Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IV (2005), 703-728

Solutions for Toda systems on Riemann surfaces

JIAYU LI AND YUXIANG LI

Dedicated to Professor Ding Weiyue on the occasion of his 60th birthday

Abstract. In this paper we study the solutions of Toda systems on Riemann surface in the critical case, proving a sufficient condition for existence.

Mathematics Subject Classification (2000): 35J60 (primary); 58G03, 35J45 (secondary).

1. Introduction

Let (Σ, g) be a compact Riemann surface with unit area 1. Ding-Jost-Li-Wang [8] studied the differential equation $\Delta u = 8\pi - 8\pi h e^u$ on (Σ, g), the so-called Kazdan-Warner problem [17] related to the Abelian Chern-Simons model (see [3, 4, 6, 5, 7, 1, 2, 9, 10, 11, 23, 13, 14, 24, 21], etc). They pursued a variational approach to the problem, trying to minimize the functional

$$J(u)=\frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g + 8\pi\int_\Sigma u\, dV_g - 8\pi \log\int_\Sigma h e^u dV_g \geq C, \quad \text{in } H^{1,2}(\Sigma) \tag{1.1}$$

for some constant $C > 0$. Because it is the critical case of the Moser-Trudinger inequality (1.1), the analysis is subtle.

Let K denote the Cartan matrix for $\mathrm{SU}(N+1)$, *i.e.*

$$K = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \dots & \dots & & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

The research was supported by NSFC.
Pervenuto alla Redazione il 27 aprile 2005 e in forma definitiva il 2 novembre 2005.

In this paper we consider the Toda systems on (Σ, g) which are related to the non-Abelian Chern-Simons model [22]:

$$-\Delta u_i = M_i \left(\frac{\exp(\sum_{j=1}^N a_{ij} u_j)}{\int_\Sigma \exp(\sum_{j=1}^N a_{ij} u_j)} - 1 \right), \quad \text{for } 1 \le i \le N.$$

If $M_i < 4\pi$ Jost-Wang [16] proved the existence of solutions and in the case where Σ is a torus, $N = 2$, $\max\{M_1, M_2\} > 4\pi$ and $\min\{M_1, M_2\} \neq 4\pi$, Marcello-Margherita [20] proved the same result.

They studied the problem by considering for $u_1, \dots, u_N \in H^{1,2}(\Sigma)$ the functional

$$\begin{aligned} \Phi_{(M_1,\dots,M_N)}(u_1, \dots, u_N) = &\frac{1}{2} \sum_{i,j=1}^N \int_\Sigma a_{ij} (\nabla u_i \nabla u_j + 2M_i u_j) dV_g \\ &- \sum_{i=1}^N M_i \log \int_\Sigma \exp\left(\sum_{j=1}^N a_{ij} u_j \right) dV_g. \end{aligned} \tag{1.2}$$

Jost-Wang [16] proved that the functional has a lower bound if and only if

$$M_i \le 4\pi, \quad \text{for } i = 1, 2, \dots, N.$$

Marcello-Margherita [20] obtained a non-minimizing critical point of the functional motivated by an earlier paper of Struwe-Tarantello [23]. The idea was later also used by Djadli and Malchiodi [12] to study the existence of conformal metrics with constant Q-curvature. It is clear that $M_i = 4\pi$ is the critical case of the functional. Whether it admits minimizer is subtle. In this paper we study this problem. For simplicity, we consider only the case that $N = 2$, the general case need only more calculations. In our case the functional is

$$\begin{aligned} \Phi(u_1, u_2) = &\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a_{ij} (\nabla u_i \nabla u_j + 8\pi u_j) dV_g \\ &- \sum_{i=1}^2 4\pi \log \int_\Sigma \exp\left(\sum_{j=1}^2 a_{ij} u_j \right) dV_g, \end{aligned}$$

and the Toda systems is

$$-\Delta u_i = 4\pi \left(\frac{\exp\left(\sum_{j=1}^2 a_{ij} u_j\right)}{\int_\Sigma \exp\left(\sum_{j=1}^2 a_{ij} u_j\right)} - 1 \right), \text{ for } 1 \le i \le 2, \tag{1.3}$$

where $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$. The systems is equivalent to

$$-\Delta u_i = 4\pi \sum_{j=1}^2 a_{ij} \left(\frac{\exp(u_j)}{\int_\Sigma \exp(u_j)} - 1 \right), \text{ for } 1 \le i \le 2,$$

where $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$.

Our main result is as follows:

Main Theorem. *Let* Σ *be a compact Riemann surface with area* 1. *If the Gauss curvature* K *of* Σ *satisfies that*

$$\max_{p\in\Sigma} K(p) < 2\pi, \tag{1.4}$$

then $\Phi(u_1, u_2)$ *has a minimizer.*

We consider the sequence of minimizers $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)$ of $\Phi_{(4\pi-\epsilon,4\pi-\epsilon)}$ for small $\epsilon > 0$. Then u^ϵ satisfies a Toda type systems. If u^ϵ converges to $u^0 = (u_1^0, u_2^0)$ in $H_2 := H^{1,2}(\Sigma) \times H^{1,2}(\Sigma)$, then it is clear that $\Phi(u^0) = \inf_{u\in H_2} \Phi(u)$, *i.e.*, u^0 is a minimizer of Φ. If u^ϵ does not converge in H_2, in this case, we say that u^ϵ blows up. Then there are two cases happened according to Jost-Wang's result. For each case, we derive a delicate lower bound of Φ which is one of the main points in this paper. We apply capacity to calculate the lower bound, so that we need not know details in the neck. Such a trick has been used by the second author of this paper in [18], [19] to prove the existence of extremal functions for the classical Moser-Trudinger inequality on a compact manifold. Another main point of this paper is the delicate constructions of blowing up sequences ϕ^ϵ in both cases, so that $\Phi(\phi^\epsilon)$ are strictly less than the lower bound derived before, and consequently we get a contradiction to the assumption that u^ϵ blows up, which proves our main theorem.

After submitting the present article we were informed of the existence of the paper [15], where a result similar to our one is established. As a matter of fact, Jost, Lin and Wang consider the more general equation

$$-\Delta u_i = 4\pi \sum_{j=1}^{2} a_{ij} \left(\frac{h_j \exp(u_j)}{\int_\Sigma h_j \exp(u_j)} - 1 \right) \quad \text{for } 1 \le i \le 2,$$

where $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$, and h_1, h_2 are positive smooth functions on Σ. Our equation corresponds to the case where $h_1 \equiv h_2$ is constant. Our approach to the proof is completely different from theirs, and it appears that the more general case considered by them could also be treated with our method. In addition, there are hopes that our techniques could be used to face other non-linear existence problems.

ACKNOWLEDGEMENTS. We thank the referee for his many helpful comments.

2. Review of known results

For any $u = (u_1, u_2) \in H^{1,2}(\Sigma) \times H^{1,2}(\Sigma)$, we set

$$\Phi_\epsilon(u) = \frac{1}{3}\int_\Sigma \Big(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2 + 3(4\pi - \epsilon)u_1 + 3(4\pi - \epsilon)u_2\Big)\, dV_g$$
$$-(4\pi - \epsilon)\log\int_\Sigma e^{u_1} dV_g - (4\pi - \epsilon)\log\int_\Sigma e^{u_2} dV_g.$$

It is not difficult to check that

$$\Phi_{(4\pi-\epsilon, 4\pi-\epsilon)}(v) = \Phi_\epsilon(u),$$

if we set $v_1 = \frac{2u_1+u_2}{3}$ and $v_2 = \frac{u_1+2u_2}{3}$.

By Jost-Wang's result ([16] Corollary 4.6), one sees that Φ_ϵ has a minimizer u^ϵ of the functional $\Phi_\epsilon(u)$, *i.e.* we can find $u^\epsilon \in H^{1,2}(\Sigma) \times H^{1,2}(\Sigma)$ such that

$$\Phi_\epsilon(u^\epsilon) = \inf \Phi_\epsilon(u).$$

Without loss of generality, we may assume that

$$\int e^{u_1^\epsilon} dV_g = \int e^{u_2^\epsilon} dV_g = 1.$$

Then we have the following equations:

$$\begin{cases} -\Delta u_1^\epsilon = (8\pi - 2\epsilon)e^{u_1^\epsilon} - (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) \\ -\Delta u_2^\epsilon = (8\pi - 2\epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon)e^{u_1^\epsilon} - (4\pi - \epsilon) \end{cases}$$

For $i = 1, 2$, let

$$S_i = \{x \in \Sigma : \text{there is a sequence } y^\epsilon \to x \text{ s.t. } u_i^\epsilon(y^\epsilon) \to +\infty\}.$$

Jost-Wang [16] (section 5) proved that there will be two possibilities:

Case 1. $S_1 = \{p_1\}$, and $S_2 = \{p_2\}$, where p_1 p_2 are two different points in Σ.

In this case, we set, for $i = 1, 2$,

$$m_i^\epsilon = u_i^\epsilon(x_i^\epsilon) = \max u_i^\epsilon, \quad (r_i^\epsilon)^2 = e^{-m_i^\epsilon}, \quad \bar{u}_i^\epsilon = \int_\Sigma u_i^\epsilon dV_g.$$

Let $(\Omega_i, x = (x^1, x^2))$ be an isothermal coordinate system around p_i $(i = 1, 2)$, and we assume the metric to be

$$g|_{\Omega_i} = e^{\varphi_i}((dx^1)^2 + (dx^2)^2)$$

with $\varphi_i(0) = 0$, $i = 1, 2$.

We set, for $i = 1, 2$, $\Omega_i^\epsilon = \{x \in \mathbb{R}^2 : x_i^\epsilon + r_i^\epsilon x \in \Omega_i\}$, which expands to the whole $\mathbb{R}^2$. In Ω_1^ϵ, we have the equations:

$$-\Delta_0(u_1^\epsilon(x_1^\epsilon + r_1^\epsilon x) - m_1^\epsilon) = e^{-\varphi_1(x_1^\epsilon + r_1^\epsilon x)}\Big((8\pi - 2\epsilon)e^{u_1^\epsilon(x_1^\epsilon + r_1^\epsilon x) - m_1^\epsilon} - (r_1^\epsilon)^2(4\pi - \epsilon)e^{u_2^\epsilon(x_1^\epsilon + r_1^\epsilon x)} - (r_1^\epsilon)^2(4\pi - \epsilon)\Big),$$

where $-\Delta_0 = \frac{\partial^2}{\partial^2 x^1} + \frac{\partial^2}{\partial^2 x^2}$. Since u_2^ϵ are bounded from above in Ω_1^ϵ, it follows from the Harnack inequality and the elliptic estimates that u_1^ϵ converges in $C^k_{\mathrm{loc}}(\mathbb{R}^2)$ for any k to the function w which satisfies the equation

$$\begin{cases} -\Delta w = 8\pi e^w, & \forall x \in \mathbb{R}^2 \\ w(x) \le w(0) = 0, & \text{and } \displaystyle\int_{\mathbb{R}^2} e^w dx \le 1. \end{cases}$$

Hence, by the result in [7], we know that

$$w = -2\log(1 + \pi|x|^2).$$

In the same way, $u_2^\epsilon(x_2^\epsilon + r_2^\epsilon x) - m_2^\epsilon$ converges to w.

Setting $\bar{u}_i^\epsilon = \int_\Sigma u_i^\epsilon dV_g$, we have the following proposition (see Lemma 5.6, and the proof of Theorem 3.1 in [16]).

Proposition 2.1. *We have $\bar{u}_j^\epsilon \to -\infty$ for $j = 1, 2$. Furthermore, for any $q \in (1, 2)$, we have*

$$u_j^\epsilon - \bar{u}_j^\epsilon \text{ converges to } G_j \text{ in } H^{1,q}(\Sigma),$$

where G_1 and G_2 satisfy

$$\begin{cases} -\Delta G_1 &= 8\pi\delta_{p_1} - 4\pi\delta_{p_2} - 4\pi, \\ -\Delta G_2 &= 8\pi\delta_{p_2} - 4\pi\delta_{p_1} - 4\pi, \\ \displaystyle\int_\Sigma G_j dV_g &= 0, \quad \text{for } j = 1, 2 \end{cases}$$

where δ_y is the Dirac distribution. Moreover,

$$u_j^\epsilon - \bar{u}_j^\epsilon \text{ converges to } G_j \text{ in } C^2_{\mathrm{loc}}(\Sigma \backslash \{p_1, p_2\}).$$

Remark 2.2. It is easy to see that, in Ω_1,

$$G_1 = -4\log r + A_1(p_1) + f_1, \quad \text{and} \quad G_2 = 2\log r + A_2(p_1) + g_1 \tag{2.1}$$

where $r^2 = x_1^2 + x_2^2$, $A_i(p_1)$ $(i = 1, 2)$ are constants, and f_1, g_1 are smooth functions which are zero at 0. Similarly, in Ω_2, we can write

$$G_1 = 2\log r + A_1(p_2) + f_2, \quad \text{and} \quad G_2 = -4\log r + A_2(p_2) + g_2 \tag{2.2}$$

where $A_i(p_1)$ $(i = 1, 2)$ are constants, and f_2, g_2 are smooth functions which are zero at 0.

Case 2. $S_1 = \{p\}$, and $S_2 = \emptyset$.

In this case, u_2^ϵ are bounded from above. Let $(\Omega; x)$ be an isothermal coordinate system around p, similar to the Case 1, we have

$$u_1^\epsilon(x_1^\epsilon + r_1^\epsilon x) - m_1^\epsilon \to -2\log(1 + \pi|x|^2).$$

We also have the following proposition (*c.f.* [16]):

Proposition 2.3. *Let $\bar{u}_1^\epsilon$ be the average of u_1^ϵ. We have $\bar{u}_1^\epsilon \to -\infty$. Furthermore, for any $q \in (1, 2)$, we have*

$$u_1^\epsilon - \bar{u}_1^\epsilon \text{ converges to } G_1 \text{ in } H^{1,q}(\Sigma),$$

and

$$u_2^\epsilon \text{ converges to } G_2 \text{ in } H^{1,q}(\Sigma),$$

where G_1 and G_2 satisfy

$$\begin{cases} -\Delta G_1 &= 8\pi\delta_p - 4\pi e^{G_2} - 4\pi, \\ -\Delta G_2 &= 8\pi e^{G_2} - 4\pi\delta_p - 4\pi, \\ \int_\Sigma G_1 dV_g = 0, & \int_\Sigma e^{G_2} dV_g = 1, \quad \sup_{x\in\Sigma} G_2 < +\infty \end{cases} \tag{2.3}$$

where δ_y is the Dirac distribution. Moreover,

$$u_1^\epsilon - \bar{u}_1^\epsilon \text{ converges to } G_1, \text{ and } u_2^\epsilon \text{ converges to } G_2 \text{ in } C^2_{\text{loc}}(\Sigma\backslash\{p\}).$$

Since G_2 is bounded from above, we can deduce from the equation (2.3) that $G_2 = 2\log r + h$ in Ω, where $h \in H^{2,q}_{\text{loc}}(\Omega)$ for any $q > 0$. Then $e^{G_2} = r^2 e^h \in C^1_{\text{loc}}(\Omega)$, and then $\Delta_0 h \in C^1_{\text{loc}}(\Omega)$. Therefore, by the standard elliptic estimates, $G_2 - 2\log r$ is smooth in Ω. So, we can write

$$G_1 = -4\log r + A_1(p) + f, \quad \text{and} \quad G_2 = 2\log r + A_2(p) + g \tag{2.4}$$

where $r^2 = x_1^2 + x_2^2$, $A_i(p)$ $(i = 1, 2)$ are constants and f, g are smooth functions which are zero at 0.

3. The lower bound for Case 1

We assume that $\Omega_1 \cap \Omega_2 = \emptyset$, and $B_r(p_1) \subset \Omega_1$. We set $v_2^\epsilon = \frac{1}{3}(2u_2^\epsilon + u_1^\epsilon) - \frac{1}{3}(2\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)$. Then, in $B_r(p_1)$, we have

$$\begin{cases} -\Delta v_2^\epsilon = (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) \in L^\infty(B_r(p_1)) \\ v_2^\epsilon|_{\partial B_r(p_1)} \to \dfrac{1}{3}(2G_2 + G_1). \end{cases}$$

So $\|v_2^\epsilon\|_{C^1} \le C$, where C is a constant depending only on r.

By a direct calculation one gets

$$\frac{1}{3}\int_{B_\delta(x_1^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g = \frac{1}{4}\int_{B_\delta(x_k)}(|\nabla u_1^\epsilon|^2+3|\nabla v_2^\epsilon|^2)dV_g$$
$$= \frac{1}{4}\int_{B_\delta(x_1^\epsilon)}|\nabla u_1^\epsilon|^2 dV_g + O(\delta^2).$$

Recall that $u_1(x_1^\epsilon + r_1^\epsilon x) - m_1^\epsilon \to w$ in $C^k(B_L(0))$, for any k, we have

$$\frac{1}{4}\int_{B_\delta(x_1^\epsilon)}|\nabla u_1^\epsilon|^2 dV_g = \frac{1}{4}\int_{B_L}|\nabla w|^2 dx$$
$$+\frac{1}{4}\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla u_1^\epsilon|^2 dV_g + o(1) + O(\delta^2).$$

Let

$$a_1^\epsilon = \inf_{\partial B_{Lr_1^\epsilon}(x_1^\epsilon)} u_1^\epsilon, \quad b_1^\epsilon = \sup_{\partial B_\delta(x_1^\epsilon)} u_1^\epsilon.$$

We set $a_1^\epsilon - b_1^\epsilon = m_1^\epsilon - \bar{u}_1^\epsilon + d_1^\epsilon$. It is clear that, for fixed L and δ,

$$d_1^\epsilon \to w(L) - \sup_{\partial B_\delta(p_1)} G_1 \quad as \quad \epsilon \to 0.$$

Let $f_1^\epsilon = \max\{\min\{u_1^\epsilon, a_1^\epsilon\}, b_1^\epsilon\}$. We get

$$\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla u_1^\epsilon|^2 dV_g \ge \int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla f_1^\epsilon|^2 dV_g$$
$$= \int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla_0 f_1^\epsilon|^2 dx$$
$$\ge \inf_{\Psi|_{\partial B_{Lr_1^\epsilon}(0)}=a_1^\epsilon,\Psi|_{\partial B_\delta(0)}=b_1^\epsilon}\int_{B_\delta(0)\setminus B_{Lr_1^\epsilon}(0)}|\nabla_0\Psi|^2 dx\,.$$

Here, $|\nabla_0 g|^2 = |\frac{\partial g}{\partial x_1}|^2 + |\frac{\partial g}{\partial x_2}|^2$. It is well-known that

$$\inf_{\Psi|_{\partial B_{Lr_1^\epsilon}}=a_1^\epsilon,\Psi|_{\partial B_\delta}=b_1^\epsilon}\int_{B_\delta\setminus B_{Lr_1^\epsilon}}|\nabla_0\Psi|^2 dx$$

is uniquely attained by the function ϕ which satisfies the equation

$$\begin{cases} -\Delta_0\phi = 0 \\ \phi|_{\partial B_{Lr_1^\epsilon}} = a_1^\epsilon, \phi|_{\partial B_\delta} = b_1^\epsilon. \end{cases}$$

Hence,

$$\phi = \frac{a_1^\epsilon - b_1^\epsilon}{-\log Lr_1^\epsilon + \log\delta}\log r + \frac{a_1^\epsilon \log\delta - b_1^\epsilon \log Lr_k}{-\log Lr_1^\epsilon + \log\delta},$$

and then

$$\int_{B_\delta(0)\setminus B_{Lr_1^\epsilon}(0)} |\nabla_0\phi|^2 dx = \frac{4\pi(a_1^\epsilon - b_1^\epsilon)^2}{-\log(Lr_1^\epsilon)^2 + \log\delta^2}.$$

Therefore, we have

$$\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} |\nabla u_1^\epsilon|^2 dV_g \geq \frac{4\pi(m_1^\epsilon - \bar{u}_1^\epsilon + d_1^\epsilon)^2}{-\log L^2 - \log(r_1^\epsilon)^2 + \log\delta^2}.$$

Recalling that $-\log(r_1^\epsilon)^2 = m_1^\epsilon$, we get

$$\begin{aligned}\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} |\nabla u_1^\epsilon|^2 dV_g &\geq 4\pi\frac{(m_1^\epsilon - \bar{u}_1^\epsilon + d_1^\epsilon)^2}{m_1^\epsilon}\left(1 - \frac{\log L^2 - \log\delta^2}{m_1^\epsilon}\right)^{-1} \\ &\geq 4\pi\frac{(m_1^\epsilon - \bar{u}_1^\epsilon + d_1^\epsilon)^2}{m_1^\epsilon}\left(1 + \frac{\log L^2 - \log\delta^2}{m_1^\epsilon} + \frac{A}{(m_1^\epsilon)^2}\right) \\ &\geq 4\pi\frac{(m_1^\epsilon - \bar{u}_1^\epsilon)^2}{m_1^\epsilon} + 8\pi d_1^\epsilon\left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right) \\ &\quad + 4\pi\left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right)^2(\log L^2 - \log\delta^2) + \frac{A'\bar{u}_1^\epsilon}{(m_1^\epsilon)^2},\end{aligned}$$

where A and A' are constants which depend only on δ and L.

Then we have

$$\begin{aligned}\frac{1}{3}\int_{B_\delta(x_1^\epsilon)} (|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2) dV_g &\geq \frac{1}{4}\int_{B_L} |\nabla w|^2 dx \\ +\pi\frac{(m_1^\epsilon - \bar{u}_1^\epsilon)^2}{m_1^\epsilon} + 2\pi d_1^\epsilon\left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right) &+ \pi\left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right)^2(\log L^2 - \log\delta^2) \\ + \frac{A'(\delta, L)\bar{u}_1^\epsilon}{(m_1^\epsilon)^2} + o(1) + O(\delta^2).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{3}\int_{B_\delta(x_2^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g\\
&\geq \frac{1}{4}\int_{B_L}|\nabla w|^2dx+\pi\frac{(m_2^\epsilon-\bar{u}_2^\epsilon)^2}{m_2^\epsilon}+2\pi d_1^\epsilon\left(1-\frac{\bar{u}_2^\epsilon}{m_2^\epsilon}\right)\\
&\quad+\pi\left(1-\frac{\bar{u}_2^\epsilon}{m_1^\epsilon}\right)^2(\log L^2-\log\delta^2)+\frac{A'\bar{u}_2^\epsilon}{(m_2^\epsilon)^2}+o(1)+O(\delta^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{3}\int_{B_\delta(x_1^\epsilon)\cup B_\delta(x_2^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g+(4\pi-\epsilon)\bar{u}_1^\epsilon+(4\pi-\epsilon)\bar{u}_2^\epsilon\\
&\geq\frac{1}{3}\int_{B_\delta(x_1^\epsilon)\cup B_\delta(x_2^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g+4\pi\bar{u}_1^\epsilon+4\pi\bar{u}_2^\epsilon\\
&\geq\frac{1}{2}\int_{B_L}|\nabla w|^2dx\\
&\quad+\sum_{i=1,2}\left(\pi\frac{(m_i^\epsilon+\bar{u}_i^\epsilon)^2}{m_i^\epsilon}+2\pi d_i^\epsilon\left(1-\frac{\bar{u}_i^\epsilon}{m_i^\epsilon}\right)+\pi\left(1-\frac{\bar{u}_2^\epsilon}{m_1^\epsilon}\right)^2(\log L^2-\log\delta^2)\right)\\
&\quad+\sum_{i=1,2}\frac{A'\bar{u}_i^\epsilon}{(m_i^\epsilon)^2}+o(1)+O(\delta^2)\\
&\geq\frac{1}{2}\int_{B_L}|\nabla w|^2dx\\
&\quad+\sum_{i=1,2}\left(\pi m_i^\epsilon\left(1+\frac{\bar{u}_i^\epsilon}{m_i^\epsilon}\right)^2+2\pi d_i^\epsilon\left(1-\frac{\bar{u}_i^\epsilon}{m_i^\epsilon}\right)+\pi\left(1-\frac{\bar{u}_2^\epsilon}{m_1^\epsilon}\right)^2(\log L^2-\log\delta^2)\right)\\
&\quad+\sum_{i=1,2}\frac{A'\bar{u}_i^\epsilon}{(m_i^\epsilon)^2}+o(1)+O(\delta^2).
\end{aligned}$$

We now set $s_i^\epsilon=1+\frac{\bar{u}_i^\epsilon}{m_i^\epsilon}$. Then, for fixed L, δ, we have

$$\begin{aligned}
&\frac{1}{3}\int_\Sigma(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g+(4\pi-\epsilon)\bar{u}_1^\epsilon+(4\pi-\epsilon)\bar{u}_2^\epsilon\\
&\qquad\qquad\geq\sum_i m_i^\epsilon\left(s_i^\epsilon+O\left(\frac{1}{m_i^\epsilon}\right)\right)^2+C.
\end{aligned}$$

Since $\Phi_\epsilon(u_\epsilon)\leq C$, we see that

$$|s_i^\epsilon|=O\left(\frac{1}{m_i^\epsilon}\right).$$

Hence for both $i = 1, 2$, $s_i^\epsilon \to 0$ as $\epsilon \to 0$. So,

$$\frac{1}{3}\int_{B_\delta(x_1^\epsilon)\cup B_\delta(x_2^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g+(4\pi-\epsilon)\bar{u}_1^\epsilon+(4\pi-\epsilon)\bar{u}_2^\epsilon$$

$$\geq \frac{1}{2}\int_{B_L}|\nabla w|^2dx+4\pi d_1^\epsilon+4\pi d_2^\epsilon+8\pi(\log L^2-\log\delta^2)+o(1)+O(\delta^2)$$

$$=\frac{1}{2}\int_{B_L}|\nabla w|^2dx+8\pi w(L)+8\pi(\log L^2-\log\delta^2)$$

$$-4\pi\sup_{\partial B_\delta(p_1)}G_1-4\pi\sup_{\partial B_\delta(p_2)}G_2+o(1)+O(\delta^2). \tag{3.1}$$

By a direct calculation, we obtain

$$\int_{B_L}|\nabla w|^2dx=16\pi\log(1+\pi L^2)-\frac{16\pi^2L^2}{1+\pi L^2}. \tag{3.2}$$

Moreover, by (2.1) and (2.2), we have

$$\frac{1}{3}\int_{B_\delta^c(x_1^\epsilon)\cap B_\delta^c(x_2^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g$$

$$=\frac{1}{3}\int_{B_\delta^c(x_1^\epsilon)\cap B_\delta^c(x_2^\epsilon)}(|\nabla G_1|^2+|\nabla G_2|^2+\nabla G_1\nabla G_2)dV_g+o(1)$$

$$=-\frac{1}{3}\sum_{i=1,2}\int_{\partial B_\delta(p_i)}\left(G_1\frac{\partial G_1}{\partial n}+G_2\frac{\partial G_2}{\partial n}+\frac{G_1\frac{\partial G_2}{\partial n}+G_2\frac{\partial G_1}{\partial n}}{2}\right)dS_g+o(1)$$

$$+\int_{B_\delta(p_1)+B_\delta(p_2)}2\pi(G_1+G_2)dV_g+o(1)$$

$$=-\frac{1}{3}\sum_{i=1,2}\int_0^{2\pi}\left(G_1\frac{\partial G_1}{\partial r}+G_2\frac{\partial G_2}{\partial r}+\frac{G_1\frac{\partial G_2}{\partial r}+G_2\frac{\partial G_1}{\partial r}}{2}\right)rd\theta|_{r=\delta}+o(1)$$

$$+\int_{B_\delta(p_1)+B_\delta(p_2)}2\pi(G_1+G_2)dV_g+o(1)$$

$$=-16\pi\log\delta-2\pi A_1(p_1)-2\pi A_2(p_2)+o(1)+O(\delta\log\delta). \tag{3.3}$$

In the end, (3.1), (3.2) and (3.3) imply that

$$\inf\Phi_0(u)\geq-8\pi\log\pi-8\pi-2\pi(A_1(p_1)+A_2(p_2)).$$

4. Lower bound for Case 2

In this case we set $v_2^\epsilon = \frac{1}{3}(2u_2^\epsilon + u_1^\epsilon) - \frac{1}{3}(2\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)$. Then we have

$$\begin{cases} -\Delta v_2^\epsilon = (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) \\ \int v_2^\epsilon = 0 \end{cases}$$

By the standard elliptic estimates, $||v_2||_{C^1(M)} < C$.

Similarly to Case 1, we have

$$\frac{1}{3}\int_{B_\delta(x_1^\epsilon)} (|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2) dV_g = \frac{1}{4}\int_{B_\delta(x_1^\epsilon)} |\nabla u_1^\epsilon|^2 dV_g + O(\delta^2),$$

and

$$\frac{1}{4}\int_{B_\delta(x_1^\epsilon)} |\nabla u_1|^2 dV_g + (4\pi - \epsilon)\bar{u}_1^\epsilon \geq \frac{1}{4}\int_{B_L} |\nabla w|^2 dx$$

$$+\left(\pi \frac{(m_1^\epsilon + \bar{u}_1^\epsilon)^2}{m_1^\epsilon} + 2\pi d_1^\epsilon \left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right) + \pi \left(1 - \frac{\bar{u}_1^\epsilon}{m_1^\epsilon}\right)^2 (\log L^2 - \log \delta^2)\right)$$

$$+\frac{A'\bar{u}_1^\epsilon}{(m_1^\epsilon)^2} + o(1) + O(\delta^2).$$

By an argument similar to that used in Case 1, we can show that $\frac{\bar{u}_1^\epsilon}{m_1^\epsilon} \to -1$, hence

$$\frac{1}{3}\int_{B_\delta(x_1^\epsilon)} (|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2) dV_g + (4\pi - \epsilon)\bar{u}_1^\epsilon$$

$$\geq \frac{1}{4}\int_{B_L} |\nabla w|^2 dx + 4\pi w(L)$$

$$+4\pi(\log L^2 - \log \delta^2) - 4\pi \sup_{\partial B_\delta(p_1)} G_1 + o(1) + O(\delta^2).$$

Set

$$G_1 = -4\log r + A_1(p) + o(x), \quad G_2 = 2\log r + A_2(p) + o(x).$$

Applying (2.4), we get

$$\begin{aligned}
&\frac{1}{3}\int_{B_\delta^c(x_1^\epsilon)}(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\nabla u_2)dV_g\\
&=\frac{1}{3}\int_{B_\delta^c(x_1^\epsilon)}\left(|\nabla G_1|^2+|\nabla G_2|^2+\frac{\nabla G_1\nabla G_2+\nabla G_2\nabla G_1}{2}\right)dV_g+o(1)\\
&=\int_{\partial B_\delta(p)}\left(G_1\frac{\partial G_1}{\partial n}+G_2\frac{\partial G_2}{\partial n}+\frac{G_1\frac{\partial G_2}{\partial n}+G_2\frac{\partial G_1}{\partial n}}{2}\right)\\
&\quad+2\pi\int_{B_\delta(p)}(G_1+G_2)dV_g-2\pi\int_\Sigma G_2dV_g+o(1)+O(\delta\log\delta)\\
&=-8\pi\log\delta-2\pi A_1(p_1)-2\pi\int_\Sigma G_2dV_g+o(1)+O(\delta\log\delta).
\end{aligned}$$

In the end, we obtain

$$\inf\Phi_0(u)\geq-4\pi\log\pi-2\pi A_1(p)+2\pi\int G_2dV_g.$$

5. Test functions for Case 1

In this section we will construct a function $\phi=(\phi_1,\phi_2)\in H^{1,2}(M)\times H^{1,2}(M)$, such that

$$\Phi_0(\phi)<-8\pi\log\pi-8\pi-2\pi(A_2(p_1)+A_1(p_2)),$$

whenever (1.4) holds. So, under assumption (1.4), Case 1 will not occur.

Let $(\Omega_i;(x,y))$ be an isothermal coordinate system around p_i $(i=1,2)$. We set

$$r(x,y)=\sqrt{x^2+y^2},\quad \textit{and}\quad B_\delta=\{(x,y):x^2+y^2<\delta^2\}.$$

We assume that near p_i $(i=1,2)$, for each $k=1,2$,

$$\begin{aligned}
G_k&=a_k(p_i)\log r+A_k(p_i)+\lambda_k(p_i)x+\mu_k(p_i)y\\
&\quad+\alpha_k(p_i)x^2+\beta_k(p_i)y^2+\gamma_k(p_i)xy+h(x,y)+O(r^4).
\end{aligned}$$

We have $a_1(p_1)=a_2(p_2)=-4$, and $a_1(p_2)=a_2(p_1)=2$. Moreover, we assume that

$$g|_{\Omega_i}=e^{\varphi_i}(dx^2+dy^2),$$

and

$$\varphi_i=b_1(p_i)x+b_2(p_i)y+c_1(p_i)x^2+c_2(p_i)y^2+c_{12}(p_i)xy+O(r^3).$$

It is well known that

$$\begin{aligned}
K(p_i)&=-(c_1(p_i)+c_2(p_i)),\\
|\nabla u|^2dV_g&=|\nabla u|^2dxdy,
\end{aligned}$$

and
$$\frac{\partial u}{\partial n} dS_g = \frac{\partial u}{\partial r} r d\theta, \quad (S = \partial B_r).$$
For α_k and β_k, we have the following lemma:

Lemma 5.1. *For any k, i, we have*
$$\alpha_k(p_i) + \beta_k(p_i) = 2\pi.$$

Proof. Near p_i we have
$$2\alpha_k(p_i) + 2\beta_k(p_i) + O(r) = \Delta_0 G_k(x, y) = e^{-\varphi_i} 4\pi. \qquad \square$$

We choose
$$\phi_1 = \begin{cases} w\left(\dfrac{x}{\epsilon}\right) + \lambda_1(p_1) r\cos\theta + \mu_1(p_1) r \sin\theta & (x, y) \in B_{L\epsilon}(p_1) \\ G_1 - \eta_1 H_1^{p_1} + 4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p_1) & (x,y) \in B_{2L\epsilon} \backslash B_{L\epsilon}(p_1) \\ G_1 - \eta_2 H_1^{p_2} + 4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p_1) & (x,y) \in B_{2L\epsilon} \backslash B_{L\epsilon}(p_2) \\ -\dfrac{\omega(\frac{x}{\epsilon}) + 2\log(1+\pi L^2)}{2} + \lambda_1(p_2) r\cos\theta + \mu_1(p_2) r\sin\theta & \\ \quad +6\log L\epsilon - 2\log(1+\pi L^2) + A_1(p_2) - A_1(p_1) & (x, y) \in B_{L\epsilon}(p_2) \\ G_1 + 4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p_1) & \text{otherwise,} \end{cases}$$

and
$$\phi_2 = \begin{cases} w\left(\dfrac{x}{\epsilon}\right) + \lambda_2(p_2) r\cos\theta + \mu_2(p_2) r \sin\theta & (x, y) \in B_{L\epsilon}(p_2) \\ G_2 - \eta_2 H_2^{p_2} + 4\log L\epsilon - 2\log(1+\pi L^2) - A_2(p_2) & (x,y) \in B_{2L\epsilon} \backslash B_{L\epsilon}(p_2) \\ G_2 - \eta_1 H_2^{p_1} + 4\log L\epsilon - 2\log(1+\pi L^2) - A_2(p_2) & (x,y) \in B_{2L\epsilon} \backslash B_{L\epsilon}(p_1) \\ -\dfrac{\omega(\frac{x}{\epsilon}) + 2\log(1+\pi L^2)}{2} + \lambda_2(p_1) r\cos\theta + \mu_2(p_1) r\sin\theta & \\ \quad +6\log L\epsilon - 2\log(1+\pi L^2) + A_2(p_1) - A_2(p_2) & (x, y) \in B_{L\epsilon}(p_1) \\ G_2 + 4\log L\epsilon - 2\log(1+\pi L^2) - A_2(p_2) & \text{otherwise.} \end{cases}$$

Here,
$$H_k^{p_i} = G_k - a_k(p_i)\log r - A_k(p_i) - \lambda_k(p_i) r\cos\theta - \mu_k(p_i) r\sin\theta,$$

and η_i is a cut-off function which equals 1 in $B_{L\epsilon}(p_i)$, equals 0 in $B^c_{2L\epsilon}(p_i)$. We may assume that
$$|\nabla \eta_i| \leq \frac{1}{L\epsilon}.$$

Now we compute $\Phi_0(\phi)$. Firstly, we compute $\int_\Sigma |\nabla\phi_1|^2 dV_g$ and $\int_\Sigma |\nabla\phi_2|^2 dV_g$. Let $\Omega = \Sigma \setminus (B_{L\epsilon}(p_1) \cup B_{L\epsilon}(p_2))$. Then

$$\int_\Sigma |\nabla\phi_1|^2 dV_g = \int_{B_{L\epsilon}(p_1)\cup B_{L\epsilon}(p_2)} |\nabla\phi_1|^2 dxdy + \int_\Omega |\nabla G_1|^2 dV_g \\ -2\sum_{i=1,2}\int_\Sigma \nabla G_1 \nabla\eta_i H_1^{p_i} dV_g + \sum_{i=1,2}\int_\Sigma |\nabla\eta_i H_1^{p_i}|^2 dV_g.$$

It is clear that we have

$$\int_{B_{L\epsilon}(p_2)} |\nabla\phi_1|^2 dV_g = \frac{1}{4}\int_{B_L} |\nabla w|^2 dxdy + \pi(L\epsilon)^2(\lambda_1^2(p_2) + \mu_1^2(p_2)),$$

and

$$\int_{B_{L\epsilon}(p_1)} |\nabla\phi_1|^2 dV_g = \int_{B_L} |\nabla w|^2 dxdy + \pi(L\epsilon)^2(\lambda_1^2(p_1) + \mu_1^2(p_1)).$$

Calculating directly and using the fact that $\int_0^{2\pi} h d\theta = 0$, we obtain,

$$\begin{aligned}\int_\Sigma \nabla G_1 \nabla\eta_1 H_1^{p_1} dV_g &= -\int_{\partial B_{L\epsilon}(p_1)} \frac{\partial G_1}{\partial n} H_1^{p_1} dS_g - 4\pi\int_{B_{2L\epsilon}\setminus B_{L\epsilon}} \eta_1 H_1^{p_1} dV_g \\ &= -\int_0^{2\pi}\left(-\frac{4}{r} + \lambda_1(p_1)\cos\theta + \mu_1(p_1)\sin\theta + O(r)\right) \\ &\quad \times(\alpha_1(p_1)r^2\cos^2\theta + \beta_1(p_1)r^2\sin^2\theta + h + O(r^4))rd\theta \\ &= 4\pi(\alpha_1(p_1) + \beta_1(p_1))(L\epsilon)^2 + O(L\epsilon)^4 \\ &= 8\pi^2(L\epsilon)^2 + O(L\epsilon)^4.\end{aligned}$$

Similarly, we get

$$\int_\Sigma \nabla G_1 \nabla\eta_2 H_1^{p_2} dV_g = -4\pi^2(L\epsilon)^2 + O(L\epsilon)^4.$$

It is obvious that

$$\int_\Sigma |\nabla\eta_j H_i^{p_j}|^2 dV_g = \int_{B_{2L\epsilon}\setminus B_{L\epsilon}(p_j)} O(r^2) dV_g = O((L\epsilon)^4).$$

Hence

$$\int_\Sigma |\nabla\phi_1|^2 dV_g = \frac{5}{4}\int_{B_L} |\nabla w|^2 dx - 8\pi^2(L\epsilon)^2 + \int_\Omega |\nabla G_1|^2 dV_g \\ +\pi(L\epsilon)^2\sum_{i=1,2}(\lambda_1^2(p_i) + \mu_1^2(p_i)) + O(L\epsilon)^4.$$

In the same way, we can show that

$$\int_\Sigma |\nabla\phi_2|^2 dV_g = \frac{5}{4}\int_{B_L} |\nabla w|^2 dx - 8\pi^2(L\epsilon)^2 + \int_\Omega |\nabla G_2|^2 dV_g + \pi(L\epsilon)^2 \sum_{i=1,2}(\lambda_2^2(p_i)+\mu_2^2(p_i)) + O(L\epsilon)^4.$$

Next, we compute $\int_\Sigma \nabla\phi_1\nabla\phi_2 dV_g$. We have

$$\begin{aligned}\int_\Sigma \nabla\phi_1\nabla\phi_2 dV_g &= \sum_{i=1,2}\int_{B_{L\epsilon}(p_i)} \nabla\phi_1\nabla\phi_2 dV_g + \int_\Omega \nabla G_1\nabla G_2 dV_g\\ &\quad - \int_\Sigma \nabla G_1\nabla\eta_1 H_2^{p_1} dV_g - \int_\Sigma \nabla G_2\nabla\eta_1 H_1^{p_1} dV_g\\ &\quad - \int_\Sigma \nabla G_1\nabla\eta_2 H_2^{p_2} dV_g - \int_\Sigma \nabla G_2\nabla\eta_2 H_1^{p_2} dV_g\\ &\quad + \sum_{i=1,2}\int_\Sigma \nabla\eta_i H_1^{p_i}\nabla\eta_i H_2^{p_i} dV_g\\ &= -\int_{B_L}|\nabla w|^2 dx + \pi(L\epsilon)^2\sum_{i=1,2}(\lambda_1(p_i)\lambda_2(p_i)+\mu_1(p_i)\mu_2(p_i))\\ &\quad -8\pi(L\epsilon)^2 + \int_\Omega \nabla G_1\nabla G_2 dV_g + O((L\epsilon)^4).\end{aligned}$$

Then we calculate $\int_\Omega (|\nabla G_1|^2 + |\nabla G_2|^2 + \nabla G_1\nabla G_2) dV_g$. We have

$$\begin{aligned}&\int_\Omega (|\nabla G_1|^2 + |\nabla G_2|^2 + \nabla G_1\nabla G_2) dV_g\\ &= \int_\Omega \left(|\nabla G_1|^2 + |\nabla G_2|^2 + \frac{\nabla G_1\nabla G_2 + \nabla G_2\nabla G_1}{2}\right) dV_g\\ &= -\int_{\partial B_{L\epsilon}(p_1)+\partial B_{L\epsilon}(p_2)} \left(G_1\frac{\partial G_1}{\partial n} + G_2\frac{\partial G_2}{\partial n} + \frac{G_1\frac{\partial G_2}{\partial n} + G_2\frac{\partial G_1}{\partial n}}{2}\right) dS_g\\ &\quad + 6\pi\int_{B_{L\epsilon(p_1)}+B_{L\epsilon}(p_2)} (G_1+G_2) dV_g\,.\end{aligned}$$

Lemma 5.2. *For any $k, m, i = 1, 2$, we have*

$$\begin{aligned}\int_{\partial B_r(p_i)} G_k\frac{\partial G_m}{\partial n} dS_g &= 2\pi a_k(p_i)a_m(p_i)\log r + 2\pi^2 a_k(p_i) r^2\\ &\quad + \pi r^2(\lambda_k(p_i)\lambda_m(p_i) + \mu_k(p_i)\mu_m(p_i))\\ &\quad + 2\pi a_k(p_i)A_m(p_i) + 4\pi^2 r^2 A_m(p_i)\\ &\quad + 4\pi^2 r^2 a_m(p_i)\log r + O(r^4\log r).\end{aligned}$$

Proof. Since $\int_0^{2\pi} h(r,\theta)d\theta = \int_0^{2\pi} \frac{\partial h}{\partial r}(r,\theta)d\theta = 0$, we have

$$\begin{aligned}\int_{\partial B_r(p_i)} G_k \frac{\partial G_m}{\partial n} dS_g = &\int_0^{2\pi} \Big(\frac{a_k(p_i)}{r} + \lambda_k(p_i)\cos\theta + \mu_k(p_i)\sin\theta + 2r\alpha_k(p_i)\cos^2\theta \\ &+2r\beta_k(p_i)\sin^2\theta + 2r\gamma_k(p_i)\sin\theta\cos\theta)(a_m(p_i)\log r \\ &+A_m(p_i) + \lambda_m(p_i)r\cos\theta + \mu_m(p_i)r\sin\theta + r^2\alpha_m(p_i)\cos^2\theta \\ &+r^2\beta_m(p_i)\sin^2\theta + r^2\gamma_m(p_i)\sin\theta\cos\theta\Big) r\,d\theta \\ &+O(r^4\log r) \\ = &\ 2\pi a_k(p_i)a_m(p_i)\log r + \pi(\alpha_m(p_i)+\beta_m(p_i))a_k(p_i)r^2 \\ &+\pi r^2(\lambda_k(p_i)\lambda_m(p_i) + \mu_k(p_i)\mu_m(p_i)) + 2\pi a_k(p_i)A_m(p_i) \\ &+(2\pi r^2 A_m(p_i) + 2\pi r^2 a_m(p_i)\log r)(\alpha_k(p_i)+\beta_k(p_i)) \\ &+O(r^4\log r).\end{aligned}$$

□

Then

$$\begin{aligned}\int_{\partial B_r(p_1)} G_1 \frac{\partial G_1}{\partial n} dS_g = &\ 32\pi\log r - 8\pi^2 r^2 + \pi r^2\big(\lambda_1^2(p_1) + \mu_1^2(p_1)\big) \\ &-8\pi A_1(p_1) + 4\pi^2 r^2 A_1(p_1) - 16\pi^2 r^2\log r + O(r^4\log r).\end{aligned}$$

$$\begin{aligned}\int_{\partial B_r(p_1)} G_2 \frac{\partial G_2}{\partial n} dS_g = &\ 8\pi\log r + 4\pi^2 r^2 + \pi r^2(\lambda_2^2(p_1) + \mu_2^2(p_1)) \\ &+4\pi A_2(p_1) + 4\pi^2 r^2 A_2(p_1) + 8\pi^2 r^2\log r + O(r^4\log r).\end{aligned}$$

$$\begin{aligned}\int_{\partial B_r(p_1)} G_1 \frac{\partial G_2}{\partial n} dS_g = &-16\pi\log r - 8\pi^2 r^2 + \pi r^2(\lambda_1(p_1)\lambda_2(p_1) + \mu_1(p_1)\mu_2(p_1)) \\ &-8\pi A_2(p_1) + 4\pi^2 r^2 A_2(p_1) + 8\pi^2 r^2\log r + O(r^4\log r).\end{aligned}$$

$$\begin{aligned}\int_{\partial B_r(p_1)} G_2 \frac{\partial G_1}{\partial n} dS_g = &-16\pi\log r + 4\pi^2 r^2 + \pi r^2(\lambda_2(p_1)\lambda_1(p_1) + \mu_2(p_1)\mu_1(p_1)) \\ &+4\pi A_1(p_1) + 4\pi^2 r^2 A_1(p_1) - 16\pi^2 r^2\log r + O(r^4\log r).\end{aligned}$$

Hence

$$\int_\Omega (|\nabla G_1|^2 + |\nabla G_2|^2) dV_g$$
$$= -\left(80\pi \log L\epsilon - 8\pi^2 (L\epsilon)^2 \quad + \pi (L\epsilon)^2 \sum_{i,j=1,2} (\lambda_i^2(j) + \mu_i^2(j)\right)$$
$$-8\pi A_1(p_1) - 8\pi A_2(p_2) + 4\pi^2 (L\epsilon)^2 \big(A_1(p_1) + A_2(p_2) + A_2(p_1) + A_1(p_2)\big)$$
$$-16\pi^2 (L\epsilon)^2 \log L\epsilon + 4\pi A_2(p_1) + 4\pi A_1(p_2)) + 4\pi \int_\Omega G_1 dV_g + 4\pi \int_\Omega G_2 dV_g$$
$$+ O((L\epsilon)^4 \log L\epsilon),$$

and

$$\int_\Omega \nabla G_1 \nabla G_2 dV_g$$
$$= -\Big(-32 \log L\epsilon - 4\pi^2 (L\epsilon)^2 + \pi (L\epsilon)^2 \sum_{i\neq j} \big(\lambda_i(p_j)\lambda_j(p_i) + \mu_i(p_j)\mu_j(p_i)\big)$$
$$-4\pi A_2(p_1) - 4\pi A_1(p_2) + 2\pi A_1(p_1) + 2\pi A_2(p_2)$$
$$+2\pi^2 (L\epsilon)^2 (A_1(p_1) + A_1(p_2) + A_2(p_1) + A_2(p_2)) - 8\pi^2 (L\epsilon)^2 \log L\epsilon\Big)$$
$$+2\pi \textstyle\sum_{i=1,2} \displaystyle\int_{B_{L\epsilon}(p_i)} (G_1 + G_2) dV_g + O((L\epsilon)^4 \log L\epsilon).$$

It is easy to check that

$$6\pi \int_{B_{L\epsilon}(p_1) + B_{L\epsilon}(p_2)} (G_1 + G_2) dV_g = 6\pi^2 (L\epsilon)^2 \left(\sum_{i,j=1,2} A_i(p_j)\right) - 24\pi^2 (L\epsilon)^2 \log L\epsilon$$
$$+ 12\pi^2 (L\epsilon)^2 + O\big((L\epsilon)^4 \log L\epsilon\big).$$

So, we get

$$\int_\Sigma (|\nabla \phi_1|^2 + |\nabla \phi_2|^2 + \nabla \phi_1 \nabla \phi_2) dV_g = \frac{3}{2} \int_{B_L} |\nabla w|^2 dx dy - 48\pi \log L\epsilon + 6\pi A_1(p_1)$$
$$+ 6\pi A_2(p_2) + O\big((L\epsilon)^4 \log L\epsilon\big). \qquad (5.1)$$

We calculate $\int_\Sigma (\phi_1+\phi_2)dV_g$. We have

$$\begin{aligned}\int_\Sigma \phi_1 dV_g &= \epsilon^2\int_{B_L} we^{\varphi(\epsilon x,\epsilon y)}\,dxdy - \int_\Sigma (\eta_1 H_1^{p_1}+\eta_2 H_1^{p_2})\,dV_g\\ &\quad +(4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p_1))\\ &\quad \times\left(1-\int_{B_{L\epsilon}(p_1)} dV_g\right) + 2\log L\epsilon\int_{B_{L\epsilon}(p_2)} dV_g\\ &\quad +A_1(p_2)\int_{B_{L\epsilon}(p_2)} dV_g - \int_{B_{L\epsilon}(p_1)+B_{L\epsilon}(p_2)} G_1\\ &\quad -\epsilon^2\int_{B_L}\frac{w+2\log(1+\pi L^2)}{2}dV_g\\ &\quad +\sum_{i=1,2}\int_{B_{L\epsilon}(p_i)}(\lambda_1(p_i)x+\mu_1(p_i)y)dV_g.\end{aligned}$$

Since

$$\begin{aligned}\int_{B_{L\epsilon}(p_1)+B_{L\epsilon}(p_2)} G_1 dV_g &= \int_0^{L\epsilon}(-2\log r + A_1(p_1)+A_1(p_2))2\pi r dr\\ &\quad +O((L\epsilon)^4\log L\epsilon)\\ &= -2\pi(L\epsilon)^2\log L\epsilon + \pi(L\epsilon)^2\\ &\quad +(A_1(p_1)+A_1(p_2))\pi(L\epsilon)^2 + O\big((L\epsilon)^4\log L\epsilon\big),\end{aligned}$$

we have

$$\begin{aligned}\int_\Sigma \phi_1 dV_g &= \frac{\epsilon^2}{2}\int_{B_L} we^{\varphi(\epsilon x,\epsilon y)}dxdy + 4\log L\epsilon + \pi(L\epsilon)^2\log(1+\pi L^2)\\ &\quad -\pi(L\epsilon)^2 - A_1(p_1) - 2\log(1+\pi L^2) + O\big((L\epsilon)^4\log L\epsilon\big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_\Sigma \phi_2 dV_g &= \frac{\epsilon^2}{2}\int_{B_L} we^{\varphi(\epsilon x,\epsilon y)}dxdy + 4\log L\epsilon + \pi(L\epsilon)^2\log(1+\pi L^2)\\ &\quad -\pi(L\epsilon)^2 - A_2(p_2) - 2\log(1+\pi L^2) + O\big((L\epsilon)^4\log L\epsilon\big).\end{aligned}$$

Moreover, we have

$$\int_{B_L}\omega e^{\varphi(\epsilon x,\epsilon y)}dxdy = 2\pi L^2 - 2\log(1+\pi L^2) - 2\pi L^2\log(1+\pi L^2) + O(L^2\epsilon^2\log L),$$

hence, we get

$$\int_\Sigma (\phi_1+\phi_2)dV_g = -A_1(p_1) - A_2(p_2) + 8\log L\epsilon - 4\log(1+\pi L^2) -2\epsilon^2\log(1+\pi L^2) + O((L\epsilon)^4\log L\epsilon). \tag{5.2}$$

We denote $B(p_j) = \frac{(b_1(p_j)+\lambda_1(p_j))^2+(b_2(p_j)+\lambda_2(p_j))^2}{4}$ and $M_i = \frac{-\frac{K(p_i)}{2}+B(p_i)}{\pi}$. Then we have

$$\begin{aligned}
&\int_{B_{L\epsilon}(p_1)} e^{\phi_1}dV_g\\
&= \epsilon^2\int_{B_L} \frac{e^{(b_1(p_1)+\lambda_1(p_1))\epsilon x+(b_2(p_1)+\lambda_2(p_1))\epsilon y+c_1(p_1)\epsilon^2x^2+c_2(p_1)\epsilon^2y^2+c_{12}\epsilon^2xy+O((r\epsilon)^3)}}{(1+\pi r^2)^2}dxdy\\
&= \epsilon^2\int_0^L \frac{1+\epsilon^2\pi M_i r^2+O(\epsilon^3r^3)}{(1+\pi r^2)^2}2\pi r dr\\
&= \epsilon^2(1-\epsilon^2M_i)\frac{\pi L^2}{1+\pi L^2} + \epsilon^4 M_i\log(1+\pi L^2) + \epsilon^2 O(\epsilon^3\log L)\\
&= \epsilon^2\left(1-\frac{1}{1+\pi L^2} + \epsilon^2M_i\log(1+\pi L^2) + O(\epsilon^2) + O(\epsilon^3\log L)\right),
\end{aligned}$$

and we also have

$$\begin{aligned}
&\int_{B_\delta(p_1)\setminus B_{L\epsilon(p_1)}} e^{\phi_1}dV_g\\
&\quad= \frac{(L\epsilon)^4}{(1+\pi L^2)^2}\int_{L\epsilon}^\delta e^{-4\log r+(\lambda_1(p_1)+b_1(p_1))x+(\mu_1(p_1)+b_2(p_1))y}\\
&\qquad\times e^{(c_1(p_1)+\alpha_1(p_1))x^2+(c_2(p_1)+\beta_1(p_1))y^2+(c_{12}(p_1)+\gamma_1)xy+(1-\eta_1)H_1^{p_1}+O(r^3)}2\pi r dr\\
&\quad= \epsilon^2\left(\frac{\pi L^2}{(1+\pi L^2)^2} - 2(M_1+1)\epsilon^2\log L\epsilon + O(\epsilon^2) + O\left(\frac{1}{L^4}\right)\right).
\end{aligned}$$

Since G_1 is bounded above, outsid $B_\delta(p_1)$, we have

$$\int_{\Sigma\setminus B_\delta} e^{\phi_1} = O(\epsilon^4).$$

Noting that $\frac{\pi L^2}{(1+\pi L^2)^2} - \frac{1}{1+\pi L^2} = O(\frac{1}{L^4})$, we get

$$\log\int_\Sigma e^{\phi_1}dV_g = \log\epsilon^2 + \epsilon^2M_1\log(1+\pi L^2) - 2\epsilon^2(M_1+1)\log L\epsilon \quad O(\epsilon^2) + O\left(\frac{1}{L^4}\right). \tag{5.3}$$

In the same way, we can get

$$\log \int_\Sigma e^{\phi_2} dV_g = \log \epsilon^2 + \epsilon^2 M_2 \log(1+\pi L^2) - 2\epsilon^2(M_2+1)\log L\epsilon + O(\epsilon^2) + O\left(\frac{1}{L^4}\right). \tag{5.4}$$

It follows from (5.1), (5.2), (5.3) and (5.4) that

$$\begin{aligned}
\Phi_0(\phi) &= \frac{1}{2}\int_{B_L} |\nabla w|^2 dxdy + 16\log L\epsilon - 2\pi(A_1(p_1)+A_2(p_2)) \\
&\quad -16\pi\log(1+\pi L^2) - 8\pi\log\epsilon^2 - 8\pi\epsilon^2\log(1+\pi L^2) \\
&\quad -4\pi\epsilon^2\Big((M_1+M_2)\log(1+\pi L^2) - 2(M_1+M_2+2)\log L\epsilon\Big) \\
&\quad +O\left(\frac{1}{L^4}\right) + O(\epsilon^2) + O\big((L\epsilon)^4\log L\epsilon\big) + O(\epsilon^3\log L) \\
&= -8\pi\log\frac{1+\pi L^2}{L^2} - 8\pi\frac{\pi L^2}{1+\pi L^2} - 2\pi\big(A_1(p_1)+A_2(p_2)\big) \\
&\quad -4\pi\epsilon^2(M_1+M_2+2)\big(\log(1+\pi L^2) - 2\log L\epsilon\big) \\
&\quad +O\left(\frac{1}{L^4}\right) + O(\epsilon^2) + O\big((L\epsilon)^4\log L\epsilon\big) + O(\epsilon^3\log L) \\
&= -8\pi\log\pi - 8\pi - 2\pi\big(A_1(p_1)+A_2(p_2)\big) \\
&\quad -4\pi(M_1+M_2+2)\epsilon^2(\log(1+\pi L^2) - 2\log L\epsilon) \\
&\quad +O\left(\frac{1}{L^4}\right) + O(\epsilon^2) + O\big((L\epsilon)^4\log L\epsilon\big) + O(\epsilon^3\log L).
\end{aligned}$$

Under assumption (1.1) we have $M_1+M_2+2>0$. Leting $L^4\epsilon^2 = \frac{1}{\log(-\log\epsilon)}$, we get

$$\begin{aligned}
\Phi_0(\phi) &= -8\pi\log\pi - 8\pi - 2\pi(A_1(p_1)+A_2(p_2)) \\
&\quad -4\pi(M_1+M_2+2)\epsilon^2(-\log\epsilon^2) + o(\epsilon^2(-\log\epsilon^2)).
\end{aligned}$$

Then for sufficiently small ϵ we have

$$\Phi_0(\phi) < -8\pi\log\pi - 8\pi - 2\pi(A_1(p_1)+A_2(p_2)).$$

This proves our claim.

6. Test functions for Case 2

Assuming that (1.4) holds on Σ, we will construct a function $\phi = (\phi_1,\phi_2) \in H^{1,2}(M)\times H^{1,2}(M)$, such that

$$\Phi_0(\phi) < -4\pi\log\pi - 2\pi A_1(p) + 2\pi\int G_2 dV_g.$$

Let $(\Omega;\ (x,y))$ be an isothermal coordinate system around p. We assume that near p

$$G_k = a_k \log r + A_k(p) + \lambda_k x + \mu_k y + \alpha_k x^2 + \beta_k y^2 + \gamma_k xy + h_k(x,\, y) + O(r^4).$$

We have $a_1(p) = -4$, and $a_2(p_1) = 2$. Moreover we assume that

$$g|_\Omega = e^{\varphi}(dx^2 + dy^2),$$

and

$$\varphi = b_1 x + b_2 y + c_1 x^2 + c_2 y^2 + c_{12} xy + O(r^3).$$

Similarly to Case 1, there hold

$$\alpha_k + \beta_k = 2\pi, \quad i = 1, 2.$$

We choose

$$\phi_1 = \begin{cases} w(\dfrac{x}{\epsilon}) + \lambda_1 r\cos\theta + \mu_1 r\sin\theta & x \in B_{L\epsilon}(p) \\ G_1 - \eta H_1 + 4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p) & x \in B_{2L\epsilon}\setminus B_{L\epsilon}(p) \\ G_1 + 4\log L\epsilon - 2\log(1+\pi L^2) - A_1(p_1) & \text{otherwise,} \end{cases}$$

and

$$\phi_2 = \begin{cases} -\dfrac{w(\frac{x}{\epsilon}) + 2\log(1+\pi L^2)}{2} + 2\log L\epsilon & \\ \quad +\lambda_2 r\cos\theta + \mu_2 r\sin\theta + A_2(p) & x \in B_{L\epsilon}(p) \\ G_2 - \eta H_2 & x \in B_{2L\epsilon}\setminus B_{L\epsilon}(p) \\ G_2 & \text{otherwise.} \end{cases}$$

Here

$$H_k = G_k - a_k \log r - A_k + \lambda_k r\cos\theta + \mu_k r\sin\theta$$

and η_i is a cut-off function which equals 1 in $B_{L\epsilon}(p)$, equals 0 in $B^c_{2L\epsilon}(p)$.

Let $\Omega = \Sigma \setminus B_{L\epsilon}(p)$. By an argument similar that used in Section 5, we can derive that

$$\begin{aligned} \int_\Sigma |\nabla\phi_1|^2 dV_g &= \int_{B_{L\epsilon}(p)} |\nabla\phi_1|^2 dxdy + \int_\Omega |\nabla G_1|^2 dV_g \\ &\quad -2\int_\Sigma \nabla G_1 \nabla\eta H_1 dV_g + \int_\Sigma |\nabla\eta H_1|^2 dV_g \\ &= \int_{B_L} |\nabla w|^2 dxdy - 16\pi(L\epsilon)^2 + \pi(L\epsilon)^2(\lambda_1^2 + \mu_1^2) + O((L\epsilon)^4), \end{aligned}$$

$$\begin{aligned}\int_\Sigma |\nabla\phi_2|^2 dV_g &= \int_{B_{L\epsilon}(p)} |\nabla\phi_1|^2 dxdy + \int_\Omega |\nabla G_2|^2 dV_g \\ &\quad -2\int_\Sigma \nabla G_2 \nabla\eta_2 H_2 dV_g + \int_\Sigma |\nabla\eta H_2|^2 dV_g \\ &= \frac{1}{4}\int_{B_L} |\nabla w|^2 dxdy + 8\pi(L\epsilon)^2 + \pi(L\epsilon)^2(\lambda_2^2+\mu_2^2) + O((L\epsilon)^4),\end{aligned}$$

and

$$\begin{aligned}\int_\Sigma \nabla\phi_1\nabla\phi_2 dV_g &= \int_\Omega \nabla G_1 \nabla G_2 dV_g + \int_{B_{L\epsilon}(p)} \nabla\phi_1\nabla\phi_2 dV_g \\ &\quad -\int \nabla G_1 \nabla\eta H_2 dV_g - \int \nabla G_2\nabla\eta H_1 dV_g + \int_\Sigma \nabla\eta H_1 \nabla\eta H_2 dV_g \\ &= -\frac{1}{2}\int_{B_L} |\nabla w|^2 dxdy + \pi(L\epsilon)^2(\lambda_1\lambda_2 + \mu_1\mu_2) \\ &\quad -4\pi(L\epsilon)^2 + \int_\Omega \nabla G_1\nabla G_2 dV_g + O((L\epsilon)^4).\end{aligned}$$

Note that

$$\begin{aligned}&\int_\Omega (|\nabla G_1|^2 + |\nabla G_2|^2 + \nabla G_1\nabla G_2) dV_g \\ &= \int_\Omega \left(|\nabla G_1|^2 + |\nabla G_2|^2 + \frac{\nabla G_1\nabla G_2 + \nabla G_2 \nabla G_1}{2}\right) dV_g \\ &= -\int_{\partial B_{L\epsilon}(p)} \left(G_1\frac{\partial G_1}{\partial n} + G_2\frac{\partial G_2}{\partial n} + \frac{G_1\frac{\partial G_2}{\partial n} + G_2\frac{\partial G_1}{\partial n}}{2}\right) dS_g \\ &\quad + 6\pi\int_{B_{L\epsilon(p)}} (G_1+G_2) - 6\pi\int_\Sigma G_2 dV_g.\end{aligned}$$

Applying Lemma 5.2, we get

$$\begin{aligned}\int_{\partial B_r(p)} G_1\frac{\partial G_1}{\partial n} dS_g &= 32\pi\log r - 8\pi^2 r^2 + \pi r^2(\lambda_1^2+\mu_1^2) \\ &\quad -8\pi A_1(p) + 4\pi^2 r^2 A_1(p) - 16\pi^2 r^2\log r + O(r^4\log r),\end{aligned}$$

$$\begin{aligned}\int_{\partial B_r(p)} G_2\frac{\partial G_2}{\partial n} dS_g &= 8\pi\log r + 4\pi^2 r^2 + \pi r^2(\lambda_2^2+\mu_2^2) \\ &\quad +4\pi A_2(p) + 4\pi^2 r^2 A_2(p) + 8\pi^2 r^2\log r + O(r^4\log r),\end{aligned}$$

$$\begin{aligned}\int_{\partial B_r(p)} G_1\frac{\partial G_2}{\partial n} dS_g &= -16\pi\log r - 8\pi^2 r^2 + \pi r^2(\lambda_1\lambda_2+\mu_1\mu_2) \\ &\quad -8\pi A_2(p) + 4\pi^2 r^2 A_2(p) + 8\pi^2 r^2\log r + O(r^4\log r),\end{aligned}$$

$$\int_{\partial B_r(p)} G_2 \frac{\partial G_1}{\partial n} dS_g = -16\pi \log r + 4\pi^2 r^2 + \pi r^2(\lambda_2\lambda_1 + \mu_2\mu_1) \\ +4\pi A_1(p) + 4\pi^2 r^2 A_1(p) - 16\pi^2 r^2 \log r + O(r^4 \log r).$$

Noticing that

$$6\pi \int_{B_{L\epsilon}(p)} (G_1 + G_2) dV_g = 6\pi^2 (L\epsilon)^2 (A_1(p) + A_2(p)) \\ - 12\pi^2 (L\epsilon)^2 \log L\epsilon + 6\pi^2 (L\epsilon)^2 + O\big((L\epsilon)^4 \log L\epsilon\big),$$

we get

$$\int_\Sigma (|\nabla\phi_1|^2 + |\nabla\phi_2|^2 + \nabla\phi_1\nabla\phi_2) dV_g \\ = \frac{3}{4}\int_{B_L} |\nabla w|^2 dx dy - 24\pi \log L\epsilon \\ -6\pi A_1(p_1) - 6\pi \int G_2 dV_g + O\big((L\epsilon)^4 \log L\epsilon\big). \tag{6.1}$$

We have

$$\int_\Sigma \phi_1 dV_g = \epsilon^2 \int_{B_L} w dV_g - \int_\Sigma \eta H_1 dV_g + (4\log L\epsilon - 2\log(1 + \pi L^2) \\ -A_1(p))(1 - \int_{B_{L\epsilon}(p)} dV_g) + \int_{B_{L\epsilon}(p)} (\lambda_1 x + \mu_1 y) dV_g \\ - \int_{B_{L\epsilon}(p)} G_1 dV_g + O((L\epsilon)^4 \log L\epsilon) \\ = \epsilon^2 \int_{B_L} w e^{\varphi(\epsilon x, \epsilon y)} dx dy + 4\log L\epsilon + 2\pi (L\epsilon)^2 \log(1 + \pi L^2) \\ -2\pi (L\epsilon)^2 - A_1(p) - 2\log(1 + \pi L^2) + O\big((L\epsilon)^4 \log L\epsilon\big).$$

Since

$$\int_{B_{L\epsilon}(p)} G_2 dV_g = \int_0^{L\epsilon} (2\log r + A_2(p)) 2\pi L\epsilon dr + O\big((L\epsilon)^4 \log L\epsilon\big) \\ = 2\pi r^2 \log r - \pi r^2 + (A_1(p)\pi r^2 + O\big((L\epsilon)^4 \log L\epsilon\big),$$

we can see that

$$
\begin{aligned}
\int_\Sigma \phi_2 dV_g &= \int_\Sigma G_2 dV_g - \int_{B_{L\epsilon}(p)} G_2 dV_g \\
&\quad - \int_\Sigma \eta\beta_2 dV_g + (2\log L\epsilon + A_1(p)) \int_{B_{L\epsilon}} dV_g \\
&\quad -\epsilon^2 \int_{B_L} \frac{w + 2\log(1+\pi L^2)}{2} e^{\varphi(\epsilon x,\epsilon y)} dxdy + O\big((L\epsilon)^4 \log L\epsilon\big) \\
&= -\frac{\epsilon^2}{2} \int_{B_L} w e^{\varphi(\epsilon x,\epsilon y)} dxdy \\
&\quad -\pi(L\epsilon)^2 \log(1+\pi L^2) + \pi(L\epsilon)^2 + O\big((L\epsilon)^4 \log L\epsilon\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int (\phi_1 + \phi_2) dV_g &= -A_1(p) + 4\log L\epsilon - \epsilon^2 \log(1+\pi L^2) \\
&\quad -2\log(1+\pi L^2) + O\big((L\epsilon)^4 \log L\epsilon\big). \qquad (6.2)
\end{aligned}
$$

Setting $B(p) = \frac{(b_1+\lambda_1)^2+(b_2+\lambda_2)^2}{4}$ and $M = \frac{-\frac{K(p)}{2}+B(p)}{\pi}$, we have

$$
\begin{aligned}
\log \int_\Sigma e^{\phi_1} dV_g &= \log\epsilon^2 + \epsilon^2 M \log(1+\pi L^2) - 2\epsilon^2(M+1)\log L\epsilon \\
&\quad + O(\epsilon^2) + O((L\epsilon)^3 \log L) + O\left(\frac{1}{L^4}\right). \qquad (6.3)
\end{aligned}
$$

It is easy to see that

$$
\int_{B_{2L\epsilon}(0)} e^{\phi_2} dV_g = O\big((L\epsilon)^4\big),
$$

and

$$
\int_{B_{2L\epsilon}(0)} e^{G_2} dV_g = O\big((L\epsilon)^4\big).
$$

Since $\int e^{G_2} = 1$, we get

$$
\log \int_\Sigma e^{\phi_2} = \log\big(1 - O\big((L\epsilon)^4\big)\big) = O\big((L\epsilon)^4\big). \qquad (6.4)
$$

In the end, we can deduce from (6.1), (6.2), (6.3) and (6.4) that

$$
\begin{aligned}
\Phi_0(\phi) &= -4\pi - 4\pi\log\pi + 2\int G_2 dV_g - \epsilon^2\big(\log(1+\pi L^2) - 2\log L\epsilon\big)(1+M) \\
&\quad + O\left(\frac{1}{L^4}\right) + O(\epsilon^3 \log L) + O\big((L\epsilon)^4 \log L\epsilon\big) + O(\epsilon^2).
\end{aligned}
$$

Let $L^4\epsilon^2 = \frac{1}{\log(-\log\epsilon)}$. Then for ϵ sufficiently small we have

$$\Phi_0(\phi) < -4\pi - 4\pi\log\pi + 2\int G_2 dV_g.$$

This proves our claim.

Therefore, if Σ satisfies the condition

$$\max_{p\in\Sigma} K(p) < 2\pi,$$

we can see that u^ϵ converges to $u^0 = (u_1^0, u_2^0)$ in $H_2 := H^{1,2}(\Sigma)\times H^{1,2}(\Sigma)$, hence it is clear that $\Phi(u^0) = \inf_{u\in H_2}\Phi(u)$, that is, u^0 is a minimizer of $\Phi_0 = \Phi$. This completes the proof of the main theorem.

References

[1] E. CAGLIOTI, P.-L. LIONS, C. MARCHIORO and M. PULVIRENTI, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description,* Commun. Math. Phys. **143** (1992), 501–525.

[2] E. CAGLIOTI, P.-L. LIONS, C. MARCHIORO and M. PULVIRENTI, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*, Commun. Math. Phys. **174** (1995), 229–260.

[3] A. S. Y. CHANG and P. YANG, *Conformal deformation of metrics on S^2*, J. Differential Geom. **23** (1988), 259–296.

[4] A. S. Y. CHANG and P. YANG, *Prescribing Gaussian curvature on S^2*, Acta Math. **159** (1987), 214–259.

[5] C.-C. CHEN and C.-S. LIN, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces,* Comm. Pure Appl. Math. **55** (2002), 728–771.

[6] W. X. CHEN and W. Y. DING, *Scalar curvature on S^2*, Trans. Amer. Math. Soc. **303** (1987), 365–382.

[7] W. X. CHEN and C. LI, *Classification of solutions of some nonlinear elliptic equations,* Duke Math. J. **63** (1991) 615–622.

[8] W. DING, J. JOST, J. LI and G. WANG, *The differential equation $\Delta u = 8\pi - 8\pi he^u$ on a compact Riemann surface,* Asian J. Math. **1** (1997), 230–248.

[9] W. DING, J. JOST, J. LI and G. WANG,*An analysis of the two-vortex case in the Chern-Simons Higgs model,* Calc. Var. Partial Differential Equations **7** (1998), 87–97.

[10] W. DING, J. JOST, J. LI and G. WANG, *Multiplicity results for the two-vortex Chern-Simons Higgs model on the two sphere,* Comment. Math. Helv. **74** (1999), 118–142.

[11] W. DING, J. JOST, J. LI, X. PENG and G. WANG, *Self duality equations for Ginzburg-Landau and Seiberg-Witten type functionals with 6^{th} order potentials,* Commun. Math. Phys. **217** (2001), 383–407.

[12] Z. DJADLI and A. MALCHIODI, *Existence of conformal metrics with constant Q-curvature,* preprint, SISSA, 70/2004/M.

[13] J. HONG, Y. KIM and P. Y. PAC, *Multivortex solutions of the Abelian Chern-Simons theory,* Phys. Rev. Lett. **64** (1990), 2230–2233.

[14] R. JACKIW and E. WEINBERG, *Self-dual Chern-Simons vortices,* Phys. Rev. Lett. **64** (1990), 2234–2237.

[15] J. JOST, C.-S. LIN and G. WANG, *Analytic aspects of the Toda system: II. Bubbling behavior and existence of solutions,* Comm. Pure Appl. Math., to appear.

[16] J. JOST and G. WANG, *Analytic aspects of the Toda system. I. A Moser-Trudinger inequality,* Comm. Pure Appl. Math. **54** (2001), 1289–1319.
[17] J. KAZDAN and F. WARNER, *Curvature functions for compact* 2*-manifolds,* Ann. of Math. **99** (1974), 14–47.
[18] Y. LI, *Moser-Trudinger inequality on compact Riemannian manifolds of dimension two,* J. Partial Differential Equations **14** (2001), 163–192.
[19] Y. LI, *The extremal functions for Moser-Trudinger inequality on compact Riemannian manifolds,* Sci. China Ser. A **48** (2005), 618–648.
[20] M. LUCIA and M. NOLASCO, *Chern-Simons vortes theorey and Toda systems,* J. Differential Equations **184** (2002), 443–474.
[21] M. NOLASCO and G. TARANTELLO, *Double vortex condensates in the Chern-Simons-Higgs theory,* Calc. Var. Partial Differential Equations **9** (1999), 31–94.
[22] M. NOLASCO and G. TARANTELLO, *Vortex condensates for the* SU(3) *Chern-Simons theory,* Commun. Math. Phys. **213** (2000), 599–639.
[23] M. STRUWE and G. TARANTELLO, *On multivortex solutions in Chern-Simons gauge theory,* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **1** (1998), 109–121.
[24] G. TARANTELLO, *Multiple condensate solutions for the Chern-Simons Higgs theory,* J. Math. Phys. **37** (1996), 3769–3796.

Math. Group
The abdus salam ICTP
34100 Trieste, Italy
and
Academy of Mathematics and Systems Sciences
Chinese Academy of Sciences
Beijing 100080, P. R. of China
jyli@ictp.it

Math. Group
The abdus salam ICTP
34100 Trieste, Italy
liy@ictp.trieste.it