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Approximation of holomorphic functions in Banach spaces admitting a Schauder decomposition

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Abstract. Let X be a complex Banach space. Recall that X admits a *finite-dimensional Schauder decomposition* if there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of X, such that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for every n. The finite-dimensional Schauder decomposition is said to be *unconditional* if, for every $x \in X$, the series $x = \sum_{n=1}^{\infty} x_n$, which represents x, converges unconditionally, that is, $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers. For short, we say that X admits an unconditional F.D.D.

We show that if X admits an unconditional F.D.D. then the following Runge approximation property holds:

(R.A.P.) *There is* $r \in (0, 1)$ *such that, for any* $\epsilon > 0$ *and any holomorphic function* f *on the open unit ball of* X, *there exists a holomorphic function* h *on* X *satisfying* $|f - h| < \epsilon$ *on the open ball of radius* r *centered at the origin.*

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1. Introduction

The Runge approximation problem (R.A.P.) of a holomorphic function in a complex Banach space X plays a crucial role in the study of the vanishing of the sheaf cohomology groups $H^q(\Omega, \mathcal{O})$, where $\Omega \subset X$ is an open subset, $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X, and $q \geq 1$. (See for instance [5]). This problem has been studied in particular by L. Lempert [4, 6], B. Josefson [3] and I. Patyi [9]. L. Lempert and B. Josefson show that the Runge approximation property (R.A.P.) holds if X has an unconditional Schauder basis while I. Patyi proves it for ℓ_1-sum Banach spaces. In this paper, we consider an important class of Banach spaces, namely those which admit an unconditional finite-dimensional Schauder decomposition (F.D.D.). (See [8]). We refer the interested reader to [8, page 51],

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for instance, for examples of Banach spaces which admit an unconditional F.D.D. but which do not have an unconditional Schauder basis.

We have the following theorem.

Theorem 1.1. *Let $(X, \| \|)$ be a complex Banach space admitting an unconditional* F.D.D. *Then there exists an equivalent norm $\| \|_1$ on X such that the following holds. Let $B_1(R) \subset X$ be the ball of radius $R > 0$ about the origin associated to $\| \|_1$, and let f be a holomorphic function in $B_1(R)$. Then for any $r \in (0, R)$ and any $\epsilon > 0$, there exists a holomorphic function h in X satisfying $|f - h| < \epsilon$ in $B_1(r)$.*

The following corollary is immediate.

Corollary 1.2. *Let X be a complex Banach space admitting an unconditional* F.D.D. *Then the Runge approximation property* (R.A.P.) *holds.*

Notice that among the Banach spaces which do not admit an unconditional F.D.D., there are the spaces $C[0, 1]$ and $L^1(0, 1)$ ([10]). The Runge approximation problem (R.A.P.) is still open, in particular, for these two spaces. It would be interesting to know if the Runge approximation property (R.A.P.) holds for $C[0, 1]$ since it is well known that every separable Banach space is isometric to a subspace of $C[0, 1]$.

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2. Preliminaries

Let X be a complex Banach space. Recall that a function $f : U \subset X \longrightarrow \mathbb{C}$, where U is an open subset of X, is *holomorphic* if f is continuous on U, and $f_{|U \cap X_1}$ is holomorphic, in the classical sense, as a function of several complex variables, for each finite-dimensional subspace X_1 of X. (See [1].)

The following known lemma gives a criterion of compactness in X (See [2], IV.5.4).

Lemma 2.1. *Let T_n be a uniformly bounded sequence of linear operators in X. If $\lim_n T_n x = x$ for every $x \in X$, then this limit exists uniformly on any compact set. Conversely, if $\lim_n T_n x = x$ uniformly for x in a bounded set K, and if, in addition, $T_n\{x \mid |x| \leq 1\}$ is compact for each n, then K is compact.*

Assume now that X admits an unconditional F.D.D. $\{X_n\}_{n=1}^{\infty}$. Let $x = \sum_{n=1}^{\infty} x_n$, $x_n \in X_n$, be the unique representation of x. It is known that for every sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| \leq 1$, $n \in \mathbb{N}$, the operator M_θ defined by

$$M_\theta \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \theta_n x_n$$

is a bounded linear operator. The (finite) constant $\sup_\theta \|M_\theta\|$ is called the *unconditional constant* of the decomposition. (See [8] and [10] for details). It is clear that one can always define on X an equivalent norm $\| \|_1$ so that the unconditional constant becomes 1. (Take $\|x\|_1 = \sup_\theta \|M_\theta x\|$). In other words, we have

$$\Big\| \sum_{n=1}^{\infty} \theta_n x_n \Big\|_1 \le \Big\| \sum_{n=1}^{\infty} x_n \Big\|_1, \quad |\theta_n| \le 1, \ n \in \mathbb{N}. \tag{2.1}$$

Using Lemma 2.1 and (2.1), one can prove the following proposition.

Proposition 2.2. *Let X be a complex Banach space admitting an unconditional* F.D.D. *Let $\| \|_1$ be the equivalent norm for which the unconditional constant is one. Let $B_1(R) \subset X$ be the ball of radius $R > 0$ about the origin associated to $\| \|_1$. Then the following holds.*

(1) *$M_\theta(B_1(R))$ is relatively compact in $B_1(R)$ for any sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| < 1$, which converges to* 0.

(2) *For any compact $K \subset B_1(R)$, there exist a sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| < 1$ which converges to* 0, *and a compact $L \subset B_1(R)$ so that $M_\theta L = K$.*

3. Description of the proof of Theorem 1.1

The proof given below has been inspired by a talk given by Lempert ([7]) on complex Banach spaces admitting an unconditional Schauder basis, and differs from the proofs given in [6] and [3]. Let $\| \|_1$ be the equivalent norm in X given by (2.1) and let $B_1(R) \subset X$ be the ball of radius $R > 0$ about the origin associated to $\| \|_1$. Let $x \in X$ be given by its unique representation $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for every n. As in [6], we restrict our attention to the bounded linear operators $M_{e^{2\pi i t}}$, $t = \{t_n\}$, $t_n \in \mathbb{R}$, $e^{2\pi i t} = \{e^{2\pi i t_n}\}$. $M_{e^{2\pi i t}}$ is clearly an isometry of X because of (2.1).

For $m \in \mathbb{N}$, let $[\sqrt{m}\,]$ be the integer part of $\sqrt{m}$. As in [7], we define

$$T(m) = \Big\{ e^{2\pi i t}, \ t = \{t_n\}, \ t_n \in \mathbb{R}, \ t_n = 0 \text{ for } n > [\sqrt{m}\,] \Big\} \tag{3.1}$$

and

$$K(m) = \left\{ k = \{k_n\}, \ k_n \in \mathbb{N} \cup \{0\}, \ k_n = 0 \text{ for } n > [\sqrt{m}], \ \sum_{n=1}^{[\sqrt{m}]} k_n \le m \right\}. \tag{3.2}$$

Definition 3.1. Let g be a holomorphic function on $B_1(r)$, $r > 0$. We say that g is homogeneous of degree a in x if $g(\lambda x) = \lambda^a g(x)$ for $\lambda \in \mathbb{C}$, $|\lambda| \le 1$. We say that g is homogeneous of degree b in x_N if $g(M_{\theta_\lambda} x) = \lambda^b g(x)$, where $\theta_\lambda = \{\theta_{\lambda n}\}$, $\theta_{\lambda n} = \lambda$ if $n = N$, $\theta_{\lambda n} = 1$ if $n \neq N$, $\lambda \in \mathbb{C}$, $|\lambda| \le 1$.

Let f be a holomorphic function on $B_1(R)$. For $k \in K(m)$ and $e^{2\pi is} \in T(m)$, one can define the following holomorphic functions on $B_1(R)$

$$f_m(x) = \int_{S^1} f(e^{2\pi it}x)e^{-2\pi imt}dt, \tag{3.3}$$

$$f^k(x) = \int_{T(m)} f(M_{e^{2\pi is}}x)e^{-2\pi ik.s}ds, \tag{3.4}$$

where dt (respectively ds) is the normalized Haar measure on the unit circle S^1 (respectively on $T(m)$).

Remark 3.2. f^k is homogeneous of degree k_n in x_n, for $n \leq [\sqrt{m}\ ]$ while f_m is homogeneous of degree m in x.

Consider now the formal series associated to f

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k . \tag{3.5}$$

Definition 3.3. The formal series associated to f given by (3.5) is called the Josefson series.

It is clear that when $\dim X < \infty$, the Josefson series converges to f, uniformly on compact sets of $B_1(R)$. As in [7], we set for $k \in K(m)$ and for any sequence of complex numbers $\theta = \{\theta_n\},\ 0 \leq \theta_n < 1$, which converges to 0,

$$\theta^{(k)} = \prod_{n \leq [\sqrt{m}\ ]} \theta_n^{k_n} \Big(\max_{n > [\sqrt{m}\ ]} \theta_n \Big)^{m-|k|}, \tag{3.6}$$

where $|k| = k_1 + \cdots + k_{[\sqrt{m}\ ]}$.

Proposition 3.4. *Let f be a holomorphic function on $B_1(R)$. Then the following inequality holds*

$$\overline{\lim_{m\to\infty}} \sup_{k \in K(m)} \Big(\theta^{(k)} |(f_m)^k|_{B_1(R)} \Big)^{\frac{1}{m}} < 1 \tag{3.7}$$

for any sequence of complex numbers $\theta = \{\theta_n\},\ 0 \leq \theta_n < 1$, which converges to 0.

Proof. By Proposition 2.2, $M_\theta(B_1(R))$ is relatively compact in $B_1(R)$ for any sequence of complex numbers $\theta = \{\theta_n\},\ 0 \leq \theta_n < 1$, which converges to 0. Therefore f is bounded on $M_\theta(B_1(R))$. Using Remark 3.2 and the fact that $|(f_m)^k|$ on $M_\theta(B_1(R))$ is bounded by $|f|$ on $M_\theta(B_1(R))$, we easily conclude that

$$\overline{\lim_{m\to\infty}} \sup_{k \in K(m)} \Big(\theta^{(k)} |(f_m)^k|_{B_1(R)} \Big)^{\frac{1}{m}} \leq 1 . \tag{3.8}$$

Replacing θ by $\lambda\theta$, for some good choice of a real number $\lambda > 1$, we see that the inequality (3.8) holds as a strict inequality. This achieves the proof of Proposition 3.4. □

We have the following lemma whose proof is left to the reader.

Lemma 3.5. *Let $K(m)$ be given by* (3.2)*, and let $\lambda \in \mathbb{R}$, with $\lambda < 1$. Then the series given by*

$$\sum_{m=0}^{\infty} |K(m)||\lambda^m|$$

is convergent.

Proposition 3.6. *Let h_m be a holomorphic function on $B_1(R)$ homogeneous of degree m in x. Suppose that for any sequence of complex numbers $\theta = \{\theta_n\}$, $0 \leq \theta_n < 1$, which converges to* 0*, and for any set L relatively compact in $B_1(R)$, the following holds*

$$\overline{\lim_{m\to\infty}} \sup_{k\in K(m)} \left(\theta^{(k)}|(h_m)^k|_L\right)^{\frac{1}{m}} < 1\,. \tag{3.9}$$

Then

$$\sum_{m=0}^{\infty} \sum_{k\in K(m)} (h_m)^k \tag{3.10}$$

converges, uniformly on compact sets of $B_1(R)$.

Proof. Let $K \subset B_1(R)$ be a compact set. Using Proposition 2.2, we can find $\theta = \{\theta_n\}$, with $\theta_n = \epsilon$, $0 \leq \epsilon < 1$, for $n > [\sqrt{m}\,]$, and a compact set $L \subset B_1(R)$, such that $K \subset M_\theta L$. Using Remark 3.2 and the assumption, we conclude that there exists $\lambda < 1$ such that for m large enough and $k \in K(m)$,

$$|(h_m)^k|_K < \lambda^m\,. \tag{3.11}$$

Using (3.11) and Lemma 3.5, we get that

$$\sum_{m=0}^{\infty} \sum_{k\in K(m)} |(h_m)^k| \tag{3.12}$$

converges uniformally on K. This achieves the proof of Proposition 3.6. □

Proposition 3.7. *Let h_m be a holomorphic function on $B_1(R)$ homogeneous of degree m in x. Suppose that for any sequence of complex numbers $\theta = \{\theta_n\}$, $0 \leq \theta_n < 1$, which converges to* 0*, the following holds*

$$\overline{\lim_{m\to\infty}} \sup_{k\in K(m)} \left(\theta^{(k)}|(h_m)^k|_{B_1(R)}\right)^{\frac{1}{m}} < 1\,. \tag{3.13}$$

Suppose also that there exists $t > 1$ such that

$$|(h_m)^k|_{B_1(R)} = \begin{cases} 0, \ \textit{or} \\ \geq t^m \end{cases} \tag{3.14}$$

for $k \in K(m)$.

Then

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (h_m)^k \tag{3.15}$$

is a holomorphic function on X.

Proof. Since h_m is a homogeneous holomorphic function, it implies that h_m extends holomorphically to X. Using Proposition 3.6, we see that it is enough to show that (3.9) holds for any set L relatively compact in X.

For L relatively compact in X, we choose $\delta > 0$ such that $\delta L \subset B_1(R)$.

Let $\alpha > 1$. Then there exits $\Theta = \{\Theta_n\}, \ 0 \leq \Theta_n < 1$, which converges to 0, such that

$$\overline{\lim_{m\to\infty}} \sup_{k \in K(m)} \left(\theta^{(k)} |(h_m)^k|_L\right)^{\frac{1}{m}} \leq \tag{3.16}$$

$$\delta^{-1} \overline{\lim_{m\to\infty}} \sup_{k \in K(m)} \left(\Theta^{(k)} |(h_m)^k|_{B_1(R)}\right)^{\frac{\alpha+1}{m}} \overline{\lim_{m\to\infty}} \sup_{\substack{k \in K(m), \\ |(h_m)^k|_{B_1(R)} \neq 0}} \left(|((h_m)^k)^{-1} t^m|_{B_1(R)}\right)^{\frac{\alpha}{m}} t^{-\alpha}.$$

Using Proposition 3.4 and the assumption, we conclude that

$$\overline{\lim_{m\to\infty}} \sup_{k \in K(m)} \left(\theta^{(k)} |(h_m)^k|_L\right)^{\frac{1}{m}} < \delta^{-1} t^{-\alpha}. \tag{3.17}$$

Using (3.17) for α large enough, we obtain (3.9). This achieves the proof of Proposition 3.7. □

Proof of Theorem 1.1. Let f be a holomorphic function in $B_1(R)$. Since X admits an unconditional F.D.D., f and $\sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k$ agree on a dense subset of $B_1(R)$. By Proposition 3.4 and Proposition 3.6, it is then clear that $f = \sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k$. For $t > 1$ and $M \in \mathbb{N}$, we put

$$h_{t,M} = \sum_{m > M} \sum_{\substack{k \in K(m), \\ |(f_m)^k|_{B_1(R)} \geq t^m}} (f_m)^k + \sum_{m=0}^{M} \sum_{k \in K(m)} (f_m)^k. \tag{3.18}$$

By Proposition 3.7, $h_{t,M}$ is holomorphic on X. One can check that for $r \in (0, R)$, and $\epsilon > 0$, there exist $t > 1$ and M such that $|f - h_{t,M}| < \epsilon$ in $B_1(r)$. This achieves the proof of Theorem 1.1. □

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