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Second order parabolic equations and weak uniqueness of diffusions with discontinuous coefficients

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Abstract. We prove the unique solvability of parabolic equations with discontinuous leading coefficients in $W_p^{1,2}((0,T)\times\mathbb{R}^d)$. Using this result, we establish the uniqueness of diffusion processes with time-dependent discontinuous coefficients.

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1. Introduction

The unique solvability of second order parabolic equations in non-divergence form is well established if the leading coefficients are uniformly continuous (see [8] and [9]). More precisely, the parabolic equation

$$u_t = a_{jk}(t,x)\,u_{x_jx_k} + b_j(t,x)\,u_{x_j} + c(t,x)\,u + f(t,x)$$

with appropriate initial and boundary conditions has a unique solution $u \in W_p^{1,2}((0,T)\times\Omega)$, $1 < p < \infty$, if $a_{jk}(t,x)$ are uniformly continuous with respect to x uniformly in t (a_{jk} satisfy the uniform ellipticity condition). However, even if $a_{jk}(t,x)$ are discontinuous, there are some cases where one can expect the unique solvability of the above equation. The most intensively investigated case is when the coefficients a_{jk} are in the space of VMO. As noted in [13] (also see references therein to find a list of many papers which deal with elliptic and parabolic equations with VMO coefficients), VMO contains the space of bounded uniformly continuous functions as a proper subset. However, for example, if a_{jk} are piecewise constant, then they are not in the space of VMO. This means that we may try to find another class of discontinuous coefficients a_{jk} (a class that includes piecewise constant coefficients) so that equations with leading coefficients from the class have strong unique solutions.

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In this paper we prove the existence and uniqueness of a solution u in $W^{1,2}_p((0,T)\times\mathbb{R}^d)$, $1<p<\infty$, to the above parabolic equation with coefficients $a_{jk}(t,x)$ being discontinuous at the hyper-plane $\{(t,0,x') : t\in\mathbb{R},\ x'\in\mathbb{R}^{d-1}\}$. Specifically, we assume that

$$a_{jk}(t,x)=a^+_{jk}(t,x)\ \text{ for }\ x_1>0\ \text{ and }\ a_{jk}(t,x)=a^-_{jk}(t,x)\ \text{ for }\ x_1<0,$$

where $a^+_{jk}(t,x)$ and $a^-_{jk}(t,x)$ are defined on $\overline{\mathbb{R}\times\mathbb{R}^d_+}$ and $\overline{\mathbb{R}\times\mathbb{R}^d_-}$ respectively and uniformly continuous with respect to spatial variables uniformly in t. In addition, $a^\pm_{jk}(t,0,x')$ are uniformly continuous as functions of (t,x'). We see that a^+_{jk} do not have any relation with a^-_{jk}, so the coefficients a_{jk} can be discontinuous at the hyper-plane $\{(t,0,x') : t\in\mathbb{R},\ x'\in\mathbb{R}^{d-1}\}$. Clearly, this contains piecewise constant a_{jk} with discontinuity at the hyper-plane. An a priori estimate, as usual, is the main step we achieve, and to do that, we make use of trace operators and multipliers.

When a_{jk}, b_j, and c are piecewise constant with discontinuity at the above hyper-plane, Salsa [12] solved in $W^{1,2}_2((0,\infty)\times\mathbb{R}^d)$ the equation $u_t=a_{jk}u_{x_jx_k}+b_ju_{x_j}+cu$ with a non-zero initial condition. For elliptic equations with similar leading coefficients, see [10], [11], and [5].

In this paper we also discuss the well-posedness of the martingale problem for L_t, where

$$L_t=\frac{1}{2}a_{jk}(t,\cdot)\frac{\partial^2}{\partial x_j\partial x_k}+b_j(t,\cdot)\frac{\partial}{\partial x_j}.$$

Here the coefficients $a_{jk}(t,x)$ are allow to be discontinuous at infinitely many parallel hyper-planes. For more details, see assumptions in Section 3. In fact, there are papers which investigate the well-posedness of martingale problems (*or* well-posedness of diffusion processes) when the diffusion coefficients $a_{jk}(x)$ are discontinuous in x. Some of them can be [2], [3], and [7]. However, they dealt with coefficients $a_{jk}(x)$ which are independent of t. We here consider time-dependent coefficients which are discontinuous at hyper-planes $\{(t,\gamma_j,x') : t\in\mathbb{R},\ x'\in\mathbb{R}^{d-1}\}$, where $\{\gamma_j\}$ is a subset in $\mathbb{R}$ with no limit points. As is shown in [15], the well-posedness of the martingale problem for L_t follows from the unique solvability of the corresponding parabolic equation (if the equation is uniquely solvable). Hence our result on parabolic equations in Section 2 plays a major role in the discussion of the martingale problem. For complete details about martingale problems, see the monograph [15].

This paper consists of two sections excluding this introduction. We discuss parabolic equations and martingale problems in Section 2 and 3 respectively.

A few words about notation. We denote by (t,x) a point in $\mathbb{R}^{d+1}$. That is, $(t,x)=(t,x_1,x')\in\mathbb{R}\times\mathbb{R}^d=\mathbb{R}^{d+1}$, where $t\in\mathbb{R}$, $x_1\in\mathbb{R}$, $x'\in\mathbb{R}^{d-1}$, and $x\in\mathbb{R}^d$. We write $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ for half-spaces

$$\{(x_1,x') : x_1>0,\ x'\in\mathbb{R}^{d-1}\}\quad\text{and}\quad\{(x_1,x') : x_1<0,\ x'\in\mathbb{R}^{d-1}\}$$

respectively. Throughout the paper we set $L_p=L_p(\mathbb{R}^{d+1})$.

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2. Second order parabolic equations with discontinuous coefficients

We consider parabolic equations of the form

$$\begin{cases} u_t(t,x) = Lu(t,x) + f(t,x) & \text{in} \quad (0,T)\times\mathbb{R}^d \\ u(0,x) = 0 \end{cases},$$

where

$$Lu(t,x) = a_{jk}(t,x)u_{x_jx_k}(t,x) + b_j(t,x)u_{x_j}(t,x) + c(t,x)u(t,x).$$

We prove the existence and uniqueness of solutions to the parabolic equations as above in the Sobolev space $W_p^{1,2}((0,T)\times\mathbb{R}^d)$, $1 < p < \infty$. The definition of $W_p^{1,2}((T_1,T_2)\times\Omega)$ can be found in [8] or [9], where $-\infty \le T_1 < T_2 \le \infty$ and Ω is an open set in $\mathbb{R}^d$.

Let us first state the assumptions on coefficients $a_{jk}(t,x)$, $b_j(t,x)$, and $c(t,x)$. Note that in the below the coefficients $a_{jk}(t,x)$ are not uniformly continuous in $x \in \mathbb{R}^d$. The function ω below is an increasing function defined on $[0,\infty)$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \searrow 0$.

Assumption 2.1. The coefficients a_{jk}, $j,k = 1,\cdots,d$, are defined by

$$a_{jk}(t,x) = \begin{cases} a_{jk}^+(t,x) & \text{if} \quad x_1 > 0 \\ a_{jk}^-(t,x) & \text{if} \quad x_1 < 0 \end{cases},$$

where $a_{jk}^{\pm}$ satisfy the following.

1. $a_{jk}^+(t,x)$ are defined on $\overline{\mathbb{R}\times\mathbb{R}_+^d}$ and $a_{jk}^-(t,x)$ are defined on $\overline{\mathbb{R}\times\mathbb{R}_-^d}$.
2. $a_{jk}^{\pm}(t,x) = a_{kj}^{\pm}(t,x)$, and there exists a constant $\kappa \in (0,1)$ such that, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\vartheta \in \mathbb{R}^d$,

$$\kappa|\vartheta|^2 \le \sum_{j,k=1}^d a_{jk}^{\pm}(t,x)\vartheta_j\vartheta_k \le \kappa^{-1}|\vartheta|^2.$$

3. For each $t \in \mathbb{R}$,

$$|a_{jk}^+(t,x) - a_{jk}^+(t,y)| \le \omega(|x-y|) \quad \text{for} \quad x,y \in \overline{\mathbb{R}_+^d},$$

$$|a_{jk}^-(t,x) - a_{jk}^-(t,y)| \le \omega(|x-y|) \quad \text{for} \quad x,y \in \overline{\mathbb{R}_-^d}.$$

4. For $t, s \in \mathbb{R}$ and $x', y' \in \mathbb{R}^{d-1}$,

$$|a^+_{jk}(t,0,x') - a^+_{jk}(s,0,y')| \le \omega(|t-s| + |x'-y'|),$$

$$|a^-_{jk}(t,0,x') - a^-_{jk}(s,0,y')| \le \omega(|t-s| + |x'-y'|).$$

Assumption 2.2. The coefficients $b_j(t,x)$ and $c(t,x)$ are measurable functions defined on $\mathbb{R} \times \mathbb{R}^d$ satisfying

$$|b_j(t,x)| + |c(t,x)| \le K$$

for some positive constant K.

The first case we consider is

$$u_t(t,x) = L_0 u(t,x) - \lambda u(t,x) + f(t,x) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^d$$

with

$$L_0 = \begin{cases} L_0^+ = a^+_{jk} D_{jk} & \text{in} \quad \mathbb{R} \times \mathbb{R}^d_+ \\ L_0^- = a^-_{jk} D_{jk} & \text{in} \quad \mathbb{R} \times \mathbb{R}^d_- \end{cases},$$

where a^+_{jk} and a^-_{jk} are constant. Recall that $L_p = L_p(\mathbb{R}^{d+1})$.

Theorem 2.3. *For any $\lambda \ge 0$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$,*

$$\lambda \|u\|_{L_p} + \|u_t\|_{L_p} + \|u_{xx}\|_{L_p} \le N \|L_0 u - \lambda u - u_t\|_{L_p}, \tag{2.1}$$

where N depends only on d, p, and κ.

To prove this theorem we need the following function spaces and trace theorem which can be found in [8, 17]. For a non-negative real number l, let

$$W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1}) = L_p(\mathbb{R}, W_p^l(\mathbb{R}^{d-1})) \cap W_p^{l/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1})),$$

where $W_p^s(\mathbb{R}^{d-1})$, $s = 0, 1, 2, \cdots$, are Sobolev spaces and $W_p^s(\mathbb{R}^{d-1})$, $s \ne$ integer, $s > 0$, are Slobodeckij spaces.

Theorem 2.4. *Set $l = 2 - 1/p$. There exists the trace operator T from*

$$W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d_+) \quad \textit{onto} \quad W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1}) \times W_p^{(l-1)/2,l-1}(\mathbb{R} \times \mathbb{R}^{d-1})$$

such that

$$Tu = \big(u(t,0,x'), u_{x_1}(t,0,x')\big)$$

and

$$\|u(\cdot,0,\cdot)\|_{W_p^{l/2,l}(\Gamma)} + \|u_{x_1}(\cdot,0,\cdot)\|_{W_p^{(l-1)/2,l-1}(\Gamma)} \le N \|u\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_+)} \tag{2.2}$$

for $u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d_+)$, where $\Gamma = \mathbb{R} \times \mathbb{R}^{d-1}$ and N depends only on d and p. Moreover, T has a bounded linear right inverse (extension operator).

We also need the following notations, similar to those in [10, 12]. Set

$$\alpha^{\pm}(\xi') = \mathrm{i}\sum_{j=2}^{d} a_{1j}^{\pm}\xi_j, \quad \xi' = (\xi_2, \cdots, \xi_d) \in \mathbb{R}^{d-1}, \quad \mathrm{i} = \sqrt{-1},$$

$$\beta^{\pm}(\xi') = \sum_{j,k=2}^{d} a_{jk}^{\pm}\xi_j\xi_k + 1,$$

and

$$H^{\pm}(\xi') = \left(\alpha^{\pm}(\xi')\right)^2 + a_{11}^{\pm}\beta^{\pm}(\xi')$$

$$= -\left(\sum_{j=2}^{d} a_{1j}^{\pm}\xi_j\right)^2 + a_{11}^{\pm}\left(\sum_{j,k=2}^{d} a_{jk}^{\pm}\xi_j\xi_k + 1\right).$$

Let $s \in \mathbb{R}$. We denote

$$z^{+}(s,\xi') = \frac{-\alpha^{+}(\xi') - \sqrt{H^{+}(\xi') + \mathrm{i}\,a_{11}^{+}s}}{a_{11}^{+}},$$

$$z^{-}(s,\xi') = \frac{-\alpha^{-}(\xi') + \sqrt{H^{-}(\xi') + \mathrm{i}\,a_{11}^{-}s}}{a_{11}^{-}},$$

$$z(s,\xi') = z^{+}(s,\xi') - z^{-}(s,\xi'),$$

where $\sqrt{w}$ means, from now on, that branch of the analytic function $w^{1/2}$ that has non-negative real part. Note that

$$\Re[z^{+}(s,\xi')] = -\frac{1}{a_{11}^{+}}\Re\left[\sqrt{H^{+}(\xi') + \mathrm{i}\,a_{11}^{+}s}\right]$$

and

$$\Re[z^{-}(s,\xi')] = \frac{1}{a_{11}^{-}}\Re\left[\sqrt{H^{-}(\xi') + \mathrm{i}\,a_{11}^{-}s}\right].$$

The following estimates hold true for $\Re[z^{\pm}(s,\xi')]$. One can find the same estimates in [12].

Lemma 2.5.

$$-N_2\sqrt{1 + |\xi'|^2 + |s|} \le \Re[z^{+}(s,\xi')] \le -N_1\sqrt{1 + |\xi'|^2 + |s|}$$

and

$$N_1\sqrt{1 + |\xi'|^2 + |s|} \le \Re[z^{-}(s,\xi')] \le N_2\sqrt{1 + |\xi'|^2 + |s|},$$

where N_1 and N_2 depend only on κ.

Proof. The lemma follows from the fact that $\kappa \le a_{11}^{\pm} \le \kappa^{-1}$ and

$$N_1|\xi'|^2 \le H^{\pm}(\xi') - a_{11}^{\pm} \le N_2|\xi'|^2, \tag{2.3}$$

which is proved in Lemma 3 in [10]. Here the constants N_1 and N_2 depend only on κ. □

The main step in the proof of Theorem 2.3 is to prove that an operator $\mathcal{T}$ defined by

$$\widetilde{\mathcal{T}h} = \widetilde{h}/z(s,\xi'), \quad h \in C_0^\infty(\mathbb{R}\times\mathbb{R}^{d-1}),$$

can be extended to a bounded operator from

$$W_p^{(l-1)/2,l-1}(\mathbb{R}\times\mathbb{R}^{d-1}) \quad \text{into} \quad W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1}),$$

where $l = 2 - 1/p$ and $\mathbb{R}\times\mathbb{R}^{d-1} = \{(t,0,x') : t \in \mathbb{R},\ x' \in \mathbb{R}^{d-1}\}$. Here $\widetilde{f}$ is the Fourier transform of f in $\mathbb{R}\times\mathbb{R}^{d-1}$. To prove this, we start with a lemma showing that $(1+s^2)^{1/4}/z(s,\xi')$ and $\mathrm{i}\xi_j/z(s,\xi')$ are multipliers. Note that we sometimes mean by $\mathbb{R}^d$, especially in Lemma 2.6 and 2.7, the space $\{(t,0,x') : t \in \mathbb{R},\ x' \in \mathbb{R}^{d-1}\}$.

Lemma 2.6.

$$\frac{(1+s^2)^{1/4}}{z(s,\xi')} \qquad \textit{and} \qquad \frac{\mathrm{i}\,\xi_j}{z(s,\xi')}, \quad j = 2,\cdots,d,$$

are multipliers for $L_p(\mathbb{R}^d)$. That is, if we set $\widetilde{\mathcal{T}f}(s,\xi') = m(s,\xi')\widetilde{f}$, where $f \in L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ and $m(s,\xi')$ is either of the above multipliers, then we have

$$\|\mathcal{T}f\|_{L_p(\mathbb{R}^d)} \le N\|f\|_{L_p(\mathbb{R}^d)},$$

where N depends only on d, p, and κ.

Proof. Set $s = \xi_1$ (recall that $\xi' = (\xi_2,\cdots,\xi_d)$). We prove

$$\left|\frac{\partial^m}{\partial\xi_{j_1}\cdots\partial\xi_{j_m}}\left(\frac{(1+\xi_1^2)^{1/4}}{z(\xi_1,\xi')}\right)\right| \le \frac{N}{|\xi_{j_1}\cdots\xi_{j_m}|} \tag{2.4}$$

and

$$\left|\frac{\partial^m}{\partial\xi_{j_1}\cdots\partial\xi_{j_m}}\left(\frac{\mathrm{i}\,\xi_j}{z(\xi_1,\xi')}\right)\right| \le \frac{N}{|\xi_{j_1}\cdots\xi_{j_m}|}, \quad j = 2,\cdots,d, \tag{2.5}$$

where $m = 1,\cdots,d$, $j_k \in \{1,\cdots,d\}$, and $j_k \ne j_l$ if $k \ne l$. Especially, the constant N depends only on d and κ. Once these inequalities are proved, the lemma follows from Mikhlin's theorem (Theorem 6′ on page 109 [14] or see page 289 [8]).

First, we prove that, for a multi-index α, $\alpha = (0, \alpha_2, \cdots, \alpha_d)$,

$$\left| D^\alpha \sqrt{H^\pm(\xi') + \mathrm{i}\, a_{11}^\pm \xi_1} \right| \le N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|-1}{2}}, \tag{2.6}$$

where N depends only on d, α, and κ. Using the matrix $\sqrt{[a_{jk}^+]}$ and an appropriate orthogonal change of variables, we can find a matrix $A = [\sigma_{jk}]$ satisfying the following:

(i) The mapping $x \to Ax$ is a diffeomorphism from $\mathbb{R}^d_+$ onto itself.
(ii) For a twice differentiable function $v(x)$ defined on $\mathbb{R}^d_+$, set $w(x) = v(Ax)$. Then $L_0^+ v(Ax) = \Delta w(x)$ in $\mathbb{R}^d_+$, where, as we recall, $L_0^+ v = a_{jk}^+ v_{x_j x_k}$.

Notice that from the above two properties of $A = [\sigma_{jk}]$ we have $a_{jk}^+ = \sum_{l=1}^d \sigma_{jl}\sigma_{kl}$, $k = 1, \cdots, d$, and $\sigma_{1k} = 0$, $k = 2, \cdots, d$. Denote by B the sub-matrix formed by removing the first row and column of A. Then B is a non-singular matrix. Observe that

$$|B^*\xi'|^2 = \sum_{j,k=2}^d (a_{jk}^+ - \sigma_{j1}\sigma_{k1})\xi_j\xi_k = -\sum_{j,k=2}^d \sigma_{j1}\sigma_{k1}\xi_j\xi_k + \sum_{j,k=2}^d a_{jk}^+\xi_j\xi_k$$

$$= -\left(\sum_{j=2}^d \sigma_{j1}\xi_j\right)^2 + \sum_{j,k=2}^d a_{jk}^+\xi_j\xi_k = -\frac{1}{\sigma_{11}^2}\left(\sum_{j=2}^d \sigma_{11}\sigma_{j1}\xi_j\right)^2 + \sum_{j,k=2}^d a_{jk}^+\xi_j\xi_k$$

$$= -\frac{1}{a_{11}^+}\left(\sum_{j=2}^d a_{1j}^+\xi_j\right)^2 + \sum_{j,k=2}^d a_{jk}^+\xi_j\xi_k = \frac{1}{a_{11}^+}H^+(\xi') - 1.$$

Thus

$$\sqrt{H^+(\xi') + \mathrm{i}\, a_{11}^+\xi_1} = \sqrt{a_{11}^+\left(|B^*\xi'|^2 + 1 + \mathrm{i}\,\xi_1\right)}.$$

From this and (2.3) we see that

$$\left| D^\alpha \sqrt{H^+(\xi') + \mathrm{i}\, a_{11}^+\xi_1} \right| \le N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|-1}{2}},$$

where N depends only on d, α, and κ. By the same calculation, we also have the above inequality with $H^-(\xi')$ and a_{11}^- in place of $H^+(\xi')$ and a_{11}^+. Therefore, (2.6) is proved. Moreover, it follows easily that

$$\left| D^\alpha \left[\left(H^\pm(\xi') + \mathrm{i}\, a_{11}^\pm \xi_1\right)^{-1/2} \right] \right| \le N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+1}{2}}. \tag{2.7}$$

Next, we prove that, for $\alpha = (0, \alpha_2, \cdots, \alpha_d)$,

$$\left| D^\alpha \left(\frac{1}{z(\xi_1, \xi')} \right) \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+1}{2}}, \tag{2.8}$$

where N depends only on d, α, and κ. If $\alpha = 0$, then by Lemma 2.5

$$\left| \frac{1}{z(\xi_1, \xi')} \right| \leq \frac{1}{-\Re[z^+(\xi_1, \xi')] + \Re[z^-(\xi_1, \xi')]} \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{1}{2}},$$

where N depends only on κ. For a multi-index $|\alpha| \geq 1$, the inequality (2.8) is justified using (2.6) and induction.

Finally, we prove that, for $\alpha = (0, \alpha_2, \cdots, \alpha_d)$,

$$\left| D^\alpha \left[\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) \right] \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+3}{2}}, \tag{2.9}$$

where N depends only on d, α, and κ. This follows from (2.7), (2.8), and

$$\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) = \frac{\mathrm{i}}{2z(\xi_1, \xi')^2} \left(\frac{a_{11}^+}{\sqrt{H^+(\xi') + \mathrm{i}\, a_{11}^+ \xi_1}} + \frac{a_{11}^-}{\sqrt{H^-(\xi') + \mathrm{i}\, a_{11}^- \xi_1}} \right).$$

Now we use (2.8) and (2.9) to prove (2.4) and (2.5). For example, if $\xi_{j_1} = \xi_1$ and $D^\alpha = \frac{\partial^{m-1}}{\partial \xi_{j_2} \cdots \partial \xi_{j_m}}$, then

$$\left| \frac{\partial^m}{\partial \xi_{j_1} \cdots \partial \xi_{j_m}} \left(\frac{(1 + \xi_1^2)^{1/4}}{z(\xi_1, \xi')} \right) \right|$$

$$= \left| \frac{\xi_1}{2(1 + \xi_1^2)^{3/4}} D^\alpha \left(\frac{1}{z(\xi_1, \xi')} \right) + (1 + \xi_1^2)^{1/4} D^\alpha \left[\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) \right] \right|$$

$$\leq N \left(\frac{|\xi_1|}{(1 + \xi_1^2)^{3/4} (1 + |\xi'|^2 + |\xi_1|)^{(|\alpha|+1)/2}} + \frac{(1 + \xi_1^2)^{1/4}}{(1 + |\xi'|^2 + |\xi_1|)^{(|\alpha|+3)/2}} \right)$$

$$\leq \frac{N}{|\xi_1| (1 + |\xi'|^2 + |\xi_1|)^{\frac{|\alpha|}{2}}} \leq \frac{N}{|\xi_{j_1} \cdots \xi_{j_m}|}.$$

The other cases are proved in a similar way. The lemma is proved. □

Using the above result and interpolation, we prove the following lemma.

Lemma 2.7. *Let $\mathcal{T}$ be an operator such that*

$$\widetilde{\mathcal{T}h} = \frac{1}{z(s,\xi')}\widetilde{h}, \quad h \in C_0^\infty(\mathbb{R}^d),$$

where $\widetilde{\mathcal{T}h}$ and $\widetilde{h}$ are Fourier transforms of $\mathcal{T}h$ and h in $\mathbb{R}^d$ respectively. Set $l = 2 - 1/p$, then

$$\|\mathcal{T}h\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})} \le N\|h\|_{W_p^{(l-1)/2,l-1}(\mathbb{R}\times\mathbb{R}^{d-1})}, \tag{2.10}$$

where N depends only on d, p, and κ.

Proof. Define an operator $\mathcal{S}$ by

$$\widetilde{\mathcal{S}h} = \frac{(1+s^2)^{1/4}}{z(s,\xi')}\widetilde{h},$$

where $h \in C_0^\infty(\mathbb{R}^d)$. By Lemma 2.6 the operator $\mathcal{S}$ can be extended to a bounded operator from $L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ to $L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ and from $W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ to $W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$. Here we used the fact that $\mathcal{S}(D^\alpha h) = D^\alpha \mathcal{S}(h)$. By real interpolation (see [1]) we have

$$(L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1})), W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1}))_{(l-1)/2,\,p} = W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1})).$$

Thus

$$\|\mathcal{S}h\|_{W_p^{(l-1)/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))} \le N\|h\|_{W_p^{(l-1)/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))}, \tag{2.11}$$

where N depends only on d, p, and κ. Since $(1+s^2)^{1/4}$ is an isometric multiplier from $W_p^{l/2}(\mathbb{R})$ $(=B_{p,p}^{l/2}(\mathbb{R}))$ to $W_p^{(l-1)/2}(\mathbb{R})$ $(=B_{p,p}^{(l-1)/2}(\mathbb{R}))$,

$$\|\mathcal{T}h\|^p_{W_p^{l/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))} = \int_{\mathbb{R}^{d-1}} \|\mathcal{T}h(\cdot,x')\|^p_{W_p^{l/2}(\mathbb{R})}\,dx'$$

$$= \int_{\mathbb{R}^{d-1}} \|\mathcal{F}_1^{-1}(1+s^2)^{1/4}\mathcal{F}_1(\mathcal{T}h)(\cdot,x')\|^p_{W_p^{(l-1)/2}(\mathbb{R})}\,dx'$$

$$= \|\mathcal{S}h\|^p_{W_p^{(l-1)/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))},$$

where $\mathcal{F}_1$ and $\mathcal{F}_1^{-1}$ are the Fourier transform and its inverse on $\mathbb{R}$. This and the inequality (2.11) prove

$$\|\mathcal{T}h\|_{W_p^{l/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))} \le N\|h\|_{W_p^{(l-1)/2}(\mathbb{R},L_p(\mathbb{R}^{d-1}))}, \tag{2.12}$$

where N depends only on d, p, and κ.

On the other hand, from Lemma 2.6, we know that $i\xi_j/z(s,\xi')$, $j=2,\cdots,d$, are multipliers for $L_p(\mathbb{R}^d)$. This implies that

$$\int_{-\infty}^{\infty}\|\mathcal{T}h(t,\cdot)\|^p_{W_p^{k+1}(\mathbb{R}^{d-1})}\,dt\le N\int_{-\infty}^{\infty}\|h(t,\cdot)\|^p_{W_p^{k}(\mathbb{R}^{d-1})}\,dt,$$

where $k=0,1$ and N depends only on d, p, and κ. Hence $\mathcal{T}$ can be extended to a bounded operator from

$$L_p(\mathbb{R},W_p^k(\mathbb{R}^{d-1}))\quad\text{to}\quad L_p(\mathbb{R},W_p^{k+1}(\mathbb{R}^{d-1})),$$

where $k=0,1$. Note that by real interpolation (see [1] or Theorem 1.18.4 in [16]), we have

$$\Big(L_p(\mathbb{R},L_p(\mathbb{R}^{d-1})),L_p(\mathbb{R},W_p^1(\mathbb{R}^{d-1}))\Big)_{l-1,\,p}=L_p(\mathbb{R},W_p^{l-1}(\mathbb{R}^{d-1}))$$

and

$$\Big(L_p(\mathbb{R},W_p^1(\mathbb{R}^{d-1})),L_p(\mathbb{R},W_p^2(\mathbb{R}^{d-1}))\Big)_{l-1,\,p}=L_p(\mathbb{R},W_p^{l}(\mathbb{R}^{d-1})).$$

Therefore,

$$\|\mathcal{T}h\|_{L_p(\mathbb{R},W_p^l(\mathbb{R}^{d-1}))}\le N\|h\|_{L_p(\mathbb{R},W_p^{l-1}(\mathbb{R}^{d-1}))},\tag{2.13}$$

where N depends only on d, p, and κ.

Finally, recall that

$$W_p^{s/2,s}(\mathbb{R}\times\mathbb{R}^{d-1})=L_p(\mathbb{R},W_p^s(\mathbb{R}^{d-1}))\cap W_p^{s/2}(\mathbb{R},L_p(\mathbb{R}^{d-1})),$$

where s is either l or $l-1$. From this, (2.12), and (2.13), the inequality (2.10) follows. □

We need the Fourier transform of a solution to a parabolic equation defined on a parabolic half-space (e.g. $\mathbb{R}\times\mathbb{R}^d_+$).

Lemma 2.8. *For $g(t,x')\in C_0^\infty(\mathbb{R}\times\mathbb{R}^{d-1})(=C_0^\infty(\mathbb{R}^d))$, let $v^+\in W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_+)$ and $v^-\in W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_-)$ be the solutions of the equations*

$$\begin{cases} v_t^+=L_0^+v^+-v^+ \ \ in\ \ \mathbb{R}\times\mathbb{R}^d_+\\ v^+(t,0,x')=g(t,x')\end{cases},\qquad \begin{cases} v_t^-=L_0^-v^--v^- \ \ in\ \ \mathbb{R}\times\mathbb{R}^d_-\\ v^-(t,0,x')=g(t,x').\end{cases}$$

Then

$$\widetilde{v_{x_1}^+}(s,0,\xi')=z^+(s,\xi')\widetilde{g}(s,\xi')\quad\text{and}\quad\widetilde{v_{x_1}^-}(s,0,\xi')=z^-(s,\xi')\widetilde{g}(s,\xi'),$$

where Fourier transforms are taken with respect to $(t,x')\in\mathbb{R}\times\mathbb{R}^{d-1}$.

Proof. First consider

$$\begin{cases} v_t = \Delta v - v \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^d_+ \\ v(t, 0, x') = g(t, x') \end{cases}, \tag{2.14}$$

where $g \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$. Let

$$K(t, x_1, x') = \begin{cases} \dfrac{1}{(4\pi t)^{d/2}} \dfrac{x_1}{t} e^{-\frac{1}{4t}|x|^2 - t} & \text{for} \quad t > 0, \quad x_1 > 0 \\ 0 & \text{for} \quad t \le 0, \quad x_1 > 0. \end{cases}$$

Then the solution to (2.14) is

$$v(t, x_1, x') = [K(\cdot, x_1, \cdot) * g(\cdot, \cdot)](t, x').$$

We see that

$$\int_{\mathbb{R}^d} e^{-\mathrm{i}(st + \xi' \cdot x')} K(t, x_1, x')\, dt\, dx' = e^{-x_1\sqrt{1+|\xi'|^2 + \mathrm{i}s}}.$$

Thus we have

$$\widetilde{v}(s, x_1, \xi') = \widetilde{g}(s, \xi') e^{-x_1\sqrt{1+|\xi'|^2+\mathrm{i}s}}$$

and

$$\widetilde{v_{x_1}}(s, 0, \xi') = -\widetilde{g}(s, \xi')\sqrt{1 + |\xi'|^2 + \mathrm{i}s}, \tag{2.15}$$

where the second equality can be justified as follows. Note that

$$v(t, x_1, x') = c_d \int_{\mathbb{R}^d} e^{\mathrm{i}(ts + x' \cdot \xi')} \widetilde{g}(s, \xi') e^{-x_1\sqrt{1+|\xi'|^2+\mathrm{i}s}}\, ds\, d\xi',$$

where $c_d = (2\pi)^{-d/2}$. Thus

$$\begin{aligned} v_{x_1}(t, 0, x') &= \lim_{h \to 0} c_d \int_{\mathbb{R}^d} e^{\mathrm{i}\,(ts + x' \cdot \xi')} \widetilde{g}(s, \xi') \frac{e^{-h\sqrt{1+|\xi'|^2+\mathrm{i}s}} - 1}{h}\, ds\, d\xi' \\ &= -c_d \int_{\mathbb{R}^d} e^{\mathrm{i}(ts + x' \cdot \xi')} \widetilde{g}(s, \xi')\sqrt{1 + |\xi'|^2 + \mathrm{i}s}\ \, ds\, d\xi'. \end{aligned}$$

By taking Fourier transforms, we arrive at (2.15). This, together with the definition of $z^+(s, \xi')$, proves the case $L_0^+ = \Delta$. The case $L_0^- = \Delta$ is proved similarly. For general L_0^+ and L_0^-, we use the change of variables in the proof of Lemma 2.6. □

Proof of Theorem 2.3. We prove the inequality (2.1) only for $\lambda > 0$ because the inequality with $\lambda = 0$ is obtained by continuity (with respect to λ). In the case $\lambda > 0$, since we can use a dilation argument, it is enough to prove (i.e., the inequality (2.1) with $\lambda = 1$)

$$\|u\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d)} \leq N\|L_0u - u - u_t\|_{L_p}, \tag{2.16}$$

where N depends only on d, p, and κ.

Let $f = u_t + u - L_0u$ and $g(t, x') = u(t, 0, x')$. Consider

$$\begin{cases} v_t^+ = L_0^+v^+ - v^+ \ \text{ in } \mathbb{R}\times\mathbb{R}^d_+ \\ v^+(t, 0, x') = g(t, x'), \end{cases} \quad \begin{cases} w_t^+ = L_0^+w^+ - w^+ + f \ \text{ in } \mathbb{R}\times\mathbb{R}^d_+ \\ w^+(t, 0, x') = 0, \end{cases}$$

and

$$\begin{cases} v_t^- = L_0^-v^- - v^- \ \text{ in } \mathbb{R}\times\mathbb{R}^d_- \\ v^-(t, 0, x') = g(t, x'), \end{cases} \quad \begin{cases} w_t^- = L_0^-w^- - w^- + f \ \text{ in } \mathbb{R}\times\mathbb{R}^d_- \\ w^-(0, t, x') = 0. \end{cases}$$

As is well known, the above equations have unique solutions

$$v^+,\ w^+ \in W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_+), \quad v^-,\ w^- \in W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_-)$$

satisfying (we set $l = 2 - 1/p$.)

$$\|v^+\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_+)} \leq N\|g\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})},\ \|w^+\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_+)} \leq N\|f\|_{L_p(\mathbb{R}\times\mathbb{R}^d_+)}, \tag{2.17}$$

and

$$\|v^-\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_-)} \leq N\|g\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})},\ \|w^-\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d_-)} \leq N\|f\|_{L_p(\mathbb{R}\times\mathbb{R}^d_-)}, \tag{2.18}$$

where N depends only on d, p, and κ. We see that

$$u = \begin{cases} v^+ + w^+ & \text{in} \quad \mathbb{R}\times\mathbb{R}^d_+ \\ v^- + w^- & \text{in} \quad \mathbb{R}\times\mathbb{R}^d_- \end{cases}.$$

It then follows that

$$\|u\|_{W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d)} \leq N\left(\|f\|_{L_p} + \|g\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})}\right),$$

where N depends only on d, p, and κ. Therefore, to prove (2.16) we need only prove

$$\|g\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})} \leq N\|f\|_{L_p}, \tag{2.19}$$

where N depends only on d, p, and κ. We prove this inequality as follows.

Observe that (in the trace sense)

$$u_{x_1}(t,0,x') = v^+_{x_1}(t,0,x') + w^+_{x_1}(t,0,x') = v^-_{x_1}(t,0,x') + w^-_{x_1}(t,0,x').$$

Thus

$$v^+_{x_1}(t,0,x') - v^-_{x_1}(t,0,x') = w^-_{x_1}(t,0,x') - w^+_{x_1}(t,0,x'). \tag{2.20}$$

Set

$$h(t,x') = w^-_{x_1}(t,0,x') - w^+_{x_1}(t,0,x').$$

Then by Theorem 2.4 we have $h \in W_p^{(l-1)/2,l-1}(\mathbb{R}\times\mathbb{R}^{d-1})$. In addition, from the inequality (2.2) and the estimates for $w^\pm$ in (2.17) and (2.18), we have

$$\|h\|_{W_p^{(l-1)/2,l-1}(\mathbb{R}\times\mathbb{R}^{d-1})} \le N\|f\|_{L_p},$$

where N depends only on d, p, and κ. From this inequality, we notice that the inequality (2.19) follows if we prove

$$\|g\|_{W_p^{l/2,l}(\mathbb{R}\times\mathbb{R}^{d-1})} \le N\|h\|_{W_p^{(l-1)/2,l-1}(\mathbb{R}\times\mathbb{R}^{d-1})}. \tag{2.21}$$

Notice that by Lemma 2.8, (2.20), and the definition of $z(s,\xi')$,

$$z(s,\xi')\widetilde{g}(s,\xi') = \widetilde{h}(s,\xi'), \quad \text{i.e.,} \quad \widetilde{g}(s,\xi') = \frac{1}{z(s,\xi')}\widetilde{h}(s,\xi').$$

Then Lemma 2.7 proves the inequality (2.21). The theorem is proved. □

Now we solve the first case. Specifically, for a given $f \in L_p = L_p(\mathbb{R}^{d+1})$ and $\lambda > 0$, there exists a unique $u \in W_p^{1,2}(\mathbb{R}\times\mathbb{R}^d)$ such that

$$u_t = L_0 u - \lambda u + f.$$

This can be done, as is well known, using the estimate in Theorem 2.3 and the method of continuity. In addition, we can find a unique solution $u \in W_p^{1,2}((0,T)\times\mathbb{R}^d)$ of the equation

$$\begin{cases} u_t = L_0 u - \lambda u + f & \text{in} \quad (0,T)\times\mathbb{R}^d \\ u(0,x) = 0 \end{cases},$$

where $0 < T \le \infty$.

We now consider the general case. That is, we deal with

$$u_t(t,x) = Lu(t,x) - \lambda u(t,x) + f(t,x) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^d, \tag{2.22}$$

where the coefficients of L satisfy Assumption 2.1 and 2.2. The problem can be solved by freezing the coefficients a_{jk} and using the estimate in Theorem 2.3. Indeed, we can proceed as follows.

Lemma 2.9. *There exist constants $\varepsilon > 0$, λ_0, and N, depending only on d, K, p, κ, and ω, such that if $u \in C_0^\infty(\mathbb{R}^{d+1})$ satisfies*

$$\operatorname{supp} u \subset \{(t, x_1, x') : |t - t_0| < \varepsilon,\ |x' - x_0'| < \varepsilon\}$$

for some $(t_0, x_0') \in \mathbb{R} \times \mathbb{R}^{d-1}$, then

$$\lambda \|u\|_{L_p} + \|u_t\|_{L_p} + \|u_{xx}\|_{L_p} \le N \|Lu - \lambda u - u_t\|_{L_p} \tag{2.23}$$

for any $\lambda > \lambda_0$.

Proof. We can assume that $t_0 = 0$ and $x_0' = 0$. Choose an infinitely differentiable function $\eta(x_1)$ defined on $\mathbb{R}$ satisfying $\eta \ge 0$ and

$$\eta(x_1) = \begin{cases} 1 & -\varepsilon/2 \le x_1 \le \varepsilon/2 \\ 0 & x_1 \ge \varepsilon \ \text{ or } \ x_1 \le -\varepsilon \end{cases},$$

where ε will be specified later. Set $\mu = 1 - \eta$.

First notice that $\mu u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)$, $\mu u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)$, and $(\mu u)(t, 0, x') = 0$. Thus by the well-known result (see, e.g., Chapter VII and Exercise 7.5 in [9]), there exist λ' and N, depending only on d, K, p, κ, and ω, such that, for any $\lambda > \lambda'$,

$$\begin{aligned} \lambda \|\mu u\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} + \|(\mu u)_t\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} + \|(\mu u)_{xx}\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} \\ \le N \|L(\mu u) - \lambda(\mu u) - (\mu u)_t\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)}. \end{aligned}$$

The same estimate holds true with $\mathbb{R}_-^d$ in place of $\mathbb{R}_+^d$. Hence we have

$$\lambda \|\mu u\|_{L_p} + \|(\mu u)_t\|_{L_p} + \|(\mu u)_{xx}\|_{L_p} \le N \|L(\mu u) - \lambda(\mu u) - (\mu u)_t\|_{L_p} \tag{2.24}$$

for any $\lambda > \lambda'$. Note that the above N is independent of ε.

Now consider ηu. We see that

$$\operatorname{supp}(\eta u) \subset \{(t, x_1, x') : |t| \le \varepsilon,\ |x_1| \le \varepsilon,\ |x'| \le \varepsilon\}.$$

Set

$$L_0 = \begin{cases} a_{jk}^+(0) D_{jk} & \text{if} \quad x_1 > 0 \\ a_{jk}^-(0) D_{jk} & \text{if} \quad x_1 < 0, \end{cases}$$

where $0 \in \mathbb{R}^{d+1}$. Then by Theorem 2.3, we have

$$\lambda \|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p} \le N \|L_0(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} \tag{2.25}$$

for $\lambda > 0$, where N is the same as in Theorem 2.3. Observe that

$$\begin{aligned} \|L_0(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} \\ \le \|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \|(L_0 - L)(\eta u)\|_{L_p} \end{aligned}$$

and

$$\|(L_0 - L)(\eta u)\|_{L_p} \le N\|(\eta u)_x\|_{L_p} + N\|\eta u\|_{L_p} + J_1 + J_2,$$

where N depends only on d and K, and

$$J_1 = \sup_{\substack{(t,x)\in \mathrm{supp}(\eta u)\\ x_1\ge 0}} \left(|a^+_{jk}(0) - a^+_{jk}(t,x)|\right) \|(\eta u)_{xx}\|_{L_p(\mathbb{R}\times\mathbb{R}^d_+)},$$

$$J_2 = \sup_{\substack{(t,x)\in \mathrm{supp}(\eta u)\\ x_1\le 0}} \left(|a^-_{jk}(0) - a^-_{jk}(t,x)|\right) \|(\eta u)_{xx}\|_{L_p(\mathbb{R}\times\mathbb{R}^d_-)}.$$

Also observe that

$$|a^+_{jk}(0) - a^+_{jk}(t,x_1,x')| \le |a^+_{jk}(0) - a^+_{jk}(t,0,x')| + |a^+_{jk}(t,0,x') - a^+_{jk}(t,x_1,x')|$$

$$\le \omega(|t| + |x'|) + \omega(|x_1|).$$

Similarly,

$$|a^-_{jk}(0) - a^-_{jk}(t,x_1,x)| \le \omega(|t| + |x'|) + \omega(|x_1|).$$

Hence we can find a sufficiently small $\varepsilon > 0$ depending only on d, K, p, κ, and ω, such that the following inequality holds.

$$\begin{aligned}&\lambda\|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p}\\ &\le N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \frac{1}{4}\|(\eta u)_{xx}\|_{L_p} + N\|(\eta u)_x\|_{L_p} + N\|\eta u\|_{L_p}.\end{aligned}$$

Using the interpolation inequality, we find that the right hand side of the above inequality is not greater than

$$N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \frac{1}{2}\|(\eta u)_{xx}\|_{L_p} + N'\|\eta u\|_{L_p},$$

where N is the same as in (2.25) and N' depends only on d, K, p, and κ. This implies that there exits $\lambda'' > 0$ (specifically, $\lambda'' \ge 2N'$) depending only on d, K, p, and κ, such that

$$\lambda\|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p} \le 2N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p}$$

for any $\lambda > \lambda''$, where N is the one in (2.25). From this inequality and the inequality (2.24) we can find λ_0 and N depending only on d, K, p, κ, and ω, such that the inequality (2.23) holds true. The lemma is proved. □

Lemma 2.10. *There exist λ_1 and N, depending only on d, K, p, κ, and ω, such that, for any $\lambda > \lambda_1$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$, the estimate* (2.23) *holds true.*

Proof. First note that (recall again that $L_p = L_p(\mathbb{R}^{d+1})$)

$$\begin{aligned}\|u_t\|_{L_p} &= \|Lu - \lambda u - (Lu - \lambda u - u_t)\|_{L_p} \\ &\le \|Lu - \lambda u - u_t\|_{L_p} + N\|u_{xx}\|_{L_p} + N\|u_x\|_{L_p} + (\lambda + N)\|u\|_{L_p}.\end{aligned}$$

Thus we need only to prove that there exist λ_1 and N such that, for $\lambda > \lambda_1$,

$$\lambda\|u\|_{L_p} + \|u_{xx}\|_{L_p} \le N\|Lu - \lambda u - u_t\|_{L_p}.$$

Choose an infinitely differentiable function $\zeta(t, x')$ defined on $\mathbb{R}^d$ (recall that $t \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$) such that $\|\zeta\|_{L_p(\mathbb{R}^d)} = 1$ and

$$\operatorname{supp}\zeta \subset \{(t, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : |t| < \varepsilon,\ |x'| < \varepsilon\},$$

where ε is as in Lemma 2.9. We denote

$$\zeta^{s,y'}(t, x') = \zeta(t - s, x' - y').$$

Then

$$|u_{xx}(t, x)|^p = \int_{\mathbb{R}^d} |u_{xx}(t, x)\zeta^{s,y'}(t, x')|^p \, ds\, dy'. \tag{2.26}$$

Using the fact that $u_{x_jx_k}\zeta = (u\zeta)_{x_jx_k} - u_{x_j}\zeta_{x_k} - u_{x_k}\zeta_{x_j} - u\zeta_{x_jx_k}$, we obtain

$$\begin{aligned}|u_{x_jx_k}(t, x)\zeta^{s,y'}(t, x')|^p &\le 2^p|(u(t, x)\zeta^{s,y'}(t, x'))_{x_jx_k}|^p \\ &\quad + N(|u(t, x)|^p + |u_x(t, x)|^p)\varphi(t - s, x' - y'),\end{aligned}$$

where $\varphi = |\zeta_x|^p + |\zeta_{xx}|^p \in L_1(\mathbb{R}^d)$. Combining this and (2.26), we have

$$\|u_{xx}\|_{L_p}^p \le N\int_{\mathbb{R}^d} \|(u\zeta^{s,y'})_{xx}\|_{L_p}^p \, ds\, dy' + N\left(\|u\|_{L_p}^p + \|u_x\|_{L_p}^p\right).$$

For each $(s, y') \in \mathbb{R}^d$, we see that $u(t, x)\zeta^{s,y'}(t, x')$ satisfies the assumptions in Lemma 2.9. Thus

$$\|(u\zeta^{s,y'})_{xx}\|_{L_p} \le N\left\|\left(L - \lambda - \frac{\partial}{\partial t}\right)(u\zeta^{s,y'})\right\|_{L_p}$$

for $\lambda > \lambda_0$, where λ_0 and N are as in Lemma 2.9. We then have

$$\|u_{xx}\|_{L_p}^p \le N\int_{\mathbb{R}^d} \left\|\left(L - \lambda - \frac{\partial}{\partial t}\right)(u\zeta^{s,y'})\right\|_{L_p}^p ds\, dy' + N\left(\|u\|_{L_p}^p + \|u_x\|_{L_p}^p\right). \tag{2.27}$$

Note that

$$\left(L-\lambda-\frac{\partial}{\partial t}\right)(u\zeta^{s,y'}) = \zeta^{s,y'}(Lu-\lambda u-u_t) + 2a_{jk}u_{x_j}\zeta^{s,y'}_{x_k} + u\left[a_{jk}\zeta^{s,y'}_{x_jx_k} + b_j\zeta^{s,y'}_{x_j} - \zeta^{s,y'}_t\right],$$

where, for instance, $\zeta^{s,y'}_{x_jx_k}(t,x') = \zeta_{x_jx_k}(t-s, x'-y')$. Thus

$$\int_{\mathbb{R}^d}\left\|\left(L-\lambda-\frac{\partial}{\partial t}\right)(u\zeta^{s,y'})\right\|^p_{L_p} ds\,dy' \le N\left(\|Lu-\lambda u-u_t\|^p_{L_p} + \|u\|^p_{L_p} + \|u_x\|^p_{L_p}\right).$$

From this and (2.27) we have

$$\|u_{xx}\|^p_{L_p} \le N\left(\|Lu-\lambda u-u_t\|^p_{L_p} + \|u\|^p_{L_p} + \|u_x\|^p_{L_p}\right). \tag{2.28}$$

Similarly,

$$\lambda^p\|u\|^p_{L_p} = \lambda^p\int_{\mathbb{R}^d}\|u\zeta^{s,y'}\|^p_{L_p}\,ds\,dy' \le N\left(\|Lu-\lambda u-u_t\|^p_{L_p} + \|u\|^p_{L_p} + \|u_x\|^p_{L_p}\right). \tag{2.29}$$

Therefore, from (2.28) and (2.29) we have

$$\lambda\|u\|_{L_p} + \|u_{xx}\|_{L_p} \le N_1\|Lu-\lambda u-u_t\|_{L_p} + N_2\left(\|u_x\|_{L_p} + \|u\|_{L_p}\right),$$

where N_1 and N_2 depend only on d, K, p, κ, and ω. Now we make use of the argument in the proof of Lemma 2.9 to finish the proof. □

Using the estimate proved in the above lemma and the method of continuity, we arrive at the following theorem.

Theorem 2.11. *Let $\lambda > \lambda_1$, where λ_1 is as in Lemma 2.10. For $f \in L_p(\mathbb{R}\times\mathbb{R}^d)$, there exists a unique solution $u \in W^{1,2}_p(\mathbb{R}\times\mathbb{R}^d)$ satisfying the parabolic equation* (2.22). *Moreover, the solution u satisfies the estimate* (2.23).

Remark 2.12. It follows from Theorem 2.11 that there exists a unique solution $u \in W^{1,2}_p((0,T)\times\mathbb{R}^d)$ to the equation

$$\begin{cases} u_t(t,x) = Lu(t,x) + f(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^d \\ u(0,x) = 0 \end{cases}.$$

Moreover, the solution u satisfies

$$\|u\|_{W^{1,2}_p((0,T)\times\mathbb{R}^d)} \le N\|f\|_{L_p((0,T)\times\mathbb{R}^d)},$$

where N depends on d, K, T, p, κ, and ω.

3. Uniqueness of diffusions with discontinuous coefficients

In this section we investigate the well-posedness of the martingale problem for L_t

$$L_t = \frac{1}{2} a_{jk}(t, \cdot) \frac{\partial^2}{\partial x_j \partial x_k} + b_j(t, \cdot) \frac{\partial}{\partial x_j}, \tag{3.1}$$

where the coefficient a_{jk} and b_j satisfy the following assumptions.

Assumption 3.1. $a_{jk}(t, x)$ and $b_j(t, x)$ are Borel measurable functions defined on $[0, \infty) \times \mathbb{R}^d$ such that, for all $t \in [0, \infty)$, $x, \vartheta \in \mathbb{R}^d$,

$$a_{jk} = a_{kj}, \qquad a_{jk}(t, x)\vartheta_j \vartheta_k \geq \kappa |\vartheta|^2,$$

$$|a_{jk}(t, x)| + |b_j(t, x)|^2 \leq K(1 + |x|^2).$$

Assumption 3.2. Let I be the set of all integers. There exists a sequence $\{\gamma_j, j \in I\} \subset \mathbb{R}$, $\gamma_j < \gamma_{j+1}$, with no limit points such that

(i) for $x = (x_1, x')$, $x_1 \notin \{\gamma_j, j \in I\}$,

$$\lim_{y \to x} \sup_{0 \leq t \leq T} |a_{jk}(t, y) - a_{jk}(t, x)| = 0$$

for all $T > 0$,

(ii) for $x = (\gamma_j, x')$, $j \in I$, there exist $a^+_{jk}(t, \gamma_j, x')$ and $a^-_{jk}(t, \gamma_j, x')$ defined on $[0, \infty) \times \mathbb{R}^{d-1}$ such that they are continuous (in (t, x')), and they satisfy

$$\lim_{\substack{(y_1, y') \to (\gamma_j, x') \\ y_1 > \gamma_j}} \sup_{0 \leq t \leq T} |a^+_{jk}(t, \gamma_j, x') - a_{jk}(t, y_1, y')| = 0$$

and

$$\lim_{\substack{(y_1, y') \to (\gamma_j, x') \\ y_1 < \gamma_j}} \sup_{0 \leq t \leq T} |a^-_{jk}(t, \gamma_j, x') - a_{jk}(t, y_1, y')| = 0$$

for all $T > 0$.

Let $\Omega = C([0, \infty), \mathbb{R}^d)$ and $\mathcal{F} = \mathcal{B}(\Omega)$, where $C([0, \infty), \mathbb{R}^d)$ is the Polish space of continuous $\mathbb{R}^d$-valued functions given on $[0, \infty)$ with metric

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1).$$

$\mathcal{B}(\Omega)$ is, as usual, the Borel σ-field of subsets of Ω. Define a random variable X on Ω by $X_t(\omega) = \omega(t)$, $\omega \in \Omega$.

We say that the martingale problem for L_t (*or* martingale problem for a_{jk} and b_j) is well-posed (see [15], [4]) if, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exists a unique probability measure $P_{s,x}$ on $(\Omega, \mathcal{F})$ satisfying

$$P_{s,x}(X_t = x, \quad 0 \le t \le s) = 1$$

such that

$$f(X_t) - \int_s^t L_r f(X_r)\, dr$$

is a $P_{s,x}$- martingale after time s for all $f \in C_0^\infty(\mathbb{R}^d)$.

Remark 3.3. The well-posedness of the martingale problem for the operator L_t in (3.1) is equivalent to that of the stochastic integral equation

$$\xi_t = x + \int_0^t \sigma(r, \xi_r) dW_r + \int_0^t b(r, \xi_r) dr,$$

where W_t is a Brownian motion, σ is the positive square root of the matrix $[a_{jk}]_{j,k=1}^d$, and $b = (b_j)_{j=1}^d$. The stochastic integral equation is said to be well-posed if, for every initial condition $x \in \mathbb{R}^d$, it admits a weak solution which is unique in the sense of probability law. For details, see [4].

We first note that, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exists a solution to the martingale problem for L_t. This follows from Remark 2.1 in [7] and references therein.

To prove uniqueness, we start with coefficients a_{jk} and b_j which satisfy assumptions in Section 2.

Lemma 3.4. *Let $a_{jk}(t, x)$ and $b_j(t, x)$ be Borel measurable functions defined on $[0, \infty) \times \mathbb{R}^d$ satisfying Assumption* 2.1 *and* 2.2 *in Section* 2. *Then the martingale problem for L_t is well-posed.*

Proof. Clearly, we need to prove only the uniqueness of a solution to martingale problem for a_{jk} and b_j starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$. By Corollary 6.2.6 in [15], this can be done if we prove that, for each $0 \le s < T$ and $f \in C_0^\infty([s, T] \times \mathbb{R}^d)$, there is a bounded measurable function u defined on $[0, T) \times \mathbb{R}^d$ such that

$$u(s, x) = -E^P \left[\int_s^T f(t, X_t)\, dt \right] \tag{3.2}$$

whenever P solves the martingale problem for L_t starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$.

Let $P = P_{s,x}$ is a solution to the martingale problem for L_t starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$. For a given $f \in C_0^\infty([s, T] \times \mathbb{R}^d)$, we set

$$\tilde{f} = \begin{cases} f(t, x) & \text{if} \quad s \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

We also fix a real number p such that $d+1 < p < \infty$. Then by remark 2.12 we can find a unique solution $u \in W_p^{1,2}((0,T)\times\mathbb{R}^d)$ of the equation

$$\begin{cases} u_t + L_t u(t,x) = \tilde{f} & \text{in} \quad (0,T)\times\mathbb{R}^d \\ u(T,x) = 0 \end{cases}.$$

Now we find a sequence $u_n \in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that each u_n is bounded (derivatives are bounded as well), $u_n \to u$ in $W_p^{1,2}((0,T)\times\mathbb{R}^d)$, and $u_n(T,x) = 0$. Note that we have (see Lemma 6.3.1 in [15])

$$u_n(s,x) = -E^P\left[\int_s^T (u_{nt} + L_t u_n)(t, X_t)\, dt\right]$$

for each n, where, as we recall,

$$\begin{aligned}(u_{nt} + Lu_n)(t, X_t) \\ = u_{nt}(t, X_t) + \frac{1}{2}a_{jk}(t, X_t)u_{nx_jx_k}(t, X_t) + b_j(t, X_t)u_{nx_j}(t, X_t).\end{aligned}$$

Since $u_n(s,x) \to u(s,x)$ pointwise by the Sobolev embedding theorem, to prove (3.2), it only remains to prove that

$$E^P\left[\int_s^T (u_{nt} + L_t u_n)(t, X_t)\, dt\right] \to E^P\left[\int_s^T (u_t + L_t u)(t, X_t)\, dt\right].$$

This follows easily from

$$\begin{aligned} E^P \int_s^T \left|(u_t^n + L_t u^n)(t, X_t) - (u_t + L_t u)(t, X_t)\right| dt \\ \le N\|(u_{nt} + L_t u_n) - (u_t + L_t u)\|_{L_p((0,T)\times\mathbb{R}^d)} \\ \le N\|u_n - u\|_{W_p^{1,2}((0,T)\times\mathbb{R}^d)} \to 0, \end{aligned}$$

where the first inequality is due to Theorem 2.3.4 in [6]. The lemma is proved. □

Now we can say that the martingale problem for L_t, where coefficients a_{jk} and b_j satisfying Assumption 3.1 and 3.1, is well-posed. The justification of this claim follows from Corollary 10.1.2 and Theorem 6.6.1 in [15]. Indeed, we may prove this as follows.

As noted earlier, we have the existence for the martingale problem for L_t. Thus by Corollary 10.1.2 in [15] we need only check the well-posedness of the martingale problem for L_t with *bounded* a_{jk} and b_j satisfying Assumption 3.1 and 3.2. In that case by Theorem 6.6.1 (and Exercise 6.7.4) in [15] it is enough to show that, for each $(s,x) \in [0,\infty)\times\mathbb{R}^d$, there exist coefficients $\tilde{a}_{jk}$ and $\tilde{b}_j$ defined on $[0,\infty)\times\mathbb{R}^d$

such that

(i) $\tilde{a}_{jk} = a_{jk}$ and $\tilde{b}_j = b_j$ in a neighborhood of (s, x),
(ii) the martingale for $\tilde{L}_t$ is well-posed, where

$$\tilde{L}_t = \frac{1}{2}\tilde{a}_{jk}(t, \cdot)\frac{\partial^2}{\partial x_j \partial x_k} + \tilde{b}_j(t, \cdot)\frac{\partial}{\partial x_j}.$$

Note that $\{\gamma_j, j \in I\}$ has no limit points. Thus, for points $(s, x) \in [0, \infty) \times \mathbb{R}^d$, where $x_1 \notin \{\gamma_j, j \in I\}$, we can find coefficients $\tilde{a}_{jk}$ and $\tilde{b}_j$ satisfying the above two conditions. Especially, (ii) is satisfied by results in Ch. 7 in [15].

For points $(s, x) \in [0, \infty) \times \mathbb{R}^d$, where $x_1 \in \{\gamma_j, j \in I\}$ (say, for example, $x_1 = 0$ and $0 \in \{\gamma_j, j \in I\}$), we find coefficients $\tilde{a}_{jk}$ and $\tilde{b}_j$ defined on $[0, \infty) \times \mathbb{R}^d$ such that they satisfy Assumption 2.1 and 2.2 in Section 2 as well as (i) in the above. In fact, we may set

$$\tilde{a}_{jk} = \varphi\, a_{jk} + (1 - \varphi)\delta_{jk}, \quad \tilde{b}_j = b_j,$$

where φ is a non-negative infinitely differentiable function defined on $\mathbb{R}^{d+1}$ such that $\varphi = 1$ in a sufficiently small neighborhood of $(s, 0, x')$ and supp φ dose not intersect $\{(t, \gamma_k, x') : (t, x') \in [0, \infty) \times \mathbb{R}^{d-1}\}$ for all $\gamma_k \in \{\gamma_j, j \in I\} \setminus \{0\}$. Now we prove (ii), i.e., the well-posedness of the martingale problem for $\tilde{a}_{jk}$ and $\tilde{b}_j$, using Lemma 3.4. Therefore, we have proved

Theorem 3.5. *Under Assumption* 3.1 *and* 3.2*, the martingale problem for L_t, where L_t is as in* (3.1)*, is well-posed.*

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