \Rightarrow \quad \text{ess inf}_{Q_{r/2}} u + c_1 \lambda_r \geq \frac{l}{2}.$$

*Proof.* Inequality (5.4) for the function  $l - u$  yields that

$$\begin{aligned} \text{ess sup}_{Q_{r/2}}(l - u) &\leq \text{ess sup}_{Q_{r/2}}(l - u)_+ \leq c_1 \left[ \int_{Q_r} (l - u)_+^q dx \right]^{\frac{1}{q}} + c_1 \lambda_r \\ &\leq c_1 \left[ \frac{1}{|Q_r|} \int_{D_l} l^q dx \right]^{\frac{1}{q}} + c_1 \lambda_r = c_1 l \left[ \frac{|D_l|}{|Q_r|} \right]^{\frac{1}{q}} + c_1 \lambda_r \\ &\leq \frac{1}{2} l + c_1 \lambda_r. \end{aligned}$$

Since  $\text{ess sup}_{Q_{r/2}}(l - u) = l - \text{ess inf}_{Q_{r/2}} u$ , the claim follows. □

The next lemma shows that the implication of the previous lemma holds for any constant  $\kappa$ . The previous lemma takes care of small values of  $\kappa$ .

**Lemma 6.2.** *Let  $\varphi \in \Phi_w(\Omega)$  and  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  satisfy Assumption 3.2. Suppose that  $u$  is a non-negative local quasiminimizer of  $\mathcal{F}$ . Then for every  $\kappa \in (0, 1)$  there exists  $\mu > 0$  such that*

$$|D_l| \leq \kappa |Q_r| \implies \text{ess inf}_{Q_{r/2}} u + c_1 \lambda_r \geq \mu l$$

for all  $Q_{2r} \subset \Omega$  and all  $l > 0$ . Here the constant  $c_1$  is from Lemma 6.1.

*Proof.* If  $l > \|u\|_\infty$ , then  $|D_l| = |Q_r|$ , so there is nothing to prove. Therefore, we assume that  $l \leq \|u\|_\infty$ . Abbreviate  $Q := Q_r$  and set, for  $0 < h < k < l$ ,

$$v := \begin{cases} 0 & \text{if } u \geq k \\ k - u & \text{if } h < u < k \\ k - h & \text{if } u \leq h. \end{cases}$$

Then  $v \in W_{\text{loc}}^{1,\varphi}(\Omega)$  and  $|\nabla v| = |\nabla u| \chi_{\{h < u < k\}}$  a.e. in  $\Omega$ .

Clearly,  $v = 0$  in  $Q \setminus D_l$ , and since  $|D_l| \leq \kappa |Q|$ , we have  $|Q \setminus D_l| \geq (1 - \kappa) |Q|$ . Under these circumstances, [35, Theorem 3.16, page 102] tells us that

$$\left( \int_Q v^{n'} dx \right)^{\frac{1}{n'}} \leq C(n, \kappa) \int_\Delta |\nabla v| dx$$

for  $v \in W^{1,1}(Q)$  and  $\Delta := D_k \setminus D_h$ . By Hölder’s inequality,

$$(k - h) |D_h|^{\frac{1}{n'}} = |D_h|^{-\frac{1}{n}} \int_{D_h} v dx \leq \left( \int_Q v^{n'} dx \right)^{\frac{1}{n'}} \lesssim |\Delta| \int_\Delta |\nabla v| dx.$$

Denote  $V(x) := \varphi(x, |\nabla v(x)|)$ . By Hölder’s inequality and Lemma 3.1,

$$\int_\Delta |\nabla v| dx \leq \left( \frac{|Q|}{|\Delta|} \right)^{\frac{1}{p}} \left( \int_Q |\nabla v|^p dx \right)^{\frac{1}{p}} \lesssim \left( \frac{|Q|}{|\Delta|} \right)^{\frac{1}{p}} (\varphi_Q^-)^{-1} \left( \int_Q V dx \right).$$

The Caccioppoli estimate (Lemma 4.6) for the function  $k - u$  implies that

$$\begin{aligned} \int_Q V dx &= \int_Q \varphi(x, |\nabla(k - u)_+|) dx \lesssim \int_{Q'} \varphi \left( x, \frac{(k - u)_+}{r} \right) dx + \Lambda \\ &\lesssim \varphi_{Q'}^+ \left( \frac{k}{r} \right) + \Lambda, \end{aligned} \tag{6.1}$$

where  $Q' := Q_{2r}$ . In the case (A1), we use the second expression and the assumption  $\varrho_\varphi(|\nabla u|) \leq 1$  to conclude that  $\int_Q V dx \lesssim \frac{1}{|Q|}$ . It then follows from (A1), (A0) and (aDec) that

$$(\varphi_{\bar{Q}}^-)^{-1} \left( \int_Q V dx \right) \leq (\varphi_{\bar{Q}'}^-)^{-1} \left( \int_Q V dx \right) \lesssim (\varphi_{Q'}^+)^{-1} \left( \int_Q V dx \right) + 1.$$

In the case of (A1-n), we use the last expression of (6.1),  $k \in (0, \|u\|_\infty)$ , (A0) and (aDec) to conclude that

$$\int_Q V dx \lesssim \varphi_{Q'}^+ \left( \frac{k}{r} \right) + \Lambda \lesssim \varphi_{\bar{Q}'}^- \left( \frac{k}{r} \right) + \Lambda + 1 \leq \varphi_{\bar{Q}}^- \left( \frac{k}{r} \right) + \Lambda + 1,$$

where the constant depends on  $\|u\|_\infty$ . In either case, we obtain that

$$(\varphi_{\bar{Q}}^-)^{-1} \left( \int_Q V dx \right) \lesssim \frac{k}{r} + (\varphi_{\bar{Q}}^-)^{-1}(\Lambda + 1) + 1 \approx \frac{1}{r}(k + \lambda_r),$$

where we also used (A0) and (aDec) to absorb the 1 in  $\lambda_r$ .

Combining the previous inequalities, we find that

$$(k - h)|D_h|^{\frac{1}{n'}} \lesssim |\Delta| \left( \frac{|Q|}{|\Delta|} \right)^{\frac{1}{p}} \frac{1}{r}(k + \lambda_r) \approx |\Delta|^{1 - \frac{1}{p}} r^{\frac{n}{p} - 1} (k + \lambda_r).$$

We divide the previous inequality by  $k$ , raise it to the power  $p'$  and substitute  $k := l2^{-i}$  and  $h := l2^{-i-1}$ ,  $i \in \mathbb{N}$ :

$$\left( \frac{l2^{-i} - l2^{-i-1}}{l2^{-i}} \right)^{p'} |D_{l2^{-i-1}}|^{\frac{p'}{n'}} \lesssim [ |D_{l2^{-i}}| - |D_{l2^{-i-1}}| ] r^{\frac{n-p}{p} p'} \left( 1 + 2^i \frac{1}{l} \lambda_r \right)^{p'}.$$

Set  $d_i := |D_{l2^{-i}}|$  for  $i = 0, \dots, i_0$  and note that  $\frac{l2^{-i} - l2^{-i-1}}{l2^{-i}} = \frac{1}{2}$ . Since  $d_i \geq d_{i_0} = |D_{l2^{-i_0}}|$  for  $i \leq i_0$ , this implies that

$$|D_{l2^{-i_0-1}}|^{\frac{p'}{n'}} \lesssim [d_i - d_{i+1}] r^{\frac{n-p}{p-1}} \left( 1 + 2^{i_0} \frac{1}{l} \lambda_r \right)^{p'}.$$

Adding these inequalities for  $i$  from 0 to  $i_0 - 1$ , we get

$$i_0 |D_{l2^{-i_0-1}}|^{\frac{p'}{n'}} \lesssim [d_0 - d_{i_0}] r^{\frac{n-p}{p-1}} \left( 1 + 2^{i_0} \frac{1}{l} \lambda_r \right)^{p'} \lesssim r^{n + \frac{n-p}{p-1}} \left( 1 + 2^{i_0} \frac{1}{l} \lambda_r \right)^{p'}.$$

Now  $n + \frac{n-p}{p-1} = p \frac{n-1}{p-1} = p'(n-1)$ . Hence

$$|D_{l2^{-i_0-1}}| \leq c i_0^{-\frac{n'}{p'}} r^n \left( 1 + 2^{i_0} \frac{1}{l} \lambda_r \right)^{n'} = c_2 i_0^{-\frac{n'}{p'}} |Q| \left( 1 + 2^{i_0} \frac{1}{l} \lambda_r \right)^{n'}.$$

We choose  $i_0$  such that  $c_2 i_0^{-\frac{n'}{p'}} \leq \frac{1}{2^q c_1^q}$  with  $c_1$  from Lemma 6.1.

We consider two cases. If  $2^{i_0} \frac{1}{l} \lambda_r \leq 1$ , then the previous inequality implies that  $|D_{l2^{-i_0-1}}| \leq \frac{1}{2^q c_1^q} |Q|$ , in which case it follows from Lemma 6.1 that  $\text{ess inf}_{Q_{r/2}} u + c_1 \lambda_r \geq l 2^{-i_0-1}$ , so the claim holds with  $\mu = 2^{-i_0-1}$ . If, on the other hand,  $2^{i_0} \frac{1}{l} \lambda_r \geq 1$ , then  $\text{ess inf}_{Q_{r/2}} u + c_1 \lambda_r \geq c_1 2^{-i_0} l$ , so the claim holds with  $\mu = c_1 2^{-i_0}$ .  $\square$

Now standard arguments yield the weak Harnack inequality, see, e.g., [43, Lemma 6.3].

**Corollary 6.3 (Weak Harnack inequality).** *Let  $\varphi \in \Phi_w(\Omega)$  and  $u \in W_{\text{loc}(\Omega)}^{1,\varphi}$  satisfy Assumption 3.2. Suppose that  $u$  is a non-negative local quasiminimizer of  $\mathcal{F}$ . Then there exists  $h > 0$  such that*

$$\left( \int_{Q_r} u^h dx \right)^{\frac{1}{h}} \lesssim \text{ess inf}_{Q_{r/2}} u + \lambda_r$$

when  $Q_{2r} \subset \Omega$  and  $(\Lambda + 1)|Q_{2r}| \leq 1$ .

By combining Corollaries 5.3 for the non-negative function and 6.3, we obtain Harnack’s inequality under Assumption 3.2. It remains to be shown that (aDec) can be replaced by (aDec) $^\infty$ .

*Proof of Theorem 4.1.* Let  $\varphi$  be from Theorem 4.1 and let  $\psi(x, t) := \varphi(x, t) + t$ . Then, by Lemma 2.2,  $\psi$  belongs to  $\Phi_w(\Omega)$  and satisfies Assumption 3.2. In particular, we have  $\varphi \leq \psi \lesssim \varphi + 1$ .

Since  $u$  is a local quasiminimizer of  $\mathcal{F}$ , it is a local quasiminimizer of  $\varphi + \Lambda + 1$ . Thus using Corollaries 5.3 and 6.3 with replacing  $(\varphi, F, \Lambda)$  by  $(\psi, \varphi + \Lambda + 1, \Lambda + 1)$ , we obtain Harnack’s inequality.  $\square$

**Remark 6.4.** All the results in Sections 4–6 hold also for bounded weak quasiminimizers  $u$  with bound  $\|u\|_\infty$ . This follows directly from the given proofs. We use the quasiminimizing property twice, first in the proof of the reverse Hölder inequality, Lemma 4.4, for the test function  $v := u - \eta(u - u_{Q_r}) = (1 - \eta)u + \eta u_{Q_r}$ , and then in the proof of the Caccioppoli inequality, Lemma 4.6, for the test function  $v := u - \eta(u - k)_+, k \geq 0$ . Thus in both cases  $\|v\|_\infty \leq \|u\|_\infty$ , so we have only used the weak quasiminimizing property. In fact, in the proofs that follow, only the latter is needed for weak quasiminimizers, the former is applied to the directly for cubical quasiminimizers.

### 7. Morrey estimates

It is well known that the Harnack inequality implies the following oscillation decay estimate (see [34, Theorem 8.22] or [46, Theorem 6.6, page 111]). We define the oscillation of  $u$  by

$$\text{osc}(u, r) := \text{ess sup}_{Q_r} u - \text{ess inf}_{Q_r} u.$$

**Theorem 7.1 (Oscillation decay estimate).** *Let  $\pm u - k$  satisfy Harnack’s inequality for every  $k \in \mathbb{R}$  and every  $Q_\sigma \subset Q_r$  where it is non-negative. Then there exists  $\mu \in (0, 1)$  such that for all  $0 < \sigma < r$ ,*

$$\text{osc}(u, \sigma) \lesssim \left(\frac{\sigma}{r}\right)^\mu [\text{osc}(u, r) + \lambda_r].$$

In the next theorem we could alternatively use the  $p$ -average on the left-hand side (as in earlier papers like [60]), but we use this simpler formulation since it is all we need.

**Theorem 7.2 (Morrey type estimate).** *Let  $\varphi \in \Phi_w(\Omega)$  and  $u \in W_{\text{loc}(\Omega)}^{1,\varphi}$  satisfy Assumption 3.2. Let  $u$  be a local quasiminimizer of  $\mathcal{F}$ . Then for any  $Q_{2r} \subset \Omega$  with  $(\Lambda + 1)|Q_{2r}| \leq 1$ ,*

$$\int_{Q_\sigma} |\nabla u| \, dx \lesssim \left(\frac{\sigma}{r}\right)^{n+\mu-1} \int_{Q_r} |\nabla u| + (\varphi_{Q_r}^-)^{-1}(\Lambda + 1) \, dx$$

for all  $0 < \sigma < r$ , with  $\mu$  from Theorem 7.1.

*Proof.* It is enough to consider  $\sigma \in (0, \frac{r}{4})$ . By the Caccioppoli inequality (Lemma 4.6) with  $k = u_{Q_{2\sigma}}$ ,  $r = \sigma$ ,  $R = 2\sigma$ , we have that

$$\begin{aligned} \int_{Q_\sigma} \varphi(x, |\nabla(u - u_{Q_\sigma})_+|) \, dx &\lesssim \int_{Q_{2\sigma}} \varphi\left(x, \frac{|u - u_{Q_\sigma}|}{\sigma}\right) \, dx + \Lambda \\ &\leq \int_{Q_{2\sigma}} \varphi_{Q_{2\sigma}}^+ \left(\frac{\text{osc}(u, 2\sigma)}{\sigma}\right) \, dx + \Lambda \\ &= \varphi_{Q_{2\sigma}}^+ \left(\frac{\text{osc}(u, 2\sigma)}{\sigma}\right) + \Lambda. \end{aligned}$$

Since  $u$  is a quasiminimizer of  $\mathcal{F}$ ,  $-u$  is a quasiminimizer of the functional  $\mathcal{F}$  with  $F$  replaced by  $F(x, -t, -z)$ . Hence the Caccioppoli estimate for  $-u$  similarly implies an estimate for  $|\nabla(u - u_{Q_{2\sigma}})_-|$ . Combining these two estimates we obtain

$$\begin{aligned} \int_{Q_\sigma} \varphi(x, |\nabla u|) \, dx &= \int_{Q_\sigma} \varphi(x, |\nabla(u - u_{Q_\sigma})|) \, dx \\ &\lesssim \varphi_{Q_{2\sigma}}^+ \left(\frac{\text{osc}(u, 2\sigma)}{\sigma}\right) + \Lambda. \end{aligned} \tag{7.1}$$



In the case (A1), we use Corollary 5.3 for  $u - u_{Q_{\tau/2}}$  and  $u_{Q_{\tau/2}} - u$  with  $h = 1$  and the  $W^{1,1}$ -Poincaré inequality, to derive that

$$\begin{aligned} \frac{\text{osc}(u, \tau/2)}{\tau} &\leq \frac{\sup_{Q_{\tau/2}}(u - u_{Q_{\tau/2}})_+}{\tau} + \frac{\sup_{Q_{\tau/2}}(u_{Q_{\tau/2}} - u)_+}{\tau} \\ &\lesssim \frac{1}{\tau} \int_{Q_\tau} |u - u_{Q_{\tau/2}}| dx + \frac{1}{\tau} \lambda_\tau \\ &\lesssim \int_{Q_\tau} |\nabla u| dx + (\varphi_{Q_\tau}^-)^{-1}(\Lambda + 1). \end{aligned} \tag{7.2}$$

By Lemma 3.1, (aDec),  $\varrho_{L^\varphi(Q_r)}(|\nabla u|) \leq 1$  and  $(1 + \Lambda)|Q_r| \leq 1$  it follows from this that

$$\frac{\text{osc}(u, \tau/2)}{\tau} \lesssim (\varphi_{Q_\tau}^-)^{-1} \left( \frac{1}{|Q_\tau|} \right)$$

for any  $0 < \tau \leq r$ . We first use this estimate with  $\tau = 4\sigma$ . By (A1), (A0) and (aDec), we conclude that

$$\varphi_{Q_{2\sigma}}^+ \left( \frac{\text{osc}(u, 2\sigma)}{\sigma} \right) \lesssim \varphi_{Q_{2\sigma}}^- \left( \frac{\text{osc}(u, 2\sigma)}{\sigma} \right) + 1.$$

In the case of bounded  $u$  and (A1- $n$ ), we obtain the same conclusion by (A1- $n$ ), (A0) and (aDec), since  $\frac{\text{osc}(u, 2\sigma)}{\sigma} \leq \frac{2\|u\|_{L^\infty}}{\sigma}$ . Thus (7.1) gives

$$\int_{Q_\sigma} \varphi(x, |\nabla u|) dx \lesssim \varphi_{Q_{2\sigma}}^- \left( \frac{\text{osc}(u, 2\sigma)}{\sigma} \right) + \Lambda + 1.$$

Since  $u$  is a local quasiminimizer of  $\mathcal{F}$  with  $F(x, t, z)$ , it follows that  $\pm u - k$  is a local quasiminimizer of the functional  $\mathcal{F}$  with  $F(x, \pm(t + k), \pm z)$ . Hence by Theorem 4.1 we can use Theorem 7.1. The later theorem and (7.2) with  $\tau = r$  yield:

$$\begin{aligned} \int_{Q_\sigma} \varphi(x, |\nabla u|) dx &\lesssim \varphi_{Q_{2\sigma}}^- \left( \left( \frac{2\sigma}{r/2} \right)^{\mu-1} \left[ \frac{\text{osc}(u, r/2)}{r} + (\varphi_{Q_r}^-)^{-1}(\Lambda + 1) \right] \right) + \Lambda + 1 \\ &\approx \varphi_{Q_{2\sigma}}^- \left( \left( \frac{\sigma}{r} \right)^{\mu-1} \left[ \frac{\text{osc}(u, r/2)}{r} + (\varphi_{Q_r}^-)^{-1}(\Lambda + 1) \right] \right) \\ &\lesssim \varphi_{Q_{2\sigma}}^- \left( \left( \frac{\sigma}{r} \right)^{\mu-1} \left[ \int_{Q_r} |\nabla u| dx + (\varphi_{Q_r}^-)^{-1}(\Lambda + 1) \right] \right), \end{aligned}$$

where in the second step we use (2.1). Since  $\varphi$  satisfies (aInc)<sub>1</sub>, Lemma 3.1 and (aDec) imply that

$$\varphi_{Q_{2\sigma}}^- \left( \int_{Q_\sigma} |\nabla u| dx \right) \lesssim \int_{Q_\sigma} \varphi_{Q_{2\sigma}}^- (|\nabla u|) dx \lesssim \int_{Q_\sigma} \varphi(x, |\nabla u|) dx.$$

We use this on the left-hand side of the earlier estimate together with (2.1) to obtain the claim. □

### 8. Continuity of $\omega$ -minimizers

We assume now that the function  $F$  satisfies

$$v \varphi(x, |z|) \leq F(x, t, z) \leq N (\varphi(x, |z|) + \Lambda_0)$$

for some constant  $\Lambda_0 \geq 0$ . Denote  $\psi(x, t) := \varphi(x, t) + t$ . By Lemma 2.2,  $\psi$  satisfies Assumption 3.2, provided  $\varphi$  satisfies the assumptions in Theorem 1.1. Furthermore,  $W^{1,\varphi} = W^{1,\psi}$  since we consider only bounded domains [39, Corollary 3.3.11].

The following is a well known variational principle due to Ekeland; see [29] or [35, Theorem 5.6, page 160] for its proof. Recall that  $f : X \rightarrow [-\infty, \infty]$  is *lower semicontinuous* if  $f(v) \leq \liminf_{k \rightarrow \infty} f(v_k)$  for every sequence  $v_k$  convergent to  $v \in X$ .

**Lemma 8.1 (Ekeland’s variational principle).** *Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow (-\infty, \infty]$  be lower semicontinuous with  $-\infty < \inf_X f < \infty$ . Suppose that*

$$f(u) \leq \inf_X f + \delta$$

for some  $\delta > 0$  and  $u \in X$ . Then there exists  $w \in X$  with  $d(u, w) \leq 1$  such that

$$f(w) \leq f(u) \text{ and } f(w) \leq f(v) + \delta d(w, v) \text{ for all } v \in X.$$

We use Ekeland’s variational principle in the space

$$X := \left\{ v \in u + W_0^{1,1}(Q_r) : \begin{array}{l} \int_{Q_r} \psi(x, |\nabla v|) dx \leq \int_{Q_r} \psi(x, |\nabla u|) dx \\ \text{and } \|v\|_{L^\infty(Q_r)} \leq \|u\|_{L^\infty(Q_r)} \end{array} \right\},$$

with the metric

$$d(v_1, v_2) := C_r \int_{Q_r} |\nabla v_1 - \nabla v_2| dx,$$

where  $C_r > 0$  is a constant which will be determined later. Moreover we define  $f : X \rightarrow \mathbb{R}$  by  $f(v) := \mathcal{F}(v, Q_r)$ . We first check the assumptions for Ekeland’s principle.

**Lemma 8.2.** *Let  $\varphi \in \Phi_w(\Omega)$ . Then  $(X, d)$  is a complete metric space. If  $(t, z) \rightarrow F(x, t, z)$  is continuous for every  $x$ , then  $f$  is lower semicontinuous.*

*Proof.* It is enough to prove that  $(X, d)$  is a closed subspace of  $(u + W_0^{1,1}(Q_r), d)$  since  $(u + W_0^{1,1}(Q_r), d)$  is a complete metric space. Let  $v_k$  be a sequence in  $X$  such that

$$\int_{Q_r} |\nabla v_k - \nabla v| dx \rightarrow 0 \text{ as } k \rightarrow \infty,$$

for some  $v \in u + W_0^{1,1}(Q_r)$ . Then we may assume, passing to a subsequence, if necessary, that  $v_k \rightarrow v$  and  $\nabla v_k \rightarrow \nabla v$  a.e. in  $Q_r$ . By [39, Lemma 2.1.6],  $\psi(x, \cdot)$  is lower semicontinuous. Therefore Fatou's lemma yields that

$$\begin{aligned} \int_{Q_r} \psi(x, |\nabla v|) dx &= \int_{Q_r} \psi(x, \lim_{k \rightarrow \infty} |\nabla v_k|) dx \leq \int_{Q_r} \liminf_{k \rightarrow \infty} \psi(x, |\nabla v_k|) dx \\ &\leq \liminf_{k \rightarrow \infty} \int_{Q_r} \psi(x, |\nabla v_k|) dx \leq \int_{Q_r} \psi(x, |\nabla u|) dx; \end{aligned}$$

the last step holds since  $v_k \in X$ . We also see that  $\|v\|_{L^\infty(Q_r)} \leq \liminf_{k \rightarrow \infty} \|v_k\|_{L^\infty(Q_r)} \leq \|u\|_{L^\infty(Q_r)}$ . Hence  $v \in X$ , and so  $(X, d)$  is closed.

For the same sequence we have that  $F(x, v_k(x), \nabla v_k(x)) \rightarrow F(x, v(x), \nabla v(x))$  for a.e.  $x \in Q_r$ . Then lower semicontinuity follows by Fatou's lemma.  $\square$

Notice that a weak quasiminimizer with bound  $\infty$  is the same thing as a local quasiminimizer. Thus we can cover both the bounded and unbounded case with the next lemma, where we show that there exists an approximating weak quasiminimizer in every cube  $Q_r$ .

**Lemma 8.3.** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (aDec) $^\infty$  and  $(t, z) \rightarrow F(x, t, z)$  be continuous. Let  $Q_{2r} \subset \Omega$  with  $|Q_r| \leq 1$ . Let  $u$  be an  $\omega$ -minimizer of  $\mathcal{F}$ . Then there exists a weak quasiminimizer  $w \in u + W_0^{1,\varphi}(Q_r)$  with bound  $\|u\|_{L^\infty(Q_r)}$  of the functional*

$$\int_{Q_r} \psi(x, |\nabla w|) + \Lambda dx \quad \text{with} \quad \Lambda := \int_{Q_r} \varphi(x, |\nabla u|) dx + \Lambda_0 + 1,$$

satisfying the estimates

$$\int_{Q_r} \psi(x, |\nabla w|) dx \lesssim \Lambda, \quad \|w\|_{L^\infty(Q_r)} \leq \|u\|_{L^\infty(Q_r)} \quad \text{and} \quad (8.1)$$

$$\int_{Q_r} |\nabla u - \nabla w| dx \leq \omega(r)(\psi_{Q_r}^-)^{-1}(\Lambda). \quad (8.2)$$

*Proof.* Let  $(X, d)$  and  $f$  be as above and choose  $C_r := [\omega(r)|Q_r|(\psi_{Q_r}^-)^{-1}(\Lambda)]^{-1}$ . For  $\varepsilon > 0$ , let  $v_\varepsilon \in X$  be such that  $f(v_\varepsilon) \leq \inf_X f + \varepsilon$ . Since  $u$  is an  $\omega$ -minimizer of  $\mathcal{F}$ ,

$$\begin{aligned} f(u) &\leq (1 + \omega(r))f(v_\varepsilon) \\ &\leq \inf_X f + \varepsilon + \omega(r)(\inf_X f + \varepsilon) \\ &\leq \inf_X f + \varepsilon + \omega(r) \left( N \int_{Q_r} \varphi(x, |\nabla u|) + \Lambda_0 dx + \varepsilon \right), \end{aligned}$$

from which, by letting  $\varepsilon \rightarrow 0^+$  we obtain

$$f(u) \leq \inf_X f + \omega(r)N \int_{Q_r} \varphi(x, |\nabla u|) + \Lambda_0 dx \leq \inf_X f + \omega(r)N|Q_r|\Lambda.$$

By Ekeland’s principle (Lemma 8.1), there exists  $w \in X$  with  $f(w) \leq f(u)$ ,

$$d(u, w) = C_r \int_{Q_r} |\nabla u - \nabla w| dx \leq 1$$

and

$$f(w) \leq f(v) + C_r \omega(r)N|Q_r|\Lambda \int_{Q_r} |\nabla w - \nabla v| dx$$

for all  $v \in X$ . Note that the former estimate is (8.2). Furthermore, (8.1) follows from  $w \in X$  and  $\psi \lesssim \varphi + 1$  used to estimate:

$$\int_{Q_r} \psi(x, |\nabla w|) dx \leq \int_{Q_r} \psi(x, |\nabla u|) dx \lesssim \int_{Q_r} \varphi(x, |\nabla u|) + 1 dx \leq \Lambda.$$

It remains to prove that  $w$  is a weak quasiminimizer of the  $\psi + \Lambda$  energy with bound  $\|u\|_{L^\infty(Q_r)}$ . Let  $v \in w + W_0^{1,1}(Q_r)$  with  $\|v\|_{L^\infty(Q_r)} \leq \|u\|_{L^\infty(Q_r)}$ . Assume first that  $v \notin X$ . Since  $v$  satisfies the  $L^\infty$ -bound by assumption, this means that  $\varrho_\psi(\nabla v) > \varrho_\psi(\nabla u)$ . By this and  $w \in X$ , we have

$$\int_{Q_r} \psi(x, |\nabla w|) + \Lambda dx \leq \int_{Q_r} \psi(x, |\nabla u|) + \Lambda dx < \int_{Q_r} \psi(x, |\nabla v|) + \Lambda dx.$$

We may cancel the integral over the set  $\{w = v\}$ , since  $\nabla w = \nabla v$  a.e. in it, so we have the quasiminimizing property in this case.

It remains to consider the case  $v \in X$ . By the structure conditions on  $\mathcal{F}$ , the estimate of  $f(w)$  above,  $\varphi \leq \psi$ , the definition of  $C_r$  and the triangle inequality, we conclude that

$$\begin{aligned} v \int_{Q_r} \varphi(x, |\nabla w|) dx &\leq f(w) \leq f(v) + C_r \omega(r)N|Q_r|\Lambda \int_{Q_r} |\nabla w - \nabla v| dx \\ &\leq N \int_{Q_r} \psi(x, |\nabla v|) + \Lambda_0 dx \\ &\quad + \frac{N\Lambda}{(\psi_{Q_r}^-)^{-1}(\Lambda)} \int_{Q_r} |\nabla w| + |\nabla v| dx. \end{aligned}$$

By [39, Lemma 2.2.1],  $\psi_{Q_r}^-$  is equivalent with a convex  $\xi \in \Phi_w$ . By [39, Theorem 2.4.10], we have  $\frac{\Lambda}{\xi^{-1}(\Lambda)} \approx (\xi^*)^{-1}(\Lambda)$ . It follows from Young’s inequality, (aInc)<sub>1</sub> and (2.1) that

$$\begin{aligned} \frac{\Lambda}{(\psi_{Q_r}^-)^{-1}(\Lambda)} t &\approx (\xi^*)^{-1}(\Lambda)t \leq \xi(\varepsilon t) + c_\varepsilon \xi^*((\xi^*)^{-1}(\Lambda)) \\ &\lesssim \varepsilon \xi(t) + c_\varepsilon \Lambda \approx \varepsilon \psi_{Q_r}^-(t) + c_\varepsilon \Lambda \end{aligned}$$

for any  $\varepsilon > 0$ . Using this for  $t = |\nabla w|$  and  $t = |\nabla v|$  as well as the estimate  $\frac{1}{c_1}\psi(x, |\nabla w|) - 1 \leq \varphi(x, |\nabla w|)$ , we conclude that

$$\begin{aligned} & \frac{1}{c_1} \int_{Q_r} \psi(x, |\nabla w|) dx - |Q_r| \\ & \leq \frac{N}{v} \int_{Q_r} \psi(x, |\nabla v|) + \Lambda_0 dx + \frac{N\Lambda}{v(\psi_{Q_r}^-)^{-1}(\Lambda)} \int_{Q_r} |\nabla w| + |\nabla v| dx \\ & \leq c_2 \int_{Q_r} \psi(x, |\nabla v|) + \Lambda_0 + \varepsilon \psi_{Q_r}^- (|\nabla w|) + \varepsilon \psi_{Q_r}^- (|\nabla v|) + c_\varepsilon \Lambda dx. \end{aligned}$$

We choose  $\varepsilon$  so small that  $c_2\varepsilon \leq \frac{1}{2c_1}$ . The  $\nabla w$ -term can be absorbed in the left-hand side and so it follows that

$$\frac{1}{2c_1} \int_{Q_r} \psi(x, |\nabla w|) dx \lesssim (c_2 + \frac{1}{2c_1})(c_\varepsilon + 1) \int_{Q_r} \psi(x, |\nabla v|) + \Lambda dx.$$

Hence  $w$  is a weak quasiminimizer of the  $\psi + \Lambda$  energy. □

Now we are ready to show that  $\omega$ -minimizers are locally Hölder continuous.

*Proof of Theorem 1.1.* Let  $Q_{2r} \subset \Omega$  be such that  $(\Lambda_0 + 1)|Q_{2r}| \leq 1$  and  $\varrho_{L^\varphi(Q_{2r})}(|\nabla u|) \leq 1$ . Let  $w \in W^{1,\varphi}(Q_r)$  be the weak quasiminimizer with bound  $\|u\|_{L^\infty(Q_r)}$  from Lemma 8.3.

Let us first estimate  $(\psi_{Q_r}^-)^{-1}(\Lambda)$  and denote  $\lambda_0 := (\psi_{Q_r}^-)^{-1}(\Lambda_0 + 1)$ . By the definition of  $\Lambda$ ,  $\varphi \leq \psi$  and (aDec) we have

$$(\psi_{Q_r}^-)^{-1}(\Lambda) \lesssim (\psi_{Q_r}^-)^{-1}\left(\int_{Q_r} \psi(x, |\nabla u|) dx\right) + \lambda_0.$$

By  $\psi \lesssim \varphi + 1$  and  $\varrho_{L^\varphi(Q_{2r})}(|\nabla u|) \leq 1$  we have  $\int_{Q_r} \psi(x, |\nabla u|) dx \lesssim \frac{1}{|Q_r|}$ , and hence (A0), (A1), (aDec) and (2.1) yield

$$(\psi_{Q_r}^-)^{-1}(\Lambda) \lesssim (\psi_{Q_r}^+)^{-1}\left(\int_{Q_r} \psi(x, |\nabla u|) dx\right) + \lambda_0.$$

Since  $u$  is a cubical minimizer of  $\mathcal{F}$ , we may use Lemma 4.4 and thus (4.1) holds. By Lemma 4.5, (aDec) and (2.1) we conclude that

$$(\psi_{Q_r}^-)^{-1}(\Lambda) \lesssim (\psi_{Q_r}^+)^{-1}\left(\psi_{Q_r}^+\left(\int_{Q_r} |\nabla u| dx\right) + \Lambda_0 + 1\right) + \lambda_0 \approx \int_{Q_r} |\nabla u| dx + \lambda_0.$$

In the case of (A1- $n$ ), we first use Lemma 3.1 with  $p = 1$ , then the estimate (4.3), and finally (A0) and the boundedness of  $u$ :

$$\begin{aligned} \psi_{Q_r}^- \left( \int_{Q_r} |\nabla u| dx \right) & \leq \int_{Q_r} \psi(x, |\nabla u|) dx \lesssim \int_{Q_{2r}} \psi\left(x, \frac{|u - u_{Q_{2r}}|}{r}\right) dx + \Lambda_0 \\ & \lesssim \psi_{Q_{2r}}^- \left( \frac{1}{r} \|u\|_{L^\infty(Q_r)} \right) + \Lambda_0 + 1 \lesssim (\Lambda_0 + 1) \psi_{Q_r}^- \left( \frac{1}{r} \right). \end{aligned}$$

Thus we have  $\int_{Q_r} |\nabla u| dx \lesssim \frac{1}{r}$  with implicit constant depending on  $\Lambda_0$  and hence by (A1-n), (aDec) and (A0) we have

$$\psi_{Q_r}^+ \left( \int_{Q_r} |\nabla u| dx \right) \lesssim \psi_{Q_r}^- \left( \int_{Q_r} |\nabla u| dx \right) + 1.$$

Since  $u$  is a cubical minimizer of  $\mathcal{F}$ , we obtain by Lemma 4.5, (aDec), the previous estimate and (2.1) that

$$\begin{aligned} (\psi_{Q_r}^-)^{-1}(\Lambda) &\lesssim (\psi_{Q_r}^-)^{-1} \left( \int_{Q_r} \psi(x, |\nabla u|) dx \right) + \lambda_0 \\ &\lesssim (\psi_{Q_r}^-)^{-1} \left( c\psi_{Q_r}^+ \left( \int_{Q_r} |\nabla u| dx \right) + \Lambda_0 + 1 \right) + \lambda_0 \\ &\lesssim \int_{Q_r} |\nabla u| dx + \lambda_0. \end{aligned}$$

Thus we have the same estimate for  $(\psi_{Q_r}^-)^{-1}(\Lambda)$  in both cases.

By (8.2), we obtain that

$$\begin{aligned} \int_{Q_r} |\nabla u - \nabla w| dx &\lesssim \omega(r) (\psi_{Q_r}^-)^{-1}(\Lambda) \\ &\lesssim \omega(r) \int_{Q_r} |\nabla u| + \lambda_0 dx. \end{aligned}$$

By Lemma 3.1 and (8.1),

$$\begin{aligned} \int_{Q_r} |\nabla w| dx &\lesssim (\psi_{Q_r}^-)^{-1} \left( \int_{Q_r} \psi(x, |\nabla w|) dx \right) \\ &\lesssim (\psi_{Q_r}^-)^{-1}(\Lambda) \lesssim \int_{Q_r} |\nabla u| + \lambda_0 dx. \end{aligned}$$

On the other hand, from the Morrey estimate (Theorem 7.2) and Remark 6.4, we have, for any  $0 < \sigma < r$ , that

$$\int_{Q_\sigma} |\nabla w| dx \lesssim \left( \frac{\sigma}{r} \right)^{n+\mu-1} \int_{Q_r} |\nabla w| + \lambda_0 dx.$$

Furthermore, since  $\mu \in (0, 1)$ ,  $\int_{Q_\sigma} \lambda_0 dx \lesssim \left( \frac{\sigma}{r} \right)^{n+\mu-1} \int_{Q_r} \lambda_0 dx$ . Combining these estimates, we find for  $0 < \sigma < r$ , that

$$\begin{aligned} Z(\sigma) &:= \int_{Q_\sigma} |\nabla u| + \lambda_0 dx \lesssim \int_{Q_\sigma} |\nabla u - \nabla w| + |\nabla w| + \lambda_0 dx \\ &\lesssim \left[ \omega(r) + \left( \frac{\sigma}{r} \right)^{n+\mu-1} \right] \int_{Q_r} |\nabla u| + \lambda_0 dx. \end{aligned}$$

Set  $\theta := \frac{\sigma}{r}$ . Then the previous inequality can be written as

$$Z(\theta r) \leq c_1 [\omega(r) + \theta^{n+\mu-1}] Z(r).$$

We first fix  $\theta$  such that  $c_1 \theta^{n+\mu-1} = \frac{1}{2} \theta^{n+\frac{\mu}{2}-1}$ . Then we choose  $r_0$  so small that  $c_1 \omega(r) \leq \frac{1}{2} \theta^{n-\frac{\mu}{2}+1}$  when  $r \in [0, r_0]$ . Then the inequality  $Z(\theta r) \leq \theta^{n+\frac{\mu}{2}-1} Z(r)$  holds for all  $r \leq r_0$ . Thus it follows from [35, Lemma 7.3, page 229] that

$$\int_{Q_\sigma} |\nabla u| + \lambda_0 \, dx \lesssim \left(\frac{\sigma}{r}\right)^{n+\frac{\mu}{2}-1} \int_{Q_r} |\nabla u| + \lambda_0 \, dx$$

for all  $r \leq r_0$  and  $\sigma \leq \tau r$ . This and the Poincaré inequality imply that

$$\sigma^{-n-\frac{\mu}{2}} \int_{Q_\sigma} |u - u_{Q_\sigma}| \, dx \lesssim \sigma^{-n-\frac{\mu}{2}+1} \int_{Q_\sigma} |\nabla u| \, dx \lesssim 1$$

for all cubes  $Q_\sigma \subset Q_r$  with  $\sigma \leq \tau r$ . For cubes  $Q_\sigma$  with  $\sigma > \tau r$  the claim is trivial. Thus  $u$  belongs to the Campanato space  $\mathcal{L}^{1, n+\frac{\mu}{2}}(Q_r)$ . This implies by the Campanato–Hölder embedding [35, Theorem 2.9, page 52] that  $u \in C_{\text{loc}}^{0, \frac{\mu}{2}}(\overline{Q_r})$ .  $\square$

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